Stable ∞-categories in algebraic geometry

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This is the second volume of the 'Derived Categories in Algebraic Geometry' series. It's focus will be threefold:

1. Develop the theory of stable ∞-categories in the sense of Lurie's Higher Algebra [HA], using Cisinski's framework in [HCHA] for higher category theory as a starting point.
2. Extend the standard theory in Section 2 of the first volume [DcAG] to the derived ∞-category of quasicoherent sheaves on a scheme, and discuss enhanced derived-categorical invariants in this setting.
3. Develop and discuss the theory of stacks, derived algebraic spaces, and derived stacks using ∞-categories.

1 The theory of ∞-categories

This section will be focused on reviewing the major constructions in [HCHA] related to the standard yoga of ∞-categories. Heavy citations are given to Cisinski's book, and proofs are often not fully present or self-contained. The point will be to review enough ∞-categorical material to proceed to the definition and constructions involved in stable ∞-category theory. Out of principle, we skip combinatorial arguments.

Remark 1.0.1 Our notation will differ from Cisinski's and Lurie's, though we use more of the latter than the former.

1. We write the simplex category as $\mathbf{\Delta}$, and the category of simplicial sets as $\mathbf{sSet} = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$. We try to write 'big' 1-categories in bold, and small 1-categories as $A, B, \ldots$. $\mathcal{S}$ denotes the ∞-category of spaces.

2. General simplicial sets, or things which are supposed to be Kan complexes, are denoted by $X, Y, \ldots$, while ∞-categories are denoted by $\mathcal{C}, \mathcal{D}, \ldots$. In either case we denote objects/simplices by lowercase letters.

3. The naive mapping space between two objects of an ∞-category is written $\mathcal{C}(x, y)$, the smart mapping space is written $\mathrm{Map}_{\mathcal{C}}(x, y)$. The internal Hom in simplicial sets is written $\underline{\mathrm{Hom}}(X, Y)$. The core of an ∞-category is written $\mathcal{C}^{\simeq}$. We also write $\mathrm{Map}_X(F, G)$ for the relative mapping space when X is a simplicial set and F, G are objects of $\mathbf{sSet}/X$. TODO: Fix Map_C notation

4. We denote the major model category structures by

$$\mathbf{sSet}_{\mathrm{KQ}} \qquad \mathbf{sSet}_{\mathrm{Joy}} \qquad (\mathbf{sSet}/X)_{\mathrm{con}} \qquad (\mathbf{sSet}/X)_{\mathrm{cov}},$$

   i.e. the Kan-Quillen model structure, Joyal model structure, and contravariant and covariant model structures on the slice.

5. In this section, the word 'category' refers to a 1-category, and we use ∞-category explicitly. After this section, 'category' will mean ∞-category and we say 1-category explicitly.

6. We refer to the entire package of [HCHA, Section 2.4] as the *Cisinski model structure* associated to a category $\mathrm{PSh}(A)$ of presheaves over a small category, equipped with an exact cylinder I and a class of I-anodyne extensions.

7. If M is a model category, then we write $\mathrm{Ho}(M)$ for its homotopy category. In contrast, if $\mathcal{C}$ is an ∞-category then we write $\mathrm{h}\mathcal{C}$ for its homotopy category, following Lurie in [HTT]. If X is an arbitrary simplicial set, we write $\tau(X)$ for the left adjoint to the nerve functor $\mathbf{Cat} \to \mathbf{sSet}$, so $\tau(\mathcal{C}) = \mathrm{h}\mathcal{C}$ in the case of an ∞-category.

Preparation for stable ∞-categories will need a good working knowledge of the ∞-categorical analogues of standard category-theoretic constructions (i.e. (co)limits and adjunctions), and ∞-categorical localization.

Remark 1.0.2 We write $\mathcal{H}_{\mathrm{Joy}} := \mathrm{Ho}(\mathbf{sSet}_{\mathrm{Joy}})$ and $\mathcal{H} := \mathrm{Ho}(\mathbf{sSet}_{\mathrm{KQ}})$. By [HCHA, Prop 3.3.6], $\mathrm{Id}\colon \mathbf{sSet}_{\mathrm{Joy}} \to \mathbf{sSet}_{\mathrm{KQ}}$ sends categorical anodyne extensions to anodyne extensions, so it follows from [HCHA, Prop. 2.4.40] that it is a left Quillen functor. In particular it preserves weak equivalences, and induces the adjunction

$$(-)^{\mathrm{gpd}}\colon \mathcal{H}_{\mathrm{Joy}} \rightleftarrows \mathcal{H} : i.$$

We usually regard the objects of $\mathcal{H}_{\mathrm{Joy}}$ and $\mathcal{H}$ as being ∞-categories and ∞-groupoids respectively. We recall that the two model structures have shared model cylinders $J = \Delta^1[f^{-1}]$, thus $\mathrm{Hom}_{\mathcal{H}}(A, X) = \mathrm{Hom}_{\mathcal{H}_{\mathrm{Joy}}}(A, X)$ whenever X is a Kan complex. In particular, i is fully faithful. We also often regard $(-)^{\mathrm{gpd}}$ as Ex^{∞}.

Remark 1.0.3 The term 'category'/'1-category' loosely means a conventional category, and we usually omit nerve notation $N(-)$ unless there is a good reason to keep it. However, when we want to consider 1-categories in the context of ∞-categories or simplicial sets we use the following observation: The following are equivalent for an ∞-category $\mathcal{C}$:

(a) There exists a category C and an equivalence of ∞-categories $\mathcal{C} \simeq N(C)$.

(b) The unit map $\mathcal{C} \to N(\mathrm{h}\mathcal{C})$ is an equivalence of ∞-categories.

(c) The unit map $\mathcal{C} \to N(\mathrm{h}\mathcal{C})$ is a trivial fibration.

$(a) = (b)$ comes from naturality of the unit, and the fact that a map $N(C) \to N(D)$ is an equivalence of ∞-categories iff the corresponding map $C \to D$ is an equivalence of categories ([HCHA, Rmk. 1.6.11]). $(b) = (c)$ follows from the fact that $\mathcal{C} \to N(\mathrm{h}\mathcal{C})$ is always a (conservative) isofibration: it is an inner fibration as a map from a 1-category to an ∞-category ([HCHA, Ex. 3.2.7]), and it is conservative and surjective on morphisms, hence it is an isofibration. Thus it is a Joyal fibration by [HCHA, Thm. 3.6.1] and the equivalence follows.

Thus we are justified in defining a 1-category in this context to be an ∞-category satisfying either of the equivalent conditions above. Especially since this map is a trivial fibration, this condition passes well between model-theoretic notions much better than, say, arbitrary equivalences of categories. It is clear that the corresponding statement holds if we replace ∞-category with ∞-groupoid.

Example 1.0.4 The definitions above also have the following pleasant consequences:

1. An ∞-category $\mathcal{C}$ is a 1-category iff $\pi_n(\mathcal{C}(x, y), f) = 0$ for all $x, y \in \mathcal{C}$ and morphisms $f\colon x \to y$, and $n \geq 1$.
2. An ∞-groupoid $\mathcal{D}$ is a 1-groupoid iff $\pi_n(\mathcal{D}, x) = 0$ for all $x \in \mathcal{D}$ and $n \geq 2$.

For (1), note that nerves have discrete mapping spaces. Conversely, consider the map $\mathcal{C} \to N(\mathrm{h}\mathcal{C})$, and induces the map

$$\mathcal{C}(x,y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y) = \pi_0(\mathcal{C}(x,y))$$

by [HCHA, Prop. 3.7.2], regarding the target as discrete. This induces an isomorphism on all homotopy groups by assumption, hence is an equivalence of ∞-groupoids by [HCHA, Cor. 3.8.14]. Thus $\mathcal{C} \to N(\mathrm{h}\mathcal{C})$ is fully faithful, and it is the identity on objects, hence an equivalence ([HCHA, Thm. 3.9.7]). For (2), note that an ∞-groupoid which is a 1-category is a 1-groupoid, so it suffices to prove that the criteria of (1) holds. This follows immediately from the fact that $\pi_{k+1}(\mathcal{D},x) \cong \pi_k(\mathcal{D}(x,y),f)$, see e.g. Example 1.1.11. Lastly, the proof of (1) shows that there is a third well-defined notion we can set, which is that of an ∞-groupoid being discrete, which occurs when $\pi_i(\mathcal{D},x) = 0$ for all $i \geq 1$ and x, i.e. the map $\mathcal{D} \to N(\mathrm{h}\mathcal{D})$ gives an equivalence of $\mathcal{D}$ with a discrete groupoid.

Example 1.0.5 We also have decent behavior regarding pullbacks and 1-categories. First, suppose we have a Cartesian diagram in **sSet**

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{D}' \\ \downarrow & \lrcorner & \downarrow p \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array}$$

where $\mathcal{C},\mathcal{D},\mathcal{D}'$ are 1-categories and p is an isofibration. Then $\mathcal{C}' \to \mathcal{C}$ is an isofibration and thus $\mathcal{C}'$ is an ∞-category. Moreover, we have a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}' & \xrightarrow{p} & \mathcal{D} & \longleftarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{h}\mathcal{D}' & \xrightarrow{\mathrm{h}(p)} & \mathrm{h}\mathcal{D} & \longrightarrow & \mathrm{h}\mathcal{C} \end{array}$$

with vertical maps all trivial fibrations. It is clear that $\mathrm{h}(p)$ is also an isofibration of the underlying categories by definition, hence its nerve is an isofibration. Thus by [HCHA, Cor. 2.3.29], we deduce that

$$\mathcal{C}' \cong \mathcal{D}' \times_{\mathcal{D}} \mathcal{C} \to N(\mathrm{h}\mathcal{D}') \times_{N(\mathrm{h}\mathcal{D})} N(\mathrm{h}\mathcal{C}) \cong N(\mathrm{h}\mathcal{D}' \times_{\mathrm{h}\mathcal{D}} \mathrm{h}\mathcal{C})$$

is an equivalence of ∞-categories, so $\mathcal{C}'$ is a 1-category.

Next suppose that the same square above is a homotopy cartesian diagram in $\mathbf{sSet}_{\mathrm{Joy}}$, where all objects are ∞-categories, and $\mathcal{C},\mathcal{D},\mathcal{D}$ are 1-categories and p is arbitrary. Factor p as

$$\mathcal{D}' \xrightarrow{i} \widetilde{\mathcal{D}} \xrightarrow{q} \mathcal{D}$$

where i is a trivial cofibration and q is a fibration. Thus $\widetilde{D}$ is an ∞-category, and i is an equivalence of categories so $\widetilde{\mathcal{D}}$ is a 1-category. Moreover, we have weak equivalences

$$\mathcal{C}' \to \mathcal{C} \times^{\mathbf{R}}_{\mathcal{D}} \mathcal{D}' \leftarrow \mathcal{C} \times^{\mathbf{R}}_{\mathcal{D}} \widetilde{\mathcal{D}},$$

where the RHS can be computed as $\mathcal{C} \times_{\mathcal{D}} \widetilde{\mathcal{D}}$ since q is a fibration of fibrant objects ([HCHA, Cor. 2.3.28]). This is a 1-category by the isofibration case, so we conclude that $\mathcal{C}'$ is a 1-category. Note that:

1. In the second case, we had to assume $\mathcal{C}'$ is an ∞-category, otherwise we have the conclusion that any Joyal fibrant replacement of $\mathcal{C}'$ is a 1-category.
2. The second case is especially nice because Joyal homotopy pullbacks are ∞-categorical pullbacks in $\mathbf{Cat}_\infty$, so the 1-categories form a full replete subcategory of that category closed under pullbacks. It is also closed under all such limits, but we haven't demonstrated this yet.

There is also the following classical lifting property which doesn't strictly appear in [HCHA].

Lemma 1.0.6 *The following conditions are equivalent for a morphism $p\colon X \to Y$ of simplicial sets:*

(*i*) *p has the RLP against $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n > 0$.*

(*ii*) *In the factorization $X \to Y_{p(X_0)} \hookrightarrow Y$, where $Y_{p(X_0)}$ is the full simplicial subset spanned by $p(X_0) \subset Y_0$, $X \to Y_{p(X_0)}$ is a trivial fibration.*

(*iii*) *p has the RLP against all monomorphisms which are bijective on objects.*

If p satisfies any of these equivalent conditions then it is called a positive fibration. In particular:

1. *The smallest saturated class containing $\{\partial\Delta^n \to \Delta^n : n > 0\}$ is the class of monomorphisms which are bijective on objects, and it is equal to the set with the LLP against the positive fibrations.*
2. *An inclusion $Y_S \hookrightarrow Y$ of a full simplicial subset is a positive fibration.*

Proof (*iii*) $\Rightarrow$ (*i*) is clear. (*i*) $\Rightarrow$ (*ii*) Since $Y_{p(X_0)} \hookrightarrow Y$ is a monomorphism, this implies that $X \to Y_{p(X_0)}$ has the RLP against $\partial\Delta^n \to \Delta^n$ for all $n > 0$. On the other hand it is surjective on objects, hence lifting against $\varnothing \to \Delta^0$.

(*ii*) $\Rightarrow$ (*iii*) Let $A \to B$ be a monomorphism containing all vertices and suppose

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{\beta} & Y \end{array}$$

is a lifting problem. For each $\sigma \in B_n$, we have $\beta(\sigma)(k) = \beta(\sigma(k)) = \beta(i(a)) = p\alpha(a)$ for some $a \in A_0$. Thus β factors as $B \to Y_{p(X_0)} \hookrightarrow Y$, but then the trivial fibration $X \to Y_{p(X_0)}$ lifts against $A \to B$.

Consequence (1) is clear from the small object argmuent and the fact that the class of monomorphisms containing all vertices is a saturated class, as the intersection of two saturated classes. (2) it clear from (*ii*) because $1_{Y_S}\colon Y_S \to Y_S$ is a trivial fibration. □

Lemma 1.0.7 *Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories.*

1. *For any monomorphism $A \hookrightarrow B$ which is bijective on objects, the induced map*
$$\underline{\mathrm{Hom}}(B, \mathcal{C}) \to \underline{\mathrm{Hom}}(A, \mathcal{C}) \times_{\underline{\mathrm{Hom}}(A,\mathcal{D})} \underline{\mathrm{Hom}}(B, \mathcal{D})$$
is an isofibration of ∞-categories.

2. *For objects x, y of $\mathcal{C}$, the induced map*
$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}\big(p(x), p(y)\big)$$
is a Kan fibration.

Proof (1) This map is an inner fibration between ∞-categories by [HCHA, Cor. 3.2.8]. There is a lifting correspondence of the form:

$$\begin{array}{ccc} A & \longrightarrow & h(\Delta^1, \mathcal{C}) \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & \mathcal{C} \times_{\mathcal{D}} h(\Delta^1, \mathcal{D}) \end{array} \qquad \begin{array}{ccc} \{1\} & \longrightarrow & \underline{\mathrm{Hom}}(B, \mathcal{C})^{\simeq} \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 & \longrightarrow & \underline{\mathrm{Hom}}(A, \mathcal{C})^{\simeq} \underset{\underline{\mathrm{Hom}}(A,\mathcal{D})^{\simeq}}{\times} \underline{\mathrm{Hom}}(B, \mathcal{D})^{\simeq} \end{array}$$

By [HCHA, Thm. 3.5.9], the right-hand map of the left-hand square is a positive fibration, so by our assumption squares of the form on the left admit lifts. Thus the right-hand square does too; but an inner fibration q is an isofibration iff $q^{\simeq}$ has the RLP against $\{1\} \to \Delta^1$.

(2) We have a cartesian diagram:

$$\begin{array}{ccccc} \mathrm{Map}_{\mathcal{C}}(x, y) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C}) & & \\ \downarrow & \lrcorner & \downarrow & & \\ \mathrm{Map}_{\mathcal{D}}\big(p(x), p(y)\big) & \longrightarrow & P & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{(x,y)} & \underline{\mathrm{Hom}}(\partial\Delta^1, \mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(\partial\Delta^1, \mathcal{D}) \end{array}$$

We just saw that $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{C}) \to P$ is an isofibration because $\partial\Delta^1 \to \Delta^1$ is bijective on objects. Thus the upper left-hand map is an isofibration between Kan complexes, hence a Kan fibration. □

Remark 1.0.8 Let $\mathcal{C}$ be an ∞-category. We make the convention that a *subcategory* or *∞-subcategory* of $\mathcal{C}$ is a simplicial subset $\mathcal{B} \subset \mathcal{C}$ such that $\mathcal{B} \to \mathcal{C}$ is an inner fibration. In particular it is an ∞-category. By the previous lemma,
$$\mathrm{Map}_{\mathcal{B}}(x, y) \hookrightarrow \mathrm{Map}_{\mathcal{C}}(x, y)$$
is a monomorphism and a Kan fibration, or in other words it is a union of connected components of the ∞-groupoid $\mathrm{Map}_{\mathcal{C}}(x, y)$. This is also equivalent to being a full subcategory inclusion of $\mathrm{Map}_{\mathcal{C}}(x, y)$ spanned by a set of morphisms $x \to y$ which is stable under the existence of 2-morphisms.

Here is an often-used lemma which is a simple consequence of the previous lemma.

Lemma 1.0.9 *Let $p\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration of ∞-categories, and $i\colon A \to B$ a monomorphism of simplicial sets which is bijective on objects. Then for any cartesian square*

$$\begin{array}{ccc} \mathcal{E}' & \longrightarrow & \underline{\mathrm{Hom}}(B, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{E} & \longrightarrow & \underline{\mathrm{Hom}}(A, \mathcal{C}) \underset{\underline{\mathrm{Hom}}(A,\mathcal{D})}{\times} \underline{\mathrm{Hom}}(B, \mathcal{D}) \end{array}$$

in which $\mathcal{E}$ is an ∞-groupoid, the map $\mathcal{E}' \to \mathcal{E}$ is a Kan fibration of ∞-groupoids.

Proof Denote by P the lower right-hand corner of the square. The fact that $A \to B$ is surjective on objects directly implies that $\underline{\mathrm{Hom}}(B, \mathcal{C}) \to P$ is conservative. Since $\mathcal{E}$ is Kan, the map $\mathcal{E} \to P$ factors through $P^{\simeq}$, and we can form a diagram:

$$\begin{array}{ccccc} \mathcal{E}' & \dashrightarrow & \underline{\mathrm{Hom}}(B, \mathcal{C})^{\simeq} & \longrightarrow & \underline{\mathrm{Hom}}(B, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathcal{E} & \longrightarrow & P^{\simeq} & \longrightarrow & P \end{array}$$

The right-hand square being cartesian is an expression of conservativity, so there is a canonical map $\mathcal{E}' \to \underline{\mathrm{Hom}}(B, \mathcal{C})^{\simeq}$ making the left-hand square cartesian. The previous lemma shows that the right-hand map is an isofibration, so the middle vertical map is an isofibration between Kan complexes, hence a Kan fibration. Thus the same is true for the left-hand vertical map. □

Remark 1.0.10 This remark discusses canonical factorizations in the Joyal model structure. Though often we want to leave this as an abstract existence statement, sometimes it is nice to have concrete and functorial factorizations. Normally, we choose the following model: if $F\colon \mathcal{C} \to \mathcal{D}$ is a functor between ∞-categories, we have the factorization

$$\mathcal{C} \to \mathcal{C} \times_{\mathcal{D}, \mathrm{ev}_0} h(\Delta^1, \mathcal{D}) \xrightarrow{\mathrm{ev}_1} \mathcal{D}, \tag{$*$}$$

where $h(\Delta^1, \mathcal{D}) \subset \mathrm{Fun}(\Delta^1, \mathcal{D})$ is the full subcategory spanned by invertible morphisms in $\mathcal{D}$. It follows from [HCHA, Cor. 3.6.7] that the map

$$h(\Delta^1, \mathcal{D}) \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} \mathcal{D} \times \mathcal{D}$$

is a Joyal fibration, and the evaluations ev_0 and ev_1 are trivial fibrations ([HCHA, Cor. 3.5.10]). Therefore $\mathcal{C} \times_{\mathcal{D}, \mathrm{ev}_0} h(\Delta^1, \mathcal{D}) \to \mathcal{C}$ is a trivial fibration, and the first map of $(*)$ is a section of this, hence a trivial cofibration. Moreover, we have a cartesian diagram

$$\begin{array}{ccccc} \mathcal{C} \times_{\mathcal{D}, \mathrm{ev}_0} h(\Delta^1, \mathcal{D}) & \longrightarrow & \mathcal{C} \times \mathcal{D} & \longrightarrow & \mathcal{C} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ h(\Delta^1, Y) & \xrightarrow{(\mathrm{ev}_0, \mathrm{ev}_1)} & \mathcal{D} \times \mathcal{D} & \xrightarrow{\mathrm{pr}_0} & \mathcal{D} \end{array}$$

by the pasting lemma, and since $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ is a Joyal fibration, we conclude that $\mathcal{C} \times_{\mathcal{D}} h(\Delta^1, \mathcal{D}) \to \mathcal{D}$ is a Joyal fibration. Therefore $(*)$ provides a canonical factorization for F. We usually at least have this model in mind when taking homotopy fibers, and we will generally use the notion $\mathrm{hofib}_y(F)$ to mean the pullback of $\{y\} \to \mathcal{D}$ along the second map of $(*)$. In particular, its objects are pairs (x, γ) where x is an object of $\mathcal{C}$ and γ is an invertible map $F(x) \xrightarrow{\sim} y$. We e.g. often use that in the case F is an equivalence, this is a contractible ∞-groupoid.

More generally, given a cospan $\mathcal{C} \to \mathcal{D} \leftarrow \mathcal{E}$ of ∞-categories, we can use this factorization to compute the homotopy pullback

$$\mathcal{C} \times^{\mathbf{R}}_{\mathcal{D}} \mathcal{E} = (\mathcal{C} \times_{\mathcal{D},\mathrm{ev}_0} h(\Delta^1, \mathcal{D})) \times_{\mathcal{D},\mathrm{ev}_1} \mathcal{E} = (\mathcal{C} \times \mathcal{E}) \times_{\mathcal{D}\times\mathcal{D}} h(\Delta^1, \mathcal{D})$$

in $\mathbf{sSet}_{\mathrm{Joy}}$. The objects of this model consist of triples (x, z, γ) where $x \in \mathcal{C}, z \in \mathcal{E}$, and γ is an equivalence $F(x) \xrightarrow{\sim} G(z)$ in $\mathcal{D}$. The previous paragraphs can be repeated using $\underline{\mathrm{Hom}}(J, \mathcal{D})$ or $\underline{\mathrm{Hom}}(I_{\simeq}, \mathcal{D})$ instead of $h(\Delta^1, \mathcal{D})$; in this case the model additionally remembers either a choice of right and left inverse to γ, or even all of the higher homotopy data associated to the inverse, respectively.

1.1 Homotopy theory of ∞-categories

The references for this section are [HCHA, 4.1-4.5,5.2,5.4,5.8]. We take special care to write down arguments when Cisinski's is extra terse or missing a step. Main results consist of characterizations of terminal objects, Quillen's theorem A, the construction of the ∞-category of spaces $\mathcal{S}$, and the simplified version of straightening/unstraightening for presheaves of spaces.

1.1.1 Right fibrations and terminal objects

In what follows we will make many statements depending on a parity (i.e. right/left, terminal/initial, etc.), and we always assume the obvious op definition is made as well. We use various results of [HCHA, Ch. 1-3] without mention; e.g. we will use ∞-groupoid and Kan complex interchangeably. For maps between ∞-groupoids, we use 'equivalence', 'homotopy equivalence', and 'weak homotopy equivalence' interchangeably. For maps between ∞-categories, we use 'equivalence' and 'weak categorical equivalence' interchangeably.

Recall that a right fibration in **sSet** is a morphism with the RLP against horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 < k \leq n$, and a right anodyne extension is a map with the LLP against right fibrations. Thus a right fibration is also a map with the RLP against right anodyne extensions, and by the small object argument, the right anodyne extensions form the smallest saturated class containing these inclusions. If $f\colon X \to Y$ is a right fibration and Y is a Kan complex, then f is in fact a Kan fibration ([HCHA, Prop. 3.5.5]). Thus if f is arbitrary, for each $y \in Y$, the fiber $X \times_Y \{y\} \to \{y\}$ is a right fibration, and thus $X \times_Y \{y\}$ is a Kan complex.

Remark 1.1.1 We recall how the contravariant model structure is defined on the slice category $\mathbf{sSet}/C$. A right fibrant object over C is an object $F \in \mathbf{sSet}/C$ such that $F \to C$ is a right fibration.

1. We identify $\mathbf{sSet}/C$ as the presheaf category of the category of elements $\mathbf{\Delta}/C$. It is easy to see that a map in $\mathbf{sSet}/C$ has the RLP against monomorphisms of the slice iff it is a trivial fibration inside $\mathbf{sSet}$.
2. Denote by $I_\sim$ the (nerve of the) walking isomorphism, i.e. the 1-category $\{0 \rightleftarrows 1\}$ where both compositions are the identity. Since this is a contractible groupoid, $I_\sim \to \Delta^0$ is a trivial fibration, so in particular $I_\sim \times X \to X$ is a weak equivalence in either of the model structures on $\mathbf{sSet}$, for all X. It gives an exact cylinder, and forms an interval object which detects homotopies in either of the model structures.
3. The *contravariant model structure on* $\mathbf{sSet}/C$ is the Cisinski model structure (via the identification of (1)) associated to $I_\sim$ and the smallest class of $I_\sim$-anodyne extensions containing the right horn inclusions. This class of $I_\sim$-anodyne extensions in fact coincides with the right anodyne extensions ([HCHA, 4.1.7]), so it follows from the Cisinski model construction that:
   (a) The cofibrations are the monomorphisms; the fibrant objects are the right fibrant presheaves over C.
   (b) The fibrations between fibrant objects are maps over C which are right fibrations (of $\mathbf{sSet}$), and the trivial fibrations are maps over C which are trivial fibrations (of $\mathbf{sSet}$).
   (c) Weak equivalences are detected by $[Y, F]_{I_\sim} \to [X, F]_{I_\sim}$ being an isomorphism for all right fibrant F, i.e. equivalence classes under $I_\sim$-homotopy, and weak equivalences of right fibrants is $I_\sim$-homotopy equivalence.
4. We write
$$\mathrm{RFib}(C) \coloneqq \mathrm{Ho}((\mathbf{sSet}/C)_{\mathrm{con}}),$$
and usually assume the presentation where $\mathrm{RFib}(C)$ consists of right fibrant presheaves over C, and $\mathrm{Hom}_{\mathrm{RFib}(C)}(F, G) = [F, G]_{I_\sim}$.

As explained, the corresponding definitions are made for the covariant model structure on the slice $\mathbf{sSet}/C$ via left fibrations, and with $\mathrm{LFib}(C)$ its homotopy category.

Example 1.1.2 If $C = X$ is an ∞-groupoid, then the contravariant model structure on $\mathbf{sSet}/X$ coincides with the slice model structure of $\mathbf{sSet}_{\mathrm{KQ}}$, because right fibrations to a Kan complex are Kan fibrations between Kan complexes. If $C = \mathcal{C}$ is an ∞-category, then right fibrant objects are in particular inner fibrations over $\mathcal{C}$, and hence ∞-categories themselves. Moreover, the following are equivalent for a morphism $\mathcal{D} \to \mathcal{D}'$ of right fibrants over $\mathcal{C}$:

(*i*) $\mathcal{D} \to \mathcal{D}'$ is a weak equivalence in $(\mathbf{sSet}/\mathcal{C})_{\mathrm{con}}$.

(*ii*) $\mathcal{D} \to \mathcal{D}'$ is a weak equivalence in the slice model category $\mathbf{sSet}_{\mathrm{Joy}}/\mathcal{C}$.

(*iii*) $\mathcal{D} \to \mathcal{D}'$ is an equivalence of ∞-categories.

Indeed, (*iii*) is equivalent to being a weak equivalence in $\mathbf{sSet}_{\mathrm{Joy}}$ ([HCHA, Cor. 3.6.6]), which is equivalent to (*ii*) by definition of the slice model structure. Since $\mathcal{D} \to \mathcal{C}$ of ∞-categories is a right fibration, it is a conservative isofibration ([HCHA, Prop. 3.4.8]), and hence a Joyal fibration ([HCHA, Thm. 3.6.1]). Thus $\mathcal{D}$ and $\mathcal{D}'$ are fibrant objects in $\mathbf{sSet}_{\mathrm{Joy}}/\mathcal{C}$. By Remark 1.1.1(2), $I_\sim \times \mathcal{C} \to \mathcal{C}$ is a trivial fibration, and so $I_\sim \times \mathcal{C}$ defines an interval object compatible with both model structures. Thus equivalence between the ∞-categories is equivalent $(I_\sim \times \mathcal{C})$-homotopy equivalence in $\mathbf{sSet}/\mathcal{C}$ in both (*i*) and (*ii*).

Let C be arbitrary and $F, G \in \mathbf{sSet}/C$. We define $\underline{\mathrm{Hom}}_C(F, G)$ as the pullback:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_C(F,G) & \hookrightarrow & \underline{\mathrm{Hom}}(F,G) \\ \downarrow & \lrcorner & \downarrow \\ \{F \to C\} & \hookrightarrow & \underline{\mathrm{Hom}}(F,C) \end{array}$$

This is evidently bifunctorial $(\mathbf{sSet}/C)^{\mathrm{op}} \times \mathbf{sSet}/C \to \mathbf{sSet}$. Since $\Delta^0 = \{F \to C\}$ is the terminal object, it is invariant under taking any limits, so we deduce that $\underline{\mathrm{Hom}}_C(-,-)$ preserves limits in both variables. If G is right fibrant over C, then the right-hand (and thus left-hand) map is a right fibration by [HCHA, Prop. 3.4.4], meaning $\underline{\mathrm{Hom}}_C(F, G)$ is a Kan complex. If $c \in C$ is an object, we write $c\colon \Delta^0 \to C$ for the corresponding object of $\mathbf{sSet}/C$. By definition, $\underline{\mathrm{Hom}}_C(c, G) = \{c\} \times_C G = G_c$ is the fiber at c.

Lemma 1.1.3 *Let $G \in \mathbf{sSet}/C$ be right fibrant, suppose $f\colon F \to F'$ is a morphism in $\mathbf{sSet}/C$, and consider the induced map*

$$f^*\colon \underline{\mathrm{Hom}}_C(F', G) \to \underline{\mathrm{Hom}}_C(F, G)$$

of ∞-groupoids.

(a) If f is a monomorphism then f^* is a Kan fibration.

(b) If f is a right anodyne extension, then f^* is a trivial fibration.

(c) If f is a weak equivalence of $(\mathbf{sSet}/C)_{con}$, then f^* is an equivalence of ∞-groupoids.

Proof (a) Since $\underline{\mathrm{Hom}}_C(F, G)$ is a Kan complex, it suffices to show f^* is a right Kan fibration. It follows from Fubini's theorem that f^* is a pullback of

$$\underline{\mathrm{Hom}}(F', G) \to \underline{\mathrm{Hom}}(F, G) \underset{\underline{\mathrm{Hom}}(F,C)}{\times} \underline{\mathrm{Hom}}(F', C),$$

which by [HCHA, Prop. 3.4.4] is a right Kan fibration. (b) If moreover f is right anodyne, then the same proposition shows that the map above is a trivial fibration.

(c) We apply [HCHA, Prop. 2.4.40] to $\underline{\mathrm{Hom}}_C(-,G)\colon \mathbf{sSet}/C \to \mathbf{sSet}^{\mathrm{op}}$, endowing the target with the opposite Kan-Quillen model structure. Then this functor preserves colimits, sends monomorphisms to cofibrations, and sends the specified class of $I_\sim$-anodyne extensions (= right anodyne extensions) to trivial cofibrations. Thus it is a left Quillen functor, and hence preserves trivial cofibrations, and thus preserves weak equivalences between cofibrant (= all) objects by Ken Brown's lemma. □

Corollary 1.1.4 *Let C be a simplicial set. The natural map*

$$\pi_0\big(\underline{\mathrm{Hom}}_C(F,G)\big) \to \mathrm{Hom}_{\mathrm{RFib}(C)}(F,G)$$

is an isomorphism for each $F \in \mathbf{sSet}/C$ and right fibrant G over C.

Remark 1.1.5 Let us recall how to construct this map. Our convention is that if X is a simplicial set then $\pi_0(X)$ is the coequalizer $X_1 \rightrightarrows X_0 \to \pi_0(X)$. Applied to hom-spaces, this is induced by the natural localization functor

$$\mathrm{Hom}_{\mathbf{sSet}/C}(F,G) \to \mathrm{Hom}_{\mathrm{RFib}(C)}(F,G).$$

We have to show that given $\Delta^1 \times F \to G$ satisfying the additional condition that the diagram

$$\begin{array}{ccc} \Delta^1 \times F & \longrightarrow & G \\ \downarrow & & \downarrow \\ F & \longrightarrow & C \end{array}$$

commutes, the two composites $i_0, i_1\colon F \to \Delta^1 \times F$ coincide in $\mathrm{RFib}(C)$. Indeed, these are sections of $\Delta^1 \times F \to F$, so it suffices to show that this is a weak equivalence, for which it in fact suffices to see that $i_1\colon F \to \Delta^1 \times F$ is a right anodyne extension. This follows from the fact that $i_1\colon \Delta^0 \to \Delta^1$ is a right anodyne extension and [HCHA, Prop. 3.4.3].

Proof (of Cor. 1.1.4) Choose a cylinder $F \sqcup F \hookrightarrow IF \to F$ in $(\mathbf{sSet}/C)_{\mathrm{con}}$ where the first map is mono and $IF \to F$ is a weak equivalence. Applying $\underline{\mathrm{Hom}}_C(-,G)$ where G is right fibrant, we obtain the diagonal

$$\underline{\mathrm{Hom}}_C(F,G) \to \underline{\mathrm{Hom}}_C(IF,G) \to \underline{\mathrm{Hom}}_C(F,G) \times \underline{\mathrm{Hom}}_C(F,G)$$

where the first map is a weak equivalence and the second is a Kan fibration, by Lemma 1.1.3(a,b). Therefore this is a cocylinder in the Kan-Quillen model structure. On one hand, we have

$$\pi_0\,\underline{\mathrm{Hom}}_C(F,G) = \mathrm{Hom}_{\mathcal{H}}(\Delta^0, \underline{\mathrm{Hom}}_C(F,G)) = \mathrm{Hom}_{\mathbf{sSet}/C}(F,G)/\!\sim$$

by [HCHA, Prop 3.1.31] since $\underline{\mathrm{Hom}}_C(F,G)$ is a Kan complex, where $\sim$ is the relation induced by $\underline{\mathrm{Hom}}_C(IF,G)$-homotopy of maps. But by construction this is the same as the relation of IF-homotopy on $\mathrm{Hom}_{\mathbf{sSet}/C}(F,G)$, so again by fibrancy of G we conclude this is equal to $\mathrm{Hom}_{\mathrm{RFib}(C)}(F,G)$ □

Remark 1.1.6 We will use the following trick a few times in some proofs. We use the language in [HCHA, Sec. 1.3] of Eilenberg-Zilber categories, and our case will be either $A = \mathbf{\Delta}$ or the category of elements $A = \mathbf{\Delta}/C$ over a presheaf. Suppose we have a model category M, a natural transformation $F \to G$ of colimit-preserving functors $\widehat{A} \to M$, and we want to test if this is componentwise a weak equivalence. Make the following assumptions:

1. $F(X) \to G(X)$ is a weak equivalence for each representable X.
2. F and G send all monomorphisms to cofibrations between cofibrant objects.

Then $F \to G$ is a componentwise weak equivalence. Indeed, by [HCHA, Cor. 1.3.10] it suffices to see that the class of presheaves where it is a weak equivalence is saturated by monomorphisms. By (2), for all X, $F(X)$ and $G(X)$ are cofibrant, so by [HCHA, Cor. 2.3.16] this class is closed under coproducts. By (2) and [HCHA, Cor. 2.3.19], it is closed under transfinite compositions of monomorphisms. By (2) and [HCHA, Cor. 2.3.29], it is closed under pushouts. This proves that it is saturated under monomorphisms. *Note:* In this case of $\mathbf{\Delta}$, (1) means checking on Δ^n for all n, and in the case of $\mathbf{\Delta}/C$, (1) means checking on all maps $\Delta^n \to C$, i.e. all simplices of C.

Corollary 1.1.7 *Let $F \to G$ be a morphism of right fibrant presheaves over C. The following conditions are equivalent:*

(i) *$F \to G$ is a weak equivalence in $(\mathbf{sSet}/C)_{\mathrm{con}}$.*

(ii) *$F \to G$ is an isomorphism in $\mathrm{RFib}(C)$.*

(iii) *For each $c \in C$, the induced map on the fibers*

$$F_c \to G_c$$

is an equivalence of ∞-groupoids.

(iv) *For each $X \in \mathbf{sSet}/C$, the map*

$$\underline{\mathrm{Hom}}_C(X, F) \to \underline{\mathrm{Hom}}_C(X, G)$$

is an equivalence of ∞-groupoids.

(v) *For each right fibrant presheaf H over C, the map*

$$\underline{\mathrm{Hom}}_C(G, H) \to \underline{\mathrm{Hom}}_C(F, H)$$

is an equivalence of ∞-groupoids.

Proof $(i) = (ii)$ as F and G are fibrant objects of $(\mathbf{sSet}/C)_{\mathrm{con}}$. $(i) \Rightarrow (v)$ by Lemma 1.1.3, and by taking π_0 of (v), we see $(v) \Rightarrow (ii)$ by the Yoneda lemma.

Instead taking π_0 of (iv) shows that $(iv) \Rightarrow (ii)$ by the Yoneda lemma. Conversely, by the pasting lemma and definition, and map $H \to H'$ gives a

Cartesian square:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_C(X,H) & \hookrightarrow & \underline{\mathrm{Hom}}(X,H) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}_C(X,H') & \hookrightarrow & \underline{\mathrm{Hom}}(X,H') \end{array}$$

If $H \to H'$ is a trivial fibration, then $\underline{\mathrm{Hom}}(X,H) \to \underline{\mathrm{Hom}}(X,H')$ is a trivial fibration (because $(-) \times H'$ preserves monomorphisms and [HCHA, Prop. 2.1.12]), so the left-hand vertical map is a trivial fibration. Thus $\underline{\mathrm{Hom}}_C(X,-)$ preserves trivial fibrations, hence weak equivalences between fibrant objects by Ken Brown's lemma. Thus $(i) \Rightarrow (iv)$.

$(iv) \Rightarrow (iii)$ holds by taking $X = c$ for each $c \in C_0$. Conversely, we are going to apply Remark 1.1.6 to the natural transformation $\underline{\mathrm{Hom}}_C(-,F) \to \underline{\mathrm{Hom}}_C(-,G)$ of colimit-preserving functors $\mathbf{sSet}/C \to \mathbf{sSet}^{\mathrm{op}}_{\mathrm{KQ}}$. Indeed, by Lemma 1.1.3 both functors send monomorphisms to Kan fibrations between Kan complexes. Thus we reduce to the case that $X = s\colon \Delta^n \to C$ is a simplex of C. By naturality, we have a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_C(s,F) & \longrightarrow & F_{s(n)} \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_C(s,G) & \longrightarrow & G_{s(n)} \end{array}$$

of ∞-groupoids induced by the map $\{n\} = \Delta^0 \to \Delta^n$ and $F \to G$. This former map is a right anodyne extension ([HCHA, Lem. 4.1.15]), hence both horizontal maps are trivial fibrations by Lemma 1.1.3. The right-hand map is an equivalence, so we deduce that the left-hand map is an equivalence. □

Example 1.1.8 Let $p\colon X \to Y$ be a right fibration of simplicial sets. Then p is a trivial fibration iff the ∞-groupoids $p^{-1}(y) = X_y$ are contractible for all $y \in Y$. Indeed, if this holds then $X \to Y$ is a morphism of right fibrant objects over Y and $X_y \to \{y\}$ is an equivalence for all y, so $X \to Y$ is a weak contravariant equivalence over Y. Thus it is a trivial fibration.

Our next goal is to study the fat join and fat slices. Observe that if D is a 1-category and $d' \in D$, then the fiber of $D/d' \to D$ over d is the class of objects of the form $(d, d \to d')$, i.e. $\mathrm{Hom}_D(d,d')$. We prove the ∞-categorical analogue of this, which holds up to canonical equivalence.

Definition 1.1.9 Let X and Y be simplicial sets. We define $X \diamond Y$ by the pushout square

$$\begin{array}{ccc} X \times \partial\Delta^1 \times Y & \longrightarrow & X \sqcup Y \\ \downarrow & \ulcorner & \downarrow \\ X \times \Delta^1 \times Y & \longrightarrow & X \diamond Y \end{array}$$

where the upper map is induced by the two projections

$$X \times \partial\Delta^1 \times Y = (X \times Y) \sqcup (X \times Y) \xrightarrow{p_X \sqcup p_Y} X \sqcup Y.$$

This is evidently functorial as $\mathbf{sSet} \times \mathbf{sSet} \to \mathbf{sSet}$, and comes equipped with a natural map from the functor $(X, Y) \mapsto X \sqcup Y$. We call this construction the *fat join*.

Note that $(-) \times T \times Y$ preserves colimits, and $(-) \sqcup Y$ preserves connected colimits, so it is clear that $X \diamond (-)$ preserves connected colimits. It also preserves monomorphisms since products do, and pushouts of monomorphisms are monomorphisms in $\mathbf{sSet}$. On the other hand, $\varnothing \diamond Y$ is the pushout of $\varnothing \to Y$ along the identity, hence is just Y. Therefore, if we write this as a functor

$$(-) \diamond Y \colon \mathbf{sSet} \to Y \backslash \mathbf{sSet},$$

then it preserves colimits, and hence has a right adjoint which we denote by $(-)//Y \colon Y \backslash \mathbf{sSet} \to \mathbf{sSet}$. If T is a sheaf with a map $t \colon Y \to T$, the explicit description of adjoints from presheaf categories says that

$$(T//Y)_n = \mathrm{Hom}_{Y \backslash \mathbf{sSet}}(\Delta^n \diamond Y, T),$$

and we call the construction the *fat slice*.

Example 1.1.10 In the construction above, take $Y = \Delta^0$, so that the map to T is just a vertex $t_1 \in T_0$. A map $X \to T//t_1$ corresponds to a map $X \diamond \Delta^0 \to T$ sending the vertex (from $X \sqcup \Delta^0$) to t_1. In other words, this is a map $X \times \Delta^1 \to T$ such that the restriction to $X \times 1$ is the constant map at t_1. By adjunction, this universal property implies that we have a cartesian square

$$\begin{array}{ccc} T//t_1 & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1, T) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle ev_1} \\ \Delta^0 & \xrightarrow{t_1} & T \end{array}$$

The counit map $T//t_0 \to T$ can also be described as the one given by composing the top map above with $\underline{\mathrm{Hom}}(\Delta^1, T) \xrightarrow{ev_0} T$. Moreover, we have another cartesian pasting diagram as follows:

$$\begin{array}{ccccc} T(t_0, t_1) & \longrightarrow & T//t_1 & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1, T) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{t_0} & T & \xrightarrow{(t_1, 1)} & T \times T \\ & & \downarrow & \lrcorner & \downarrow{\scriptstyle p_1} \\ & & \Delta^0 & \xrightarrow{t_0} & T \end{array}$$

On the right-hand side, the bottom and outer diagrams are pullbacks, hence the upper-right diagram is cartesian. On the top, the outer and right-hand diagrams are pullbacks, so there is a canonical map $T(t_0, t_1) \to T//t_1$ such

that the left-hand diagram is cartesian. In particular, $T(t_0, t_1)$ is the fiber of $T//t_1 \to T$, so we want to study this map $T//t_1 \to T$ in more detail.

Example 1.1.11 Here's a simple application of the cartesian diagram in the previous example; suppose $\mathcal{D}$ is an ∞-groupoid, then by [HCHA, Cor. 3.1.7] the source map $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \to \mathcal{D}$ is a trivial fibration, so $\mathcal{D}//y$ is a contractible ∞-groupoid for any $y \in \mathcal{D}$. By [HCHA, Cor. 3.1.7], $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ is a Kan fibration, so $\mathcal{D}//y \to \mathcal{D}$ is a Kan fibration. For any $f\colon x \to y$ in $\mathcal{D}$, the long exact sequence of homotopy groups gives

$$0 = \pi_{i+1}(\mathcal{D}//y, f) \to \pi_{i+1}(\mathcal{D}, x) \to \pi_i(\mathcal{D}(x, y), f) \to \pi_i(\mathcal{D}//y, f) = 0$$

for $i \geq 0$, thus $\pi_{i+1}(\mathcal{D}, x) \xrightarrow{\sim} \pi_i(\mathcal{D}(x, y), f)$ for $i \geq 0$. Note that this was known to us by definition when $f = 1_x$, so this extends the conclusion to general maps.

Since $\Delta^0 \diamond \Delta^0 = \Delta^0 * \Delta^0 = \Delta^1$, functoriality of the join and fat join give canonical maps $X \diamond Y \to \Delta^1$ and $X * Y \to \Delta^1$ (which are the projections to the added lines). We prove the following result up to its combinatorial input.

Proposition 1.1.12 *There is a unique map $\gamma_{X,Y}\colon X \diamond Y \to X * Y$ natural in X and Y making the diagram*

$$\begin{array}{ccc} X \sqcup Y & \longrightarrow & X * Y \\ \downarrow & \overset{\gamma}{\nearrow} & \downarrow \\ X \diamond Y & \longrightarrow & \Delta^1 \end{array}$$

commute. For each X and Y, $\gamma_{X,Y}$ is a weak categorical equivalence.

Proof We define this map on n-simplices. An n-simplex of $X \times \Delta^1 \times Y$ is a triple (x, i, y) where $x \in X_n, y \in Y_n$, and i is the unique integer with $-1 \leq i \leq n$ which is the largest number that the corresponding map $[n] \to [1]$ sends to $0 \in [1]$. Here, if it is the constant map at 1 then $i = -1$. Identifying $(X * Y)_n = \bigsqcup_{i+j+1} X_i \times Y_j$, we define a map $c_{X,Y}\colon X \times \Delta^1 \times Y \to X * Y$ by

$$(x, i, y) \mapsto (x|_{\Delta\{0,\dots,i\}}, y|_{\Delta\{i+1,\dots,n\}}).$$

This map makes the diagram

$$\begin{array}{ccc} X \times \partial\Delta^1 \times Y & \longrightarrow & X \sqcup Y \\ \downarrow & & \downarrow \\ X \times \Delta^1 \times Y & \xrightarrow{c_{X,Y}} & X * Y \end{array}$$

commute: Indeed, starting with an n-simplex (x, s, y) where $s = 0 \in \partial\Delta^1$, this corresponds to the constant map at 0 in D^1, so equals (x, n, y) in the notation above, which gets sent to $(x, *) \in X_n \times Y_{-1} = X_n$, and likewise if $s = 1$, this corresponds to the constant map at 1, so equals $(x, -1, y)$. This gets sent to $(*, y) \in X_{-1} \times Y_n$, which again agrees with the other composite direction.

This provides a natural map $X \diamond Y \to X * Y$ commuting with $X \sqcup Y$, and it is easy to see that that this commutes with the maps to Δ^1 using the middle factor projection in $X \times \Delta^1 \times Y$ and the definition of $c_{X,Y}$. For uniqueness, we observe that any map making the diagram in the proposition statement must also have its restriction to $X \times \Delta^1 \times Y$ agree with $c_{X,Y}$ above, so the universal property gives the result.

To show that this map is a weak categorical equivalence, we first apply Remark 1.1.6 to the natural map $(-)\diamond Y \to (-)*Y$ of functors $\mathbf{sSet} \to \mathbf{sSet}_{\mathrm{Joy}}$. These functors preserve monomorphisms, and hence send monomorphisms to cofibrations between cofibrant objects, but they don't preserve disconnected colimits. Instead, on $\varnothing$ this map is the identity $Y \to Y$, so it follows that the class of objects on which is a weak equivalence is anyway closed under coproducts. Thus the conclusion applies and we may take $Y = \Delta^n$ for some n. Likewise, we can do the same thing with X and assume $X = \Delta^m$ for some m. Showing that

$$\Delta^m \diamond \Delta^n \to \Delta^m * \Delta^n = \Delta^{m+n+1}$$

is a weak categorical equivalence is a combinatorial argument for which we defer to [HCHA, Prop. 4.2.3]. □

Corollary 1.1.13 *For $X \in \mathbf{sSet}$, the functors*

$$X \diamond (-) \qquad (-) \diamond X \qquad X * (-) \qquad (-) * X$$

all preserve weak categorical equivalences.

Proof The statement about joins follows from the one about fat joins and Prop. 1.1.12. For (e.g.) the first statement, let $Y \to Y'$ be a weak categorical equivalence, and consider the induced diagram:

$$\begin{array}{ccccc} X \times \Delta^1 \times Y & \longleftarrow & X \times \partial\Delta^1 \times Y & \longrightarrow & X \sqcup Y \\ \downarrow & & \downarrow & & \downarrow \\ X \times \Delta^1 \times Y' & \longleftarrow & X \times \partial\Delta^1 \times Y' & \longrightarrow & X \sqcup Y' \end{array}$$

The two left-hand vertical maps are cofibrations, and all three vertical maps are weak equivalences: the first two by [HCHA, Cor. 3.6.3], and the third by Ken Brown's lemma and the fact that trivial cofibrations are stable under pushout. By [HCHA, Cor. 2.3.29], we conclude that the induced map between the pushouts $X \diamond Y \to X \diamond Y'$ is a weak categorical equivalence. □

Remark 1.1.14 In the following proof, we will use the fact that if $A \to B$ and $S \to T$ are monomorphisms, then the induced map $A \diamond T \cup B \diamond S \to B \diamond T$ is a monomorphism. This follows from the fact that the intersection of $A \diamond T$ and $B \diamond S$ inside of $B \diamond T$ is $A \diamond S$, a fact which we will also use. To see this, we use the fact that $(P \diamond Q)_n = P_n \sqcup Q_n \sqcup \bigsqcup_{i=1}^{n} (P_n \times Q_n)$, where the n summands correspond to the n non-constant maps $[n] \to [1]$. Via this description and the analagous fact for $(-) \times (-)$, we deduce the claim.

Lemma 1.1.15 *Let $p\colon X \to Y$ be a Joyal fibration, $S \hookrightarrow T$ a monomorphism, and $T \in \mathbf{sSet}/X$. Then the canonical map*

$$X//T \to X//S \times_{Y//S} Y//T$$

is a right fibration. If Y is an ∞-category, then $X//T$, and $X//S \times_{Y//S} Y//T$ are ∞-categories.

Proof (1) We have to show that this map has the RLP against all right anodyne extensions $A \to B$, which by adjunction corresponds to showing that

$$A \diamond T \cup B \diamond S \to B \diamond T \tag{$*$}$$

has the LLP against $p\colon X \to Y$. Thus it suffices to show that $(*)$ is a trivial cofibration, and it is already a monomorphism by the remark above. We have a commutative diagram

$$\begin{array}{ccccc} A \diamond T & \hookleftarrow & A \diamond S & \hookrightarrow & B \diamond S \\ \downarrow & & \downarrow & & \downarrow \\ A * T & \hookleftarrow & A * S & \hookrightarrow & B * S \end{array}$$

where the vertical maps are weak equivalences by Prop. 1.1.12, so by [HCHA, Prop. 2.3.17] the induced map of pushouts

$$A \diamond T \cup B \diamond S \to A * T \cup B * S$$

is a weak equivalence. Thus the naturality square containing $(*)$ and its analogue with the standard joins implies that we may show $A * T \cup B * S \to B * T$ is a weak equivalence, or even a trivial cofibration.

To show this, we must prove left lifting against Joyal fibrations $X' \to Y'$, or equivalently that $A \to B$ has left lifting against $X'/T \to X'/S \times_{Y'/S} Y'/T$, noting that X' automatically obtains a map from T in any lifting property square. Since $X' \to Y'$ is in particular an inner fibration, this map is a right fibration by [HCHA, Thm. 3.4.22], so we conclude since $A \to B$ is right anodyne.

(2) If Y is an ∞-category, then apply (1) to the Joyal fibration $Y \to \Delta^0$ with the same $S, T \to Y$, which shows that $Y//T \to Y//S$ is a right hence inner fibration. The special case where S is empty shows that $Y//T$ is an ∞-category. On the other hand, the assumption on p implies that X is an ∞-category, so the same logic means that $X//T$ and $X//S$ are ∞-categories. By pullback stability

$$X//S \times_{Y//S} Y//T \to X//S$$

is an inner fibration to an ∞-category, so the LHS is an ∞-category. □

If T is a simplicial set, then the natural map $(-) \diamond T \to (-) * T$ induces a map $(-)/T \to (-)//T$ by adjunction. We observe that if $\mathcal{C} \in T\backslash\mathbf{sSet}$ is an ∞-category, so that $\mathcal{C} \to \Delta^0$ is a Joyal fibration, the previous lemma with $S = \varnothing$ shows that $\mathcal{C}//T \to \mathcal{C}$ is a right fibration. Likewise, the same logic

applied to [HCHA, Thm. 3.4.22] shows that $\mathcal{C}/T \to \mathcal{C}$ is a right fibration. Thus $\mathcal{C}/T \to \mathcal{C}//T$ is a morhpism of right fibrant objects.

Corollary 1.1.16 *Let $\mathcal{C}$ be an ∞-category.*

1. *For any $T \in \mathbf{sSet}/\mathcal{C}$, the natural map $\mathcal{C}/T \to \mathcal{C}//T$ is an equivalence of ∞-categories, and hence a weak equivalence in $(\mathbf{sSet}/\mathcal{C})_{\mathrm{con}}$.*
2. *Let y be an object of $\mathcal{C}$. The induced fiberwise map*

$$(\mathcal{C}/y)_x \to (\mathcal{C}//y)_x = \mathcal{C}(x, y)$$

is an equivalence of ∞-groupoids for each object x in $\mathcal{C}$.

Proof (1) The functors $(-)*T$ and $(-)\diamond T$ preserve monos, and by Cor. 1.1.13 they preserve weak equivalences. Thus, they preserve cofibrations and trivial cofibrations when they are viewed as functors

$$\mathbf{sSet}_{\mathrm{Joy}} \to T\backslash\mathbf{sSet}_{\mathrm{Joy}},$$

where the target has the coslice model structure, meaning they are left Quillen. Since $(-)\diamond T \to (-)*T$ is a componentwise weak categorical equivalence in the coslice by Prop. 1.1.12, the corresponding derived functors on the homotopy categories are isomorphic. Thus the adjoint map $(-)/T \to (-)//T$ induces an isomorphism between the right derived functors. An ∞-category is still fibrant in the coslice $T\backslash\mathbf{sSet}_{\mathrm{Joy}}$, so $\mathcal{C}/T \to \mathcal{C}//T$ induces an isomorphism in $\mathcal{H}_{\mathrm{Joy}}$. By Lemma 1.1.15 (and [HCHA, Thm. 3.4.22]), these are both fibrant in $\mathbf{sSet}_{\mathrm{Joy}}$, hence the map is an equivalence of ∞-categories. The final statement follows by Example 1.1.2.

(2) By Cor. 1.1.7, the map of (1) is a fiberwise equivalence. We saw in Example 1.1.10 that $(\mathcal{C}//y)_x = \mathcal{C}(x, y)$, which completes the proof. □

Remark 1.1.17 We record the fact demonstrated in the proof above that the functors $(-)/T$ and $(-)//T$ are right Quillen for the Joyal model structure. Taking $T = \Delta^0$, this shows the important fact that if $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then $\mathcal{C}/x \to \mathcal{D}/F(x)$ is an equivalence of ∞-categories for each $x \in \mathcal{C}$.

The next result has the same style as the prior one: Something holds by definition for $(-)//(-)$, so we can deduce it up to equivalence for $(-)/(-)$.

Proposition 1.1.18 *Let $\mathcal{C}$ be an ∞-category and x be an object of $\mathcal{C}$. The canonical map*

$$\underline{\mathrm{Hom}}(A, \mathcal{C}/x) \to \underline{\mathrm{Hom}}(A, \mathcal{C})/x$$

is an equivalence of ∞-categories for all $A \in \mathbf{sSet}$.

Proof The map $\mathcal{C}/x \to \mathcal{C}$ induces $\underline{\mathrm{Hom}}(A, \mathcal{C}/x) \to \underline{\mathrm{Hom}}(A, \mathcal{C})$, and this sends the constant map at $1_x \in \mathcal{C}/x$ to the constant map at x, inducing the canonical

map in the proposition statement by adjunction. The same logic works for $(-)//(-)$, so we have a commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A, \mathcal{C}/x) & \longrightarrow & \underline{\mathrm{Hom}}(A, \mathcal{C})/x \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(A, \mathcal{C}//x) & \longrightarrow & \underline{\mathrm{Hom}}(A, \mathcal{C})//x \end{array}$$

The right-hand vertical map is an equivalence by Prop. 1.1.12, and the left-hand one is too by the same result and [HCHA, Thm. 3.6.9]. We claim the bottom map is an isomorphism: Indeed by example Example 1.1.2, we have

$$\begin{aligned} \underline{\mathrm{Hom}}(A, \mathcal{C}//x) &= \underline{\mathrm{Hom}}(A, \{x\} \times_{\mathcal{C}} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})) \\ &= \underline{\mathrm{Hom}}(A, \{x\}) \times_{\underline{\mathrm{Hom}}(A,\mathcal{C})} \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})) \\ &= \{x\} \times_{\underline{\mathrm{Hom}}(A,\mathcal{C})} \underline{\mathrm{Hom}}(\Delta^1, \underline{\mathrm{Hom}}(A, \mathcal{C})) \\ &= \underline{\mathrm{Hom}}(A, \mathcal{C})//x \end{aligned}$$

as desired. □

Our last goal in this section is to discuss terminal objects. Note about terminology: We will use final map/functor in the colimit-preserving sense, and cofinal map/functor in the limit preserving sense. We use '(co)final functor' if the source and target are known to be ∞-categories. We use terminal and initial for objects of simplicial sets.

Remark 1.1.19 A map $f\colon X \to Y$ of simplicial sets is *final* if for any C and map $Y \to C$, the map $X \to Y$ is a weak equivalence of $(\mathbf{sSet}/C)_{\mathrm{con}}$. Note this is the notion of absolute weak equivalence given in [HCHA, Ch. 2.5], so we can port the entire theory of absolute weak equivalences to this setting. The results of that section say the following basic things:

1. A monomorphism is a right anodyne extension iff it is final.
2. A morphism $X \to Y$ is final iff it factors as $X \to Y' \to Y$ where $X \to Y'$ is a right anodyne extension and $Y' \to Y$ is a trivial fibration.
3. The final maps are the smallest class of morphisms containing the right anoydyne extensions which is closed under composition, and so that if f and gf are in this class then g is too.
4. Suppose $X \to Y$ is a morphism and Y is right fibrant over C. Then $X \to Y$ is final iff it is a weak equivalence in $(\mathbf{sSet}/C)_{\mathrm{con}}$.

Note that, since $1\colon Y \to Y$ is a right fibration and is the terminal object of $\mathrm{RFib}(Y)$, we conclude from the last fact that $f\colon X \to Y$ is final iff $(X, f) \in \mathrm{RFib}(Y)$ is a terminal object.

Let X be a simplicial set. An object $x \in X$ is *terminal* if the morphism $x\colon \Delta^0 \to X$ is final. Since this is a monomorphism, this is equivalent to x being

a right anodyne extension. If $f\colon X \to Y$ is a map and $x \in X$ is a terminal object, then f is final iff $f(x) \in Y$ is terminal, which follows from (3) of the remark above. Thus, just as in ordinary category theory, final functors are not too interesting when the source has a terminal object. As usual, we implicitly make the corresponding definition for cofinal map and initial object.

Consider the monoidal category $(\mathbf{sSet}, *, \varnothing)$ with the join operation. This has a right action on the category of pointed simplicial sets $\mathbf{sSet}_* = \Delta^0/\mathbf{sSet}$, where given S and a pair (X, x_0), we define $(X, x_0) * S \equiv X *_{x_0} S$ by the pushout

$$\begin{array}{ccc} \Delta^0 * S & \longrightarrow & \Delta^0 \\ {\scriptstyle x_0 * 1}\downarrow & \ulcorner & \downarrow{\scriptstyle \overline{x}_0} \\ X * S & \longrightarrow & X *_{x_0} S \end{array}$$

which is canonically pointed as $(X *_{x_0} S, \overline{x}_0)$. The functoriality in both variables is clear from the definition and the same property for $*$. To see that this is associative, note that $\Delta^0 * S$ is always nonempty, hence the pushout square is a connected diagram. Thus, observe that

$$\begin{aligned} (X *_{x_0} S) *_{\overline{x}_0} S' &= (X *_{x_0} S) * S' \underset{\Delta^0 * S'}{\amalg} \Delta^0 \\ &= (X * S \underset{\Delta^0 * S}{\amalg} \Delta^0) * S' \underset{\Delta^0 * S'}{\amalg} \Delta^0 \\ &= X * S * S' \underset{\Delta^0 * S * S'}{\amalg} \Delta^0 * S' \underset{\Delta^0 * S'}{\amalg} \Delta^0 \\ &= X * S * S' \underset{\Delta^0 * S * S'}{\amalg} \Delta^0 \\ &= X *_{x_0} (S * S'), \end{aligned}$$

because $(-) * S'$ preserves connected colimits.

Remark 1.1.20 Fix a simplicial set S, and consider the functor $\mathbf{sSet}_* \to \mathbf{sSet}_*$ sending $(X, x) \mapsto X *_x S$. by construction, a pointed map $X *_x S \to (Y, y)$ is a map $X * S \to Y$ sending $x_0 * 1 \mapsto y$, or equivalently a pointed map $X \to Y/S$ where $S \to Y$ is the constant map at y, and the base point of Y/S is determined by the constant map $\Delta^0 * S \to Y$ at y. Thus this construction, written $(Y, y) \mapsto (Y, y)/S$, determines a right adjoint $\mathbf{sSet}_* \to \mathbf{sSet}_*$.

Definition 1.1.21 We define the cone functor $C(-,-)\colon \mathbf{sSet}_* \to \mathbf{sSet}_*$ as the action functor $(X, x_0) \mapsto X *_{x_0} \Delta^0 \equiv C(X, x_0)$ of the object $\Delta^0 \in \mathbf{sSet}$, with its canonical base point $\overline{x}_0 \in C(X, x_0)$.

Since $\Delta^0 * \Delta^0 = \Delta^1$, we can also describe $C(X, x_0)$ as $(X * \Delta^0)/(x_0 * 1)$ where $x_0 * 1$ is now identified with the line joining x_0 to the apex, which is analagous to topology: $X * \Delta^0 = CX$ is the usual cone, and for the pointed cone we quotient the line joining the basepoint to the apex. By Remark 1.1.20, the cone functor is a left adjoint to the functor $(Y, y) \mapsto (Y/y, 1_y)$.

Lemma 1.1.22

1. *A morphism $p\colon X \to Y$ is a right fibration iff it is an inner fibration and for each object x of X, the map $X/x \to Y/p(x)$ is a trivial fibration.*
2. *If $U \to V$ is a monomorphism, then $U * \Delta^0 \to V * \Delta^0$ is a right anodyne extension.*

Proof (1) A right fibration is an inner fibration with the RLP against $\Lambda^n_n \to \Delta^n$. Writing $\Lambda^n_n = \partial\Delta^{n-1} * \Delta^0 \subset \Delta^{n-1} * \Delta^0 = \Delta^n$, the RLP against this inclusion is equivalent to $\partial\Delta^{n-1} \to \Delta^n$ having the LLP against $X/\Delta^0 \to Y/\Delta^0$ for all maps $\Delta^0 \to X$, i.e. $X/x \to Y/p(x)$ is a trivial fibration for all $x \in X$.

(2) To see that $U * \Delta^0 \to V * \Delta^0$ having the LLP against right fibrations, we can show that for each right fibration $X \to Y$, $U \to V$ has the LLP against $X/\Delta^0 \to Y/\Delta^0$, which we just showed. □

Example 1.1.23 If $(X, x) \in \mathbf{sSet}_*$, then $\overline{x} \in C(X, x)$ is a terminal object. Indeed, $\overline{x} \to C(X, x)$ is a right anodyne extension because it is the pushout of $\Delta^0 * \Delta^0 \to X * \Delta^0$, which is a right anodyne extension since $\Delta^0 \to X$ is a monomorphism, by Lemma 1.1.22(2).

An application of this is the following criterion: Let X be a simplicial set with an object x. If the canonical map $(X/x, 1_x) \to (X, x)$ has a pointed section, then x is a terminal object. Indeed, a section of this map in $\mathbf{sSet}_*$ would give, by adjunction, a pointed retract of $(X, x) \to C(X, x)$. Thus $x\colon \Delta^0 \to X$ is a retract of the right anodyne extension $\overline{x}\colon \Delta^0 \to C(X, x)$.

Proposition 1.1.24 *For any object x of X, 1_x is a terminal object of X/x. In particular, if $\mathcal{C}$ is an ∞-category and $x \in \mathcal{C}$, then the commutative diagram*

$$\begin{array}{ccc} \{x\} & \xrightarrow{1_x} & \mathcal{C}/x \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

exhibits $\mathcal{C}/x$ as a canonical fibrant replacement for x in $(\mathbf{sSet}/\mathcal{C})_{con}$.

Proof The second statement follows immediately because $\mathcal{C}/x$ is right fibrant over $\mathcal{C}$, and $1_x\colon \Delta^0 \to \mathcal{C}/x$ is a right anodyne extension, hence trivial cofibration. For the first part, we use the criterion explained above, along with the following trick: Since Δ^0 is a terminal object, it has the (unique) structure of a monoid object in $(\mathbf{sSet}, *, \varnothing)$. Therefore the action functor $C(-,-)\colon \mathbf{sSet}_* \to \mathbf{sSet}_*$ defined by Δ^0 has the structure of a monoid object in $\mathrm{Fun}(\mathbf{sSet}_*, \mathbf{sSet}_*)$, or in other words is a monad. Hence its right adjoint $(-)/(-)$ is a comonad.

Let's unwind the definitions: the unit of Δ^0 is $\varnothing \to \Delta^0$, so the monad unit is, component-wise on (X, x), given by $X = X *_x \varnothing \to C(X, x)$. Taking right adjoints shows that the comonad counit is $X/x \to X$. The counit axioms say that the component at X/x, namely $(X/x)/1_x \to X/x$, has a section given by comultiplication. Thus $1_x \in X/x$ is a terminal object. □

Example 1.1.25 Let $\mathcal{D}$ be an ∞-groupoid. Then $\mathcal{D}/y \to \mathcal{D}$ and $\mathcal{D}//y \to \mathcal{D}$ are right fibrations to a Kan complex, hence Kan fibrations. Then we have

$$\{x\} \times^{\mathbf{R}}_{\mathcal{D}} \{y\} \xrightarrow{\sim} \{x\} \times^{\mathbf{R}}_{\mathcal{D}} \mathcal{D}/y \xrightarrow{\sim} \{x\} \times^{\mathbf{R}}_{\mathcal{D}} \mathcal{D}//y = \mathcal{D}(x, y)$$

because the final map $y \to \mathcal{D}/y$ is in particular a weak equivalence in $\mathbf{sSet}_{\mathrm{KQ}}$, hence of ∞-groupoids, as is $\mathcal{D}/y \to \mathcal{D}//y$. The final equality is because $\mathcal{D}//y \to \mathcal{D}$ is a Kan fibration and the pullback is of Kan complexes. Note that the same steps also hold in the Joyal model structure because everything is an ∞-groupoid. We often use the case $\Omega^1(\mathcal{D}, x) = \{x\} \times^{\mathbf{R}}_{\mathcal{D}} \{x\}$.

Corollary 1.1.26 *Let $\mathcal{C}$ be an ∞-category. There is a canonical equivalence*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}/x, \mathcal{C}/y) \to \mathcal{C}(x, y)$$

of ∞-groupoids for all objects x and y of $\mathcal{C}$.

Proof Consider the maps $1_x \colon \{x\} \to \mathcal{C}/x$ and $\mathcal{C}/y \to \mathcal{C}//y$, which induce a commutative diagram:

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}/x, \mathcal{C}/y) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{C}}(x, \mathcal{C}/y) & = & (\mathcal{C}/y)_x \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}/x, \mathcal{C}//y) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{C}}(x, \mathcal{C}//y) & = & \mathcal{C}(x, y)
\end{array}$$

The previous proposition shows that $x \to \mathcal{C}/x$ is a right anodyne extension, hence the upper left map is a trivial fibration by Lemma 1.1.3. The right-most vertical map is an equivalence of ∞-groupoids by Cor. 1.1.16, which completes the proof. □

Theorem 1.1.27 *Let $\mathcal{C}$ be an ∞-category and t be an object of $\mathcal{C}$. The following conditions are equivalent:*

(i) t is a terminal object of $\mathcal{C}$.

(ii) The ∞-groupoids $\mathcal{C}(x, t)$ are contractible for all $x \in \mathcal{C}$.

(iii) The map $\mathcal{C}/t \to \mathcal{C}$ is a trivial fibration.

(iv) The map $\mathcal{C}/t \to \mathcal{C}$ is an equivalence of ∞-categories.

(v) The map $(\mathcal{C}/t, 1_t) \to (\mathcal{C}, t)$ has a pointed section.

(vi) For $n \geq 1$, any map $u \colon \partial\Delta^n \to \mathcal{C}$ such that $u(n) = t$ extends to Δ^n.

Proof *(iii)* = *(vi)* Note that $\partial\Delta^{n+1} = \Delta^n * \varnothing \cup \partial\Delta^n * \Delta^0$ by [HCHA, Prop. 3.4.17] as a subobject of Δ^n. We thus have the correspondence of lifting properties

$$\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & \mathcal{C}/t \\
\downarrow & \nearrow & \downarrow \\
\Delta^n & \longrightarrow & \mathcal{C}/\varnothing
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\partial\Delta^n * \Delta^0 \cup \Delta^n * \varnothing & = & \partial\Delta^{n+1} & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow & \nearrow & \downarrow \\
\Delta^n * \Delta^0 & = & \Delta^{n+1} & \longrightarrow & *
\end{array}$$

where the correspondence is given by pointed maps $\partial\Delta^{n+1} \to \mathcal{C}$, i.e. sending the second $(-) * \Delta^0$ factor to t, but this corresponds to sending $n+1$ to t.

$(i) = (v)$ We showed that $(v) \Rightarrow (i)$ in Example 1.1.23. Conversely, if $t \to \mathcal{C}$ is a right anodyne extension, then $\mathcal{C}/t \to \mathcal{C}$ is a right fibration so it has the RLP against it, giving the section.

$(iii) = (iv)$ This is map is already a right fibration, hence Joyal fibration, between fibrant objects in $\mathbf{sSet}_{\text{Joy}}$. Thus the two conditions are equivalent.

$(ii) = (iii)$ Likewise, $\mathcal{C}/t \to \mathcal{C}$ is a fibration between fibrant objects in $(\mathbf{sSet}/\mathcal{C})_{\text{con}}$, so being a trivial fibration is equivalent to being a weak equivalence. By Cor. 1.1.7, this is equivalent to $(\mathcal{C}/t)_x \to \mathcal{C}_x = \Delta^0$ being an equivalence of ∞-groupoids for all $x \in \mathcal{C}$, i.e. equivalent to $(\mathcal{C}/t)_x$ being contractible. But this is in turn equivalent to $\mathcal{C}(x,t)$ being contractible for all x by Cor. 1.1.16.

$(iii) \Rightarrow (i)$ A trivial fibration is final by Remark 1.1.19(2), and the terminal object $1_t \in \mathcal{C}/t$ has image t in $\mathcal{C}$, so t is terminal in $\mathcal{C}$.

$(i)/(v) \Rightarrow (iv)$ If t is terminal and we have a section $\mathcal{C} \to \mathcal{C}/t$ sending $t \mapsto 1_t$, then this is a final monomorphism, and hence a right anodyne extension by Remark 1.1.19. Therefore it is a weak equivalence in $(\mathbf{sSet}/\mathcal{C})_{con}$, so $\mathcal{C}/t \to \mathcal{C}$ is one too, which implies that it is an equivalence of ∞-categories by Example 1.1.2. □

This theorem will be behind many subsequent statements about ∞-categories, such as characterizations of (co)limits and their corresponding uniqueness statements.

Example 1.1.28 For instance, we see immediately that if $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories, then $t \in \mathcal{C}$ is terminal iff $F(t)$ is terminal. Indeed, by Remark 1.1.17, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}/t & \xrightarrow{\simeq} & \mathcal{D}/F(t) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\simeq} & \mathcal{D} \end{array}$$

where the horizontal maps are equivalences of ∞-categories, so the left-hand map is one iff the right-hand map is.

Corollary 1.1.29 *Let $\mathcal{C}$ be an ∞-category and t be an object of $\mathcal{C}$. If t is terminal, then t is a terminal object of $\mathrm{h}\mathcal{C}$. If $\mathcal{C}$ is a 1-category, then the converse holds.*

Proof For each object x of $\mathcal{C}$, we have $\pi_0\big(\mathcal{C}(x,t)\big) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,t)$ by [HCHA, Prop. 3.7.2]. If t is terminal in $\mathcal{C}$, then this is the one-point set for all t, thus t is terminal in the homotopy category. If $\mathcal{C}$ is a 1-category as in Remark 1.0.3,

then $\pi_n\big(\mathcal{C}(x,t)\big) = 0$ for all $n \geq 1$ already, and the prior equality shows it for π_0 as well. Thus $\mathcal{C}(x,t)$ is a contractible (1-)groupoid. □

Corollary 1.1.30 *Let $\mathcal{C}$ be an ∞-category. If $\mathcal{C}$ has a terminal object, then the full subcategory spanned by the terminal objects is a contractible ∞-groupoid.*

Proof Let $\mathcal{T} \subset \mathcal{C}$ be this (nonempty) full subcategory ([HCHA, Ex. 3.9.5]), then we claim that the map $\mathcal{T} \to \Delta^0$ is an equivalence of ∞-categories (hence of ∞-groupoids). It is surjective on objects, and by assumption it's mapping spaces $\mathcal{T}(t,t') = \mathcal{C}(t,t')$ are contractible ∞-groupoids, so the maps $\mathcal{T}(t,t') \to \Delta^0(*,*) = \Delta^0$ are equivalences. Thus $\mathcal{T} \to \Delta^0$ is an equivalence. □

We usually phrase this result by saying that terminal objects in an ∞-category are unique up to a contractible choice. Concretely, it follows from Cor. 1.1.29 that any map $t \to t'$ of terminal objects in $\mathcal{C}$ is invertible, so the mapping space $\mathcal{C}(t,t') \simeq \Delta^0$ is the space of equivalences between these two objects. The choice of equivalence is unique up to homotopy, moreover given two such maps we have two homotopies witnessing that these are equivalences. The choice of such homotopies are unique up to homotopy, and so on.

Corollary 1.1.31 *Let $\mathcal{C}$ be an ∞-category and A a simplicial set. If t is a terminal object of $\mathcal{C}$, then the constant map t in $\underline{\mathrm{Hom}}(A,\mathcal{C})$ is a terminal object.*

Proof We have a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A,\mathcal{C}/t) & \xrightarrow{\simeq} & \underline{\mathrm{Hom}}(A,\mathcal{C})/t \\ & \searrow^{\simeq} & \downarrow \\ & & \underline{\mathrm{Hom}}(A,\mathcal{C}) \end{array}$$

where the upper map is an equivalence by Prop. 1.1.18, and the left-hand map is $\underline{\mathrm{Hom}}(A,-)$ applied to the equivalence of $\mathcal{C}/t \to \mathcal{C}$. Thus the right-hand map is an equivalence. □

Example 1.1.32 This example discusses the right fibrations between ordinary categories. Let $F\colon C \to D$ be a functor of small categories; recall that F is a trivial fibration iff it is a surjective-on-objects equivalence. Indeed one direction is clear; for the other direction, note that it is automatically an inner fibration, and surjectivity on objects + fully faithful gives the isofibration condition. Thus F is a trivial fibration.

Now we apply this to the induced map on slices. By Lemma 1.1.22 and what we just said, F is a right fibration if and only if $C/c \to D/F(c)$ is a surjective-on-objects equivalence for each $c \in C$. This consists of two conditions:

1. Surjectivity on objects means that for each $c \in C$ and map $d \to F(c)$, there should be a map $c' \to c$ in C such that $F(c' \to c) = d \to F(c)$.

2. Full faithfulness says the following for each $c \in C$: given a pair of maps $v\colon z \to c$ and $u\colon c' \to c$, for any $g\colon F(z) \to F(c')$ satisfying $F(u) \circ g = F(v)$, there is a unique $w\colon z \to c'$ with $u \circ w = v$ and $F(w) = g$.

Such a functor F is also called a category over D fibered in groupoids, as we know that its fibers $F_d \subset C$ are subgroupoids of C.

In the setting of the above example, usually when one is constructing such a functor, we have in mind the desired groupoids F_d along with some notion of pullback functor $u^*\colon F_{d'} \to F_d$ for every map $u\colon d \to d'$ in D. Then in this case we should define C as the following category of elements construction: an object is a pair (d, x) where $d \in D$ and $x \in F_d$, and a morphism $(d, x) \to (d', x')$ should consist of a map $u\colon d \to d'$ and an isomorphism $\alpha\colon x \xrightarrow{\sim} u^*x'$. The composition should be given in the following way: For

$$(d, x) \xrightarrow{(u,\alpha)} (d', x') \xrightarrow{(v,\beta)} (d'', x'')$$

the composite should be given by $vu\colon d \to d''$ equipped with

$$x \xrightarrow{\alpha} u^*x' \xrightarrow{u^*\beta} u^*v^*x'' \xrightarrow{\sim} (vu)^*x''$$

satisfying associativity and unitality, which are the substantive conditions we have to check. Then this data will form a category over D fibered in groupoids: given (d, x) and a map $u\colon d' \to d$, we can always lift this to form (d', u^*x) equipped with $(d', u^*x) \to (d, x)$ witnessed by the identity map. For the second condition, start with (d, x) and $v\colon (e, y) \to (d, x)$ and $u\colon (d', x') \to (d, x)$. Given $g\colon e \to d'$ over d, then this extends uniquely to a morphism by choosing inverses of the associated isomorphism data. This shows that $F\colon C \to D$ will be a category of D fibered in groupoids.

Example 1.1.33 When using the construction described above, we just need to describe the category C properly and check that it is well-defined. There are often tricks to doing this. For instance, ignoring size issues, consider the following case: we take $D = \mathbf{Sch}$ to be the category of schemes, and C to be the category whose objects are pairs $(X, \mathcal{F})$ where X is a scheme and $\mathcal{F}$ is a quasicoherent sheaf on X. A morphism $f\colon (X, \mathcal{F}) \to (Y, \mathcal{G})$ should be a map $f\colon X \to Y$ and an isomorphism $\alpha\colon \mathcal{F} \xrightarrow{\sim} f^*\mathcal{G}$, but for simplicity we can encode this as a map f with a morphism $\alpha\colon \mathcal{G} \to f_*\mathcal{F}$ such that $\alpha^\flat\colon f^*\mathcal{G} \to \mathcal{F}$ is an isomorphism.

Let's check that this forms a category. To carry out the composition

$$(X, \mathcal{F}) \xrightarrow{(f,\alpha)} (Y, \mathcal{G}) \xrightarrow{(g,\beta)} (Z, \mathcal{H})$$

we simply take $gf\colon X \to Z$ equipped with

$$\mathcal{H} \xrightarrow{\beta} g_*\mathcal{G} \xrightarrow{g_*\alpha} g_*f_*\mathcal{F} = (gf)_*\mathcal{F},$$

then we know that this preserves the condition that $f^*g^*\mathcal{H} \to \mathcal{F}$ is an isomorphism. Moreover, it is now easy to check associativity and unitality

of composition because of the literal equalities $g_* f_* = (gf)_*$ and $1_{X,*} = \mathrm{Id}_{\mathrm{QCoh}(X)}$. We remark that we aren't assuming $f_*\mathcal{F}$ is quasicoherent here, and the morphism is taken in the category of $\mathcal{O}_X$-modules. This is also the same way one shows e.g. that $\mathcal{O}_X$-modules form a category, without having to check naturality properties of pullback.

Therefore we obtain a category $F\colon C \to \mathbf{Sch}$ over schemes fibered in groupoids. Explicitly, it is clear that $F_X = \mathrm{QCoh}(X)^{\simeq}$. With appropriate smallness restrictions, we can make this a right fibration. In the next section we will discuss how to think of this as a functor $\mathbf{Sch}^{\mathrm{op}} \to \mathcal{S}$, where $\mathcal{S}$ is the ∞-category of ∞-groupoids.

1.1.2 The ∞-category $\mathcal{S}$ of spaces

Suppose F is a right fibrant object over C. There are quite a few basic properties about F, viewed as a family of fibers $\{F_c : c \in C\}$ that we cannot yet state precisely. Here is a simple example of this, and we will give a pedestrian proof to demonstrate this annoyance.

If $u\colon x \to y$ is a morhpism in C, we can consider the Hom-space $\underline{\mathrm{Hom}}_C(u, F)$ of maps over C. The two inclusions $\Delta^0 \to \Delta^1$ determine maps

$$F_x \xleftarrow{\mathrm{ev}_0} \underline{\mathrm{Hom}}_C(u, F) \xrightarrow{\mathrm{ev}_1} F_y,$$

and ev_1 is a trivial fibration by Lemma 1.1.3, because $\{1\} \to \Delta^1$ is a right anodyne extension. Thus the space of sections of ev_1 is a nonempty contractible ∞-groupoid, and choosing one produces a map $F_y \to F_x$, which is unique in the homotopy category $\mathcal{H}$ since ev_1 is an isomorphism therein. We denote this map by $u^*\colon F_y \to F_x$ in $\mathcal{H}$.

In the case that $C = \mathcal{C}$ is an ∞-category, we would like to use F to define a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ sending $c \mapsto F_c$ on objects, where $\mathcal{S}$ is some ∞-category whose objects are ∞-groupoids. We would want $\mathrm{h}\mathcal{S} = \mathcal{H}$, so that the induced map $\mathrm{h}\mathcal{C}^{\mathrm{op}} \to \mathcal{H}$ agrees with the construction above. In this section we will do exactly this. Let us first illustrate how even basic properties we would want such a construction to satisfy take some work from the results we have so far.

Proposition 1.1.34 *Let $\mathcal{C}$ be an ∞-category, $u\colon x \to y$ be a morphism of $\mathcal{C}$, and F a right fibrant object over $\mathcal{C}$. If u is invertible in $\mathcal{C}$, then the induced map*

$$u^*\colon F_y \to F_x \quad \text{in } \mathcal{H}$$

is an isomorphism in $\mathcal{H}$. In particular, any choice of the map $F_y \to F_x$ as above, unique up to contractible choice, is an equivalence of ∞-groupoids.

Proof The second statement follows from the first because F_x and F_y are ∞-groupoids. For the first, note that by our definition it suffices to show that $\mathrm{ev}_0\colon \underline{\mathrm{Hom}}_C(u, F) \to F_x$ is an equivalence of ∞-categories, and thus that $(\Delta^0, x) \to (\Delta^1, u)$ is a weak equivalence in $(\mathbf{sSet}/\mathcal{C})_{\mathrm{con}}$, by Lemma 1.1.3.

Consider the degenerate simplex $\Delta^2 \xrightarrow{\sigma_0} \Delta^1 \xrightarrow{u} \mathcal{C}$, then corresponding to the two decompositions $\Delta^2 = \Delta^1 * \Delta^0 = \Delta^0 * \Delta^1$, this induces two two maps $0 \to \mathcal{C}/u$ and $\Delta^1 \to \mathcal{C}/y$ making the diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & \mathcal{C}/u \\ \downarrow & & \downarrow \\ \Delta^1 & \longrightarrow & \mathcal{C}/y \end{array}$$

commute. This diagrams makes the left-hand map the morphism $(\Delta^0, x) \to (\Delta^1, u)$ over $\mathcal{C}$. The map $\mathcal{C}/u \to \mathcal{C}/x$ is a trivial fibration by [HCHA, Lem. 3.4.21] with $n = 1, k = 0, Y = \Delta^0$, hence an equivalence of ∞-categories. By Example 1.1.28, the image of 0 is a terminal object of $\mathcal{C}/u$ because its image in $\mathcal{C}/x$ is one. Likewise, the image of 1 under the bottom map is $1_y \in \mathcal{C}/y$, which is terminal. Thus both horizontal maps are final, and hence weak equivalences in $(\mathbf{sSet}/\mathcal{C})_{\text{con}}$. Thus it suffices to prove that the right hand map in

$$\mathcal{C}/x \leftarrow \mathcal{C}/u \to \mathcal{C}/y$$

is a weak equivalence, where the left-hand map is a trivial fibration.

By Prop. 1.1.12 we may instead show that $\mathcal{C}//u \to \mathcal{C}//y$ is a fiberwise equivalence, given that $\mathcal{C}//x \leftarrow \mathcal{C}//u$ is a fiberwise eqiuvalence. Taking the fiber over $z \in \mathcal{C}$ produces

$$\mathcal{C}(z, x) \xleftarrow{\simeq} (\mathcal{C}//u)_z \to \mathcal{C}(z, y),$$

Choosing a homotopy inverse of the first map produces $\mathcal{C}(z, x) \to \mathcal{C}(z, y)$, which is a composition-by-u map as constructed in [HCHA, Ch. 3.7]. It remains to show this is an equivalence.

For any simplicial set K, apply $\underline{\mathrm{Hom}}(K, -)$; by the interaction of mapping spaces with internal hom we obtain

$$\underline{\mathrm{Hom}}(K, \mathcal{C})(z, x) \to \underline{\mathrm{Hom}}(K, \mathcal{C})(z, y)$$

where z, x, y denote the constant maps at these objects. Taking π_0 of this yields $\mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(K,\mathcal{C})}(z, x) \to \mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(K,\mathcal{C})}(z, y)$, which we know is an isomorphism because $x \to y$ is also invertible in $\underline{\mathrm{Hom}}(K, \mathcal{C})$. On the other hand, applying $\pi_0\,\underline{\mathrm{Hom}}(K, -)$ to the map in question also yields

$$\mathrm{Hom}_{\mathcal{H}}(K, \mathcal{C}(z, x)) \to \mathrm{Hom}_{\mathcal{H}}(K, \mathcal{C}(z, y)).$$

Since this is an isomorphism, $\mathcal{C}(z, x) \to \mathcal{C}(z, y)$ is an isomorphism in $\mathcal{H}$, hence it is an equivalence of ∞-groupoids. □

Example 1.1.35 For instance, applying this to the right fibration $\mathcal{C}//y \to \mathcal{C}$, we see that a map $u\colon x \to x'$ induces the unique composition map $\mathcal{C}(x', y) \to \mathcal{C}(x, y)$ in the homotopy category, or as a map $\mathcal{C}(x', y) \to \mathcal{C}(x, y)$ of spaces it is unique up to contractible choice. And moreover, if $x \to x'$ is an equivalence

in $\mathcal{C}$, then $\mathcal{C}(x', y) \simeq \mathcal{C}(x, y)$ is an equivalence of spaces. We also saw this in the proof, since we reduced to (the opposite of) this case.

We remark that even the simple fact above has missed a lot of homotopical information: we haven't shown that u^* in the homotopy category is dependent only on u in $\mathrm{h}\mathcal{C}$, and more importantly we don't know have any information on the higher homotopy data.

We first must do some more work with left and right fibrations. For now, we will assume the main results of [HCHA, Ch. 4.4]. We will now define some Quillen adjunctions and their derived functors. Let $p\colon A \to B$ be a morphism of simplicial sets. We first have the standard slice adjunction written

$$p_!\colon \mathbf{sSet}/A \rightleftarrows \mathbf{sSet}/B : p^*,$$

where $p_!(X \to A) = X \to A \to B$, and $p^*(Y \to B) = Y \times_B A \to A$. Then $p_!$ preserves monomorphisms and sends right anodyne extensions to right anodyne extensions (because these are measured in **sSet**), so we conclude by [HCHA, Prop. 2.4.40] that $p_!$ is left Quillen on the contravariant model structures. The same is clearly true for the covariant model structures.

Therefore we obtain derived adjunctions

$$\mathbf{L}p_!\colon \mathrm{RFib}(A) \rightleftarrows \mathrm{RFib}(B) : \mathbf{R}p^* \qquad \mathbf{L}p_!\colon \mathrm{LFib}(A) \rightleftarrows \mathrm{LFib}(B) : \mathbf{R}p^*$$

on the homotopy categories.

Remark 1.1.36 Note that colimits in $\mathbf{sSet}/B$ are computed in **sSet**, so the functor p^* preserves colimits and hence itself has a right adjoint, which we denote by $p_*\colon \mathbf{sSet}/A \to \mathbf{sSet}/B$. The condition for $p^* \dashv p_*$ to be Quillen is more subtle, however. To apply [HCHA, Prop. 2.4.40], we see that we need p to 'pull back' right anodyne extensions to right anodyne extensions (in the contravariant case). This property is not stable under base change, so it turns out that the correct notion is to *universally* pull back right anodyne extensions, namely all pullbacks of p should pull back a right anodyne extension to a right anodyne extension. If p satisfies this property we say it is a *proper map*. Dually, if $p^{\mathrm{op}}\colon A^{\mathrm{op}} \to B^{\mathrm{op}}$ is proper, we say p is a *smooth map*. Thus if p is proper then p^* is left Quillen, and we obtain a derived adjunction

$$\mathbf{L}p^*\colon \mathrm{RFib}(B) \rightleftarrows \mathrm{RFib}(A) : \mathbf{R}p_*,$$

and if p is smooth then we obtain one

$$\mathbf{L}p^*\colon \mathrm{LFib}(B) \rightleftarrows \mathrm{LFib}(A) : \mathbf{R}p_*,$$

on the covariant homotopy categories. We will study these conditions later. For now, an important example is that a left fibration is proper, and a right fibration is smooth, by [HCHA, Prop. 4.4.11].

Here are the basic observations about this definition. Firstly, left derived functors are computed on cofibrant resolutions, but since all objects are cofibrant

we may just write $\mathbf{L}p_! = p_!$. On the other hand, $\mathbf{R}p^*$ is computed on right fibrant/left fibrant presheaves over the base B. In the proper/smooth case, this also means that $\mathbf{L}p^* = p^*$, and moreover that both of these equal $\mathbf{R}p^*$, as p^* preserves weak equivalences between cofibrant (hence all) objects.

Note that if $p\colon A \to B$ and $q\colon B \to C$ are two maps, then this implies that $\mathbf{L}q_! \circ \mathbf{L}p_! = \mathbf{L}(qp)_!$, so by adjunction $\mathbf{R}(qp)^* \cong \mathbf{R}p^* \circ \mathbf{R}q^*$. In the proper/smooth case, this will in turn imply that $\mathbf{R}(qp)_* = \mathbf{R}q_* \circ \mathbf{R}p_*$. Also, if A is a simplicial set, we will denote

$$\int_A F \coloneqq \mathbf{L}p_!(F) \in \mathcal{H}$$

where $p\colon A \to \Delta^0$ is the unique map, and in either of the situations we have $\mathbf{L}p_!\colon \mathrm{LFib}(A) \to \mathcal{H}$ and $Lp_!\colon \mathrm{RFib}(A) \to \mathcal{H}$.

If A is a fixed simplicial set, we can also define a base change pairing

$$\langle -, - \rangle : \mathbf{sSet}/A \times \mathbf{sSet}/A \to \mathbf{sSet}$$

given by $\langle F \to A, G \to A \rangle = F \times_A G$. Suppose F is right fibrant over A, namely $p_F\colon F \to A$ is a right fibration; then $\langle F, - \rangle$ is the composite of

$$\mathbf{sSet}/A \xrightarrow{p_F^*} \mathbf{sSet}/F \xrightarrow{q_!} \mathbf{sSet}$$

where $q\colon F \to \Delta^0$. Since p_F is smooth, p_F^* is left Quillen and thus $\langle F, - \rangle$ is left Quillen for the covariant model structure. Likewise, if G is left fibrant over A, then $\langle -, G \rangle$ is left Quillen for the contravariant model structure. Therefore weak equivalences in one variable are preseved when the other is fibrant, so we obtain a total right derived functor

$$\mathbf{R}\langle -, - \rangle :\ \mathrm{RFib}(A) \times \mathrm{LFib}(A) \to \mathcal{H}$$

which is computed either by a right fibrant resolution in the first variable, or a left fibrant resolution in the second variable. This is recorded by a natural map $\langle -, - \rangle \to \mathbf{R}\langle -, - \rangle$, which is an isomorphism in either of these cases.

For example, if $f\colon X \to A$ is a map, then we have a canonical isomorphisms

$$\int_X \mathbf{R}f^*(F) \cong \mathbf{R}\langle f, F \rangle \qquad \int_X \mathbf{R}f^*(G) \cong \mathbf{R}\langle G, f \rangle$$

in $\mathcal{H}$ for all $F \in \mathrm{LFib}(A)$ and $G \in \mathrm{RFib}(A)$. Indeed, e.g. for the first case, represent F by a left fibration $Y \to A$, then both sides become $X \times_Y A$ as an object of $\mathbf{sSet}_{\mathrm{KQ}}$.

We use the following alternative notation if $X = \Delta^0$, i.e. f is an object $a \in A_0$. In this case, for $F \in \mathrm{LFib}(A)$ and $G \in \mathrm{RFib}(A)$ we write

$$F_{/a} \coloneqq \mathbf{R}\langle a, F \rangle = \mathbf{R}a^*(F) \qquad G_{a/} \coloneqq \mathbf{R}\langle G, a \rangle = \mathbf{R}a^*(G),$$

both objects of $\mathcal{H}$, noting that $\int_{\Delta^0}(-) = \mathrm{Id}_{\mathcal{H}}$. Here is an important special case: if $A = \mathcal{C}$ is an ∞-category, then we showed in Prop. 1.1.24, then $\mathcal{C}/x$ is

a canonical fibrant replacement in $\mathrm{RFib}(\mathcal{C})$ for the point x. Therefore

$$F_{/x} = \mathbf{R}\langle x, F\rangle = \langle \mathcal{C}/x, F\rangle = \mathcal{C}/x \times_{\mathcal{C}} X$$

where $X \to \mathcal{C}$ is any representative for F.

Lemma 1.1.37 *Let $F \to G$ be a morphism in $\mathrm{LFib}(A)$. The following conditions are equivalent:*

(i) φ is an isomorphism in $\mathrm{LFib}(A)$.

(ii) For any $E \in \mathrm{RFib}(A)$, the map $\mathbf{R}\langle E, F\rangle \to \mathbf{R}\langle E, G\rangle$ is an isomorphism in $\mathcal{H}$.

(iii) For each object a of A, the induced map $\varphi_{/a}\colon F_{/a} \to G_{/a}$ is an isomorphism in $\mathcal{H}$.

Proof $(i) \Rightarrow (ii)$ by functoriality, and $(ii) \Rightarrow (iii)$ by taking $E = \Delta^0 \xrightarrow{a} A$. Assuming (3), represent F and G by left fibrant presheaves X and Y over A. Then $F_{/a} = a^*(X) = X_a$, and likewise $G_{/a} = Y_a$, so the corresponding map $X \to Y$ is a fiberwise equivalence, hence a weak equivalence by the dual version of Cor. 1.1.7. □

Corollary 1.1.38 *A functor $\mathcal{C} \to \mathcal{D}$ of ∞-categories is cofinal iff for every object d of $\mathcal{D}$, the ∞-category*

$$\mathcal{C}/d := \mathcal{D}/d \times_{\mathcal{D}} \mathcal{C}$$

is weakly contractible.

Proof Since $\mathcal{D}/d \to \mathcal{D}$ is a right fibration, so too is $\mathcal{C}/d \to \mathcal{C}$, thus $\mathcal{C}/d$ is an ∞-category. The functor is cofinal if and only if $\mathcal{C} \to \mathcal{D}$ is an isomorphism in $\mathrm{LFib}(\mathcal{D})$, by Remark 1.1.19. By Lemma 1.1.37, this is equivalent to $\mathcal{C}_{/d} \to \mathbf{R}d^*(\mathcal{D}) = \Delta^0$ being an isomorphism in $\mathcal{H}$ for all objects d of $\mathcal{D}$. But we showed that $\mathcal{C}_{/d} = \mathcal{D}/d \times_{\mathcal{D}} \mathcal{C}$ in the case of an ∞-category. □

Example 1.1.39 The classical version of Quillen's theorem A says that if $C \to D$ is a functor such that $N(C/d)$ is weakly contractible for all $d \in D$, then $N(C) \to N(D)$ is a weak homotopy equivalence. This follows immediately because nerves commute with limits.

Remark 1.1.40 Let $\mathcal{C} \to \mathcal{D}$ be a cofinal functor of ∞-categories. If $\mathcal{C}$ and $\mathcal{D}$ are 1-categories, then because the maps to the homotopy categories are trivial fibrations, we deduce that $\mathcal{C} \to \mathrm{h}\mathcal{D}$ is cofinal, and then the same fact implies $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is cofinal. Then we deduce that $\pi_0(\mathrm{h}\mathcal{C}/d) = *$ for all $d \in \mathrm{h}\mathcal{D}$ from Cor. 1.1.38, which means that $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is cofinal in the sense of category theory. However, the converse does not hold, because a nonempty connected 1-category can have nontrivial higher homotopy groups.

Proposition 1.1.41 *Let $u\colon A \to B$ be a map and $E \in \mathrm{RFib}(A)$, $F \in \mathrm{LFib}(B)$. There are canonical isomorphisms*

$$\mathbf{R}\langle E, \mathbf{R}u^* F\rangle \cong \mathbf{R}\langle \mathbf{L}u_! E, F\rangle \qquad \mathbf{R}\langle F, \mathbf{L}u_! E\rangle \cong \mathbf{R}\langle \mathbf{R}u^* F, E\rangle$$

in $\mathcal{H}$, natural in E and F.

Proof We prove the first case, as the second follows by taking opposites. Let $Y \to B$ be a left fibrant presheaf over A representing F. Then $\mathbf{R}u^* F = A \times_B Y$ is left fibrant over A, and thus

$$\begin{aligned}\mathbf{R}\langle E, \mathbf{R}u^* F\rangle &= \langle E, A \times_B Y\rangle = X \times_A A \times_B Y \\ &= X \times_B Y = \langle \mathbf{L}u_! X, Y\rangle \\ &= \mathbf{R}\langle \mathbf{L}u_! E, F\rangle\end{aligned}$$

where $X \to A$ is a presheaf representing E. □

Let us rephrase some of our old results in terms of the new language. The purpose is to translate infinity-categorical conditions into 1-categorical conditions on homotopy categories. For example, if A is a simplicial set and $u\colon a \to b$ is a morphism of A, then for a right fibrant object X we already know that there's a unique map $u^*\colon X_b \to X_a$ in $\mathcal{H}$ making the diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_A(u, X) & \longrightarrow & X_b \\ & \searrow & \downarrow{\scriptstyle u^*} \\ & & X_a \end{array}$$

commute. It follows from this uniqueness and the naturality of evaluation maps that this induces a natural transformation on the category of right fibrations, and thus this extends to a natural transformation $\mathbf{R}b^* \to \mathbf{R}a^*$ of functors $\mathrm{RFib}(A) \to \mathcal{H}$. Likewise, the dual scenario gives a natural map $\mathbf{R}a^* \to \mathbf{R}b^*$ of functors $\mathrm{LFib}(A) \to \mathcal{H}$.

Example 1.1.42 Let $\mathcal{C}$ be an ∞-category and $u\colon a \to b$ a morphism, inducing natural maps

$$u^*\colon \mathbf{R}b^* \to \mathbf{R}a^* \qquad u_*\colon \mathbf{R}a^* \to \mathbf{R}b^*$$

of functors $\mathrm{RFib}(\mathcal{C}) \to \mathcal{H}$ and $\mathrm{LFib}(\mathcal{C}) \to \mathcal{H}$ respectively. Then the following conditions are equivalent:

1. u is invertible in $\mathcal{C}$.
2. The map $u^*\colon \mathbf{R}b^* \to \mathbf{R}a^*$ is a natural isomorphism.
3. The map $u_*\colon \mathbf{R}a^* \to \mathbf{R}b^*$ is a natural isomorphism.

Indeed, (1) ⇒ (2), (3) because this can be checked on right fibrant objects, in which case Prop. 1.1.34 gives the result (and 3 is by duality). To see e.g. (2) ⇒ (1), apply this to the right fibrant object $\mathcal{C}//x$, in which case we have seen that the resulting maps

$$\mathcal{C}(b, x) \to \mathcal{C}(a, x)$$

induce precomposition upon applying π_0. Since these are equivalences for all x, we deduce that $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(b,-) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(a,-)$, hence $a \xrightarrow{\sim} b$ in $\mathcal{C}$. Note that adjunction provides a corresponding map of the left adjoints, and it is also equivalent for either of those to be a natural isomorphism.

Theorem 1.1.43 *Let $f\colon X \to Y$ be a morphism of simplicial sets,*

(a) $\mathbf{L}f_!\colon \mathrm{RFib}(X) \to \mathrm{RFib}(Y)$ *is fully faithful.*

(b) $\mathbf{L}f_!\colon \mathrm{LFib}(X) \to \mathrm{RFib}(Y)$ *is fully faithful.*

(c) The unit map $x \to \mathbf{R}f^*\mathbf{L}f_!(x)$ is an isomorphism in $\mathrm{RFib}(X)$ for all objects x of X.

(d) The unit map $x \to \mathbf{R}f^*\mathbf{L}f_!(x)$ is an isomorphism in $\mathrm{LFib}(X)$ for all objects x of X.

Moreover, if X and Y are ∞-categories, then these conditions are equivalent to the assertion that f is a fully faithful functor.

Proof (a) is equivalent to the unit being an isomorphism, i.e.

$$\mathbf{R}\langle E, F\rangle \to \mathbf{R}\langle \mathbf{R}f^*\mathbf{L}f_!E, F\rangle = \mathbf{R}\langle \mathbf{L}f_!E, \mathbf{L}f_!F\rangle = \mathbf{R}\langle E, \mathbf{R}f^*\mathbf{L}f_!F\rangle$$

for all $E \in \mathrm{RFib}(X)$ and $F \in \mathrm{LFib}(Y)$, using Prop. 1.1.41 and (the dual of) Lemma 1.1.37. Thus the latter lemma shows that this is equivalent to $F \to \mathbf{R}f^*\mathbf{L}f_!F$ being an isomorphism for all F, i.e. (b). Applying the latter lemma again shows that we may restrict to the case E is an object or F is an object; this provides (c) and (d).

Now assume $f\colon \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then $\mathbf{L}f_!(x) = f(x)$, and we have canonical fibrant replacements

$$x \simeq \mathcal{C}/x \qquad f(x) \simeq \mathcal{D}/f(x)$$

in $\mathrm{RFib}(\mathcal{C})$ and $\mathrm{RFib}(\mathcal{D})$. Applying $\mathbf{R}f^*$ on the right produces $\mathcal{C}/f(x) = \mathcal{C} \times_{\mathcal{D}} \mathcal{D}/f(x)$, so the unit map is represented by the morphism

$$\mathcal{C}/x \to \mathcal{D}/f(x) \times_{\mathcal{D}} \mathcal{C}$$

of right fibrant objects. This is an isomorphism in $\mathrm{RFib}(\mathcal{C})$ iff the corresponding map with the fat slice is one. This is equivalent to being a fiberwise equivalence (Cor. 1.1.7), and its fibers over an object w of $\mathcal{C}$ are precisely

$$\mathcal{C}(w,x) \to \mathcal{D}/f(x) \times_{\mathcal{D}} \mathcal{C} \times_{\mathcal{C}} w = \mathcal{D}/f(x) \times_{\mathcal{D}} f(w) = \mathcal{D}(f(w), f(x)),$$

for which being a fiberwise equivalence is exactly full faithfulness. □

Corollary 1.1.44 *Let $f\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories.*

1. *If f is essentially surjective, then the functors*

$$\mathbf{R}f^*\colon \mathrm{RFib}(\mathcal{D}) \to \mathrm{RFib}(\mathcal{C}) \qquad \mathbf{R}f^*\colon \mathrm{LFib}(\mathcal{D}) \to \mathrm{LFib}(\mathcal{C})$$

are both conservative.

2. *If f is an equivalence of ∞-categories, then the functors*

$$\mathbf{L}f_!\colon \mathrm{RFib}(X) \to \mathrm{RFib}(Y) \qquad \mathbf{L}f_!\colon \mathrm{LFib}(X) \to \mathrm{RFib}(Y)$$

are equivalences of categories. In particular $f_! \dashv f^*$ is a Quillen equivalence on the contravariant and covariant model structures.

Proof 1. If $F \to G$ is a morphism in $\mathrm{RFib}(\mathcal{D})$ for which $\mathbf{R}f^*F \to \mathbf{R}f^*G$ is an isomorphism, then for an object x of $\mathcal{C}$ we can apply $\mathbf{R}x^*$ to both sides to obtain $\mathbf{R}f(x)^*F \xrightarrow{\sim} \mathbf{R}f(x)^*G$. By Example 1.1.42, we deduce that for $\mathbf{R}y^*F \xrightarrow{\sim} \mathbf{R}y^*G$ is an isomorphism for all objects y of $\mathcal{D}$, i.e. $F_{y/} \xrightarrow{\sim} G_{y/}$ is an isomorphism for all y. Thus $F \to G$ is an isomorphism by Lemma 1.1.37.

2. Then f is fully faithful and essentially surjective, so Thm. 1.1.43 and part (1) apply. We conclude because a fully faithful functor with a conservative right adjoint is an equivalence. □

Our next goal is to construct $\mathcal{S}$, the category of spaces. We will assume the technical machinery of [HCHA, Sec. 5.1] on minimal fibrations. Unfortunately, we will have to do some set theory. We will use the following conventions:

Definition 1.1.45

1. Recall that a *Grothendieck universe* is a nonempty set $\mathbf{U}$ such that $\mathbb{N} \in \mathbf{U}$ and:
   (*i*) If $x \in y$ and $y \in \mathbf{U}$, then $x \in \mathbf{U}$.
   (*ii*) If $x, y \in \mathbf{U}$, then $\{x, y\} \in \mathbf{U}$.
   (*iii*) If $x \in \mathbf{U}$ then $\mathcal{P}(x) \in \mathbf{U}$.
   (*iv*) If $(x_i)_{i \in I}$ is a family of sets such that $I \in \mathbf{U}$ and $x_i \in \mathbf{U}$ for all $i \in I$, then $\bigcup_i x_i \in \mathbf{U}$.

Fix a Grothendieck universe $\mathbf{U}$.

2. A set is $\mathbf{U}$-small if it is an element of $\mathbf{U}$, and the category $\mathbf{Set}_\mathbf{U}$ denotes the (non-strictly) full subcategory of $\mathbf{Set}$ consisting of the $\mathbf{U}$-small sets.
3. A $\mathbf{U}$-small category is a small category for which the set of objects and morphisms are elements of $\mathbf{U}$. We denote by $\mathbf{Cat}_\mathbf{U}$ the full subcategory of $\mathbf{Cat}$ consisting of $\mathbf{U}$-small categories.
4. A simplicial set X is $\mathbf{U}$-small if for each $n \in \mathbb{N}$, X_n is a $\mathbf{U}$-small set, or equivalently if $X\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ factors through $\mathbf{Set}_\mathbf{U}$. We denote by $\mathbf{sSet}_\mathbf{U}$ the (non-strictly) full subcategory of $\mathbf{sSet}$ consisting of the $\mathbf{U}$-small simplicial sets.

We assume the standard basic properties of a Grothendieck universe. In the development of the theory, we will usually not assume the universe axiom, i.e. that every set is contained in a universe. Our assumption that $\mathbb{N} \in \mathbf{U}$ implies that $\mathbf{\Delta}$ is a $\mathbf{U}$-small category. If x, y are $\mathbf{U}$-small sets, then y^x is also $\mathbf{U}$-small,

so we observe that the representable presheaves Δ^n are all **U**-small simplicial sets. We remark that X is a simplicial set for which each X_n is isomorphic to a **U**-small set, then it is easy to see that X is isomorphic to a **U**-small simplicial set.

The categories $\mathbf{Set_U}, \mathbf{sSet_U}, \mathbf{Cat_U}$ are all small but not **U**-small categories; however if **V** is a universe such that $\mathbf{U} \in \mathbf{V}$, then these categories are **V**-small. Such a **V** is called an *enlargenment* of the universe **U**. We will generally have fixed a universe **U**.A **U**-small (co)limit is a (co)limit of a functor $I \to C$ where I is a **U**-small category. A category C is **U**-(co)complete if it admits all **U**-small (co)limits. The categories $\mathbf{Set_U}, \mathbf{sSet_U}$, and $\mathbf{Cat_U}$ are all **U**-bicomplete, as is easy to see.

We construct a simplicial set U as follows: for each $n \geq 0$, U_n consists of triples (X, p, C) where X is a **U**-small simplicial set, $p\colon X \to \Delta^n$ is a morphism of simplicial sets, and C is a function assigning to each morphism $f\colon [m] \to [n]$ a triple

$$f^*(X) \qquad f^*p\colon f^*(X) \to \Delta^m \qquad \widetilde{f}\colon f^*(X) \to X$$

where $f^*(X)$ is a **U**-small simplicial set, so that the following conditions are satisfied:

1. For each morphism $f\colon [m] \to [n]$, the corresponding diagram

$$\begin{array}{ccc} f^*(X) & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle f^*p}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^m & \xrightarrow{f} & \Delta^n \end{array}$$

   of morphisms of **U**-small simplicial sets is cartesian.

2. For the map $f = 1_n\colon [n] \to [n]$, we have $1_n^*(X) = X$ and $\widetilde{1_n} = 1_X$, so $1_n^* p = p$.

We note that at least U_n is a well-defined set, because for all morphisms f there does exist at least one such choice of Cartesian square as above, as $\mathbf{sSet_U}$ has limits (which are also limits in **sSet**). Next we specify the action on maps: if $u\colon [m] \to [n]$ is a map, then we define $U_n \to U_m$ by associating by sending (X, p, C) to the $(u^*(X), u^*(p), u^*C)$, where the pullback datum u^*C is defined as follows:

1. For each map $f\colon [m'] \to [m]$, we associate the square

$$\begin{array}{ccccc} (uf)^*(X) & \longrightarrow & u^*(X) & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^{m'} & \xrightarrow{f} & \Delta^m & \longrightarrow & \Delta^n \end{array}$$

   on the left-hand side above, since the pasting lemma implies that there is a unique map $(uf)^*(X) \to u^*(X)$ making the diagram commute, and hence cartesian. In other words we are *defining* $(uf)^*X = f^*u^*X$.

2. It is easy to see from this definition that U is functorial, producing a well-defined simplicial set U in **sSet**.

Note that U is very much *not* **U**-small, however each element of U_n is an element of **U**, i.e. $U_n \subset \mathbf{U}$, and thus $U \subset \mathbf{U}$. U is the simplicial set of morphisms with small fibers. We also define a pointed version $U_\bullet$ as follows, where $(U_\bullet)_n$ is a tuple $((X, p, C), s)$ where $(X, p, C) \in U_n$, and s is a section of p. Then given a map $u\colon [m] \to [n]$, the section associated to $u^*(X) \to \Delta^m$ is the unique map $\Delta^m \to u^*(X)$ induced by 1_{Δ^m} and s. It is clear that this also defines a simplicial set, and that there is a projection $U_\bullet \to U$ forgetting the section.

Definition 1.1.46 We define the simplicial set $\mathcal{S}$ as the subobject $\mathcal{S} \subset U$ such that for each n, $\mathcal{S}_n \subset U_n$ consists of the triples (X, p, C) for which $p\colon \Delta^n \to X$ is a left fibration. We define the simplicial set $\mathcal{S}_\bullet$ to be the unique subobject $\mathcal{S}_\bullet \subset U_\bullet$ fitting into a cartesian square:

$$\begin{array}{ccc} \mathcal{S}_\bullet & \hookrightarrow & U_\bullet \\ {\scriptstyle p_{\text{univ}}}\downarrow & \lrcorner & \downarrow \\ \mathcal{S} & \hookrightarrow & U \end{array}$$

The map p_{univ} is called the *universal left fibration*.

We note that $\mathcal{S}$ is well-defined, because the pullback of left fibration is a left fibration. We will later call $\mathcal{S}$ the ∞-category of spaces.

Example 1.1.47 Let us consider the low-dimensional simplices of these simplicial sets. An object of U is a **U**-small simplicial set X with a **U**-small choice of product $\Delta^m \times X$ for each $m > 0$, so that the $m = 0$ case is the identity. We should think of this as roughly just being a **U**-small simplicial set. A morphism in U from X to Y is a 'family' of small simplicial sets $M \to \Delta^1$, where $M_0 = X, M_1 = Y$. For $U_\bullet$ and object is precisely a small simplicial set X, with the choices of product as above, and a point of X.

Now consider $\mathcal{S}$. An object of $\mathcal{S}$ is a **U**-small simplicial set (with choices of products as above) X for which $X \to \Delta^0$ is a left fibration, which precisely means that X is an ∞-groupoid. An object of $(\mathcal{S})_\bullet$ is another ∞-groupoid but equipped with a point. Now, a morphism consists of a left fibration $\mathcal{C} \to \Delta^1$ such that $\mathcal{C}_0$ and $\mathcal{C}_1$ are ∞-groupoids. We note moreover that since Δ^1 is an ∞-category, the map $0 \to 1$ induces a map $\mathcal{C}_0 \to \mathcal{C}_1$ which is unique up to contractible choice, and unique in the homotopy category.

Consider the natural map $\pi\colon U_\bullet \to U$, and suppose we have an n-simplex $\Delta^n \to U$ represented by (X, p, C). Then in fact there is a canonical map $X \to U_\bullet$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{x}} & U_\bullet \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ \Delta^n & \xrightarrow{x} & U \end{array}$$

cartesian. To see this, we first construct the upper map in a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(\Delta^m, X) & \dashrightarrow & \mathrm{Hom}(\Delta^m, U_\bullet) \\ {\scriptstyle p_*}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_*} \\ \mathrm{Hom}(\Delta^m, \Delta^n) & \xrightarrow{x_*} & \mathrm{Hom}(\Delta^m, U) \end{array}$$

Given $\varphi\colon \Delta^m \to X$, the composite $p\varphi\colon \Delta^m \to \Delta^n$ induces a pullback datum $(p\varphi)^*(X, p, C) \in \mathrm{Hom}(\Delta^m, U)$; since $(p\varphi)^*X = \Delta^m \times_{\Delta^n} X$, the maps $1_{\Delta^m}, \varphi$ induces a section $s\colon \Delta^m \to (p\varphi)^*X$, and hence an element $\mathrm{Hom}(\Delta^m, U_\bullet)$. It is clear that this association is natural in $\mathbf{\Delta}$ because pulling back is natural, giving a map $\widetilde{x}\colon X \to U_\bullet$.

It remains to check that the square of Hom's above is cartesian. Suppose we are given an m-simplex $(q\colon X' \to \Delta^m, C', t)$ of $U_\bullet$ and a map $f\colon \Delta^m \to \Delta^n$ whose images in $\mathrm{Hom}(\Delta^m, U)$ agree. This means precisely that $f^*(X, p, C) = (X', q, C')$ in U, i.e. that the square

$$\begin{array}{ccc} X' & \xrightarrow{\widetilde{f}} & X \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ \Delta^m & \xrightarrow{f} & \Delta^n \end{array}$$

is cartesian. Thus the section $t\colon \Delta^m \to X'$ induces a map $\Delta^m \to X' \to X$; applying p_* recovers f, and it follows from uniqueness that its image in $\mathrm{Hom}(\Delta^m, U_\bullet)$ is (X', q, C', t). Since $\widetilde{f}$ and t are uniquely determined, it follows that this map $\Delta^m \to X$ is unique, completing the proof.

Lemma 1.1.48 *Let $f\colon X \to Y$ be a morphism of simplicial sets. There is a canonical correspondence of the form*

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{F}} & U_\bullet \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{F} & U \end{array} \qquad\qquad \begin{array}{ccc} \varphi^*(X) & \longrightarrow & X \\ {\scriptstyle \varphi^*(f)}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow{\varphi} & Y \end{array}$$

between cartesian squares on the left, and for each $\Delta^n \to Y$ a choice of cartesian square on the right with $\varphi^*(X)$ a $\mathbf{U}$-small simplicial set. Such data exists if and only if for all $\varphi\colon \Delta^n \to Y$, the pullback $\Delta^n \times_Y X$ is isomorphic to a $\mathbf{U}$-small set.

Proof Starting with a square on the left and a map $\varphi\colon \Delta^n \to Y$, the composite $\Delta^n \to U$ is a simplex (X', p, C), and thus in the diagram

$$\begin{array}{ccccc} X' & \dashrightarrow & X & \longrightarrow & U_\bullet \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle f} & \lrcorner & \downarrow \\ \Delta^n & \xrightarrow{\varphi} & Y & \longrightarrow & U \end{array} \qquad (*)$$

there is a unique map $X' \to X$ making both squares cartesian, because of the result above. The left-hand square gives the data on the right side in the lemma statement.

Conversely, given cartesian squares for each $\Delta^n \to Y$, we obtain a map $\mathrm{Hom}(\Delta^n, Y) \to \mathrm{Hom}(\Delta^n, U)$ obtained by sending $\varphi\colon \Delta^n \to Y$ to $\varphi^*(X) \to \Delta^n$; indeed, we define the pullback datum C using, for each map $[m] \to [n]$, the chosen pullback square for the composite $\Delta^m \to \Delta^n \to Y$. By construction this is natural in n, producing a map $F\colon Y \to U$. Moreover, we have a map $X \to U_\bullet$, given by sending an n-simplex x to $F(x) \in U_n$ equipped with the section induced by $x\colon \Delta^n \to X$ and 1_{Δ^n}.

This produces a commutative diagram on the right side of $(*)$, and associated pullback squares on the left for each $\Delta^n \to Y$, such that the outer square is cartesian. This implies that the right-hand square is cartesian, because this property in a presheaf category can be checked after pulling back to all representables. It is evident from the constructions that the two directions are mutually inverse.

The last statement is clear, because the right-side data for each map $\Delta^n \to Y$ exists precisely when we can choose a **U**-small representative for the pullback $\Delta^n \times_Y X$. □

Definition 1.1.49 A morphism $f\colon X \to Y$ of simplicial sets has **U**-*small fibers* if for all simplices $\Delta^n \to Y$, the pullback $\Delta^n \times_Y X$ is isomorphic to a **U**-small set. In this case, a choice of **U**-small pullback for all simplices gives a cartesian square as in Lemma 1.1.48, in which case we call F a *classying map for* f.

In other words, f has a classifying map iff it has **U**-small fibers; it is not literally unique, nor is it unique up to isomorphism because the image of such a map depends on the choice of **U**-small pullbacks. However, we should view it as 'homotopically unique' since it corresponds to data induced by choosing **U**-small representatives among isomorphism classes. We note that if X is a **U**-small simplicial set, then any map $f\colon X \to Y$ has **U**-small fibers, and in fact we can choose a 'canonical' **U**-small representative, given by the set-theoretic cartesian product $\Delta^n \times_Y X \subset \Delta^n \times X$.

Remark 1.1.50 Let $f\colon X \to Y$ be a map. Then f has **U**-small fibers iff for all n and n-simplices $\sigma \in Y_n$, $f_n^{-1}(\sigma) \subset X_n$ is isomorphic to a **U**-small set. Indeed, on one hand

$$f_n^{-1}(\sigma) = \{1_{[n]}\} \times_{Y_n} X_n \subset (\Delta^n \times_Y X)_n.$$

Conversely, given $\sigma\colon \Delta^n \to Y$ we have

$$(\Delta^n \times_Y X)_m = \coprod_{\alpha\colon [m]\to[n]} f_m^{-1}(\sigma \circ \alpha),$$

so this is also isomorphic to a **U**-small set.

Example 1.1.51 Suppose we are given a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow{i} & B \end{array}$$

with i a monomorphism and f, f' having **U**-small fibers. If F is a classifying map for F, then we may choose a classifying map G for f such that $Gi = f$. Indeed, F corresponds to a collection of choices of pullbacks $\varphi^*(X)$ for each $\varphi\colon \Delta^n \to A$. For $\Delta^n \to B$, if this simplex is contained in A, then we may choose the pullback given prescribed by f, and otherwise we can choose any **U**-small pullback. Then it is clear from the construction that the corresponding classifying map G restricts to F on A.

Corollary 1.1.52 *The canonical map $p_{\mathrm{univ}}\colon \mathcal{S}_\bullet \to \mathcal{S}$ is a left fibration. The following are equivalent for a morphism $f\colon X \to Y$ with* **U***-small fibers:*

(*i*) *f is a left fibration.*

(*ii*) *There exists a classifying map for f which factors through $\mathcal{S} \hookrightarrow U$.*

(*iii*) *Every classifying map for f factors thorugh $\mathcal{S} \hookrightarrow U$.*

Proof Consider a map $\Delta^n \to \mathcal{S}$ associated to (X', p, C), and consider the commutative diagram:

$$\begin{array}{ccccccc} \Lambda^n_k & \dashrightarrow & X' & \longrightarrow & \mathcal{S}_\bullet & \longrightarrow & U_\bullet \\ \downarrow & & \downarrow{\scriptstyle p} & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^n & = & \Delta^n & \longrightarrow & \mathcal{S} & \longrightarrow & U \end{array}$$

Ignoring the left-most square, we have already seen that the outer rectangle is cartesian, so there's a unique map $X' \to \mathcal{S}_\bullet$ making the middle square cartesian. Now for any square involving $\Lambda^n_k \to \mathcal{S}_\bullet$ for $k < n$, and a map $\Delta^n \to \mathcal{S}$, the former map factors through X', producing a commutative square as on the left. Since p is a right fibration by assumption, there is a filler $\Delta^n \to X'$, and thus the composite to $\Delta^n \to X' \to \mathcal{S}_\bullet$ fills the original square. Thus p_{univ} is a left fibration.

Now, since f has **U**-small fibers, it admits a classifying map. (*iii*) $\Rightarrow$ (*ii*) is clear, and (*ii*) $\Rightarrow$ (*i*) is immediate because if a classifying map factors through $\mathcal{S}$, then we obtain a factorization as in:

$$\begin{array}{ccccc} X & \longrightarrow & \mathcal{S}_\bullet & \longrightarrow & U_\bullet \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ Y & \longrightarrow & \mathcal{S} & \longrightarrow & U \end{array}$$

Thus $X \to Y$ is a left fibration since p_{univ} is. (*i*) $\Rightarrow$ (*iii*) By the correspondence of Lemma 1.1.48, any classifying map F sends an n-simplex $\varphi\colon \Delta^n \to Y$ to an object in U_n of the form $\varphi^*(X) \to \Delta^n$, which is a pullback of f. Thus if f is a left fibration then this map is too, hence lies in $\mathcal{S}_n$. □

Our next goal is to show that $\mathcal{S}$ (and hence $\mathcal{S}_\bullet$, by the previous result) is an ∞-category. We will need two lemmas, and then this will be clear. We will also demonstrate that the core of the map $\mathcal{S}_\bullet \to \mathcal{S}$ should be interpreted as the universal Kan fibration.

Lemma 1.1.53 *Let $n > 0$ and $0 < k < n$, and $i\colon \Lambda^n_k \hookrightarrow \Delta^n$ be the inclusion.*

1. *The functors*

$$\mathbf{L}i_!\colon \mathrm{RFib}(\Lambda^n_k) \to \mathrm{RFib}(\Delta^n) \qquad \mathbf{L}i_!\colon \mathrm{LFib}(\Lambda^n_k) \to \mathrm{LFib}(\Delta^n)$$

   are equivalences of categories.

2. *For any commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p} & & {\scriptstyle q}\downarrow \\ \Lambda^n_k & \hookrightarrow & \Delta^n \end{array}$$

   where f is final (resp. cofinal), and p and q are right (resp. left) fibrations, the induced map $X \to \Lambda^n_k \times_{\Delta^n} Y$ is a fiberwise equivalence over Λ^n_k.

Proof (1) The LFib case will follow by dualization. First, the functor $\mathbf{R}i^*$ is conservative: because i is bijective on objects, we see that if $F \to G$ is a map and $\mathbf{R}i^*F \xrightarrow{\sim} \mathbf{R}i^*G$, then $F_{/a} \xrightarrow{\sim} G_{/a}$ for all objects a of Λ^n_k, implying $F \to G$ is an isomorphism by Lemma 1.1.37. Thus it suffices to show that $\mathbf{L}i_!$ is fully faithful, and by Thm. 1.1.43 this is equivalent to showing that $a \to \mathbf{R}i^*\mathbf{L}i_!a$ is an isomorphism in $\mathrm{RFib}(\Lambda^n_k)$, for all $a \in [n]$.

Note that $\mathbf{L}i_!(a) = a$ as an object of the ∞-category Δ^n, and this has a canonical fibrant resolution given by $D^n/a = N([n]/a) = N([a]) = \Delta^a$. Thus the unit map is represented by $\{a\} \to \Lambda^n_k \times_{\Delta^n} \Delta^a = \Lambda^n_k \cap \Delta^a$. To show this is an isomorphism, it suffices to show that this map is final, or in other words that a is terminal object of the codomain. If $a < n$, then since $k \neq n$, it follows that $\Delta^a \subset \Lambda^n_k$, and this case is clear because a is terminal in the category Δ^a. If $a = n$, then the map is $\{n\} \to \Lambda^n_k$, which is a right anodyne extension by [HCHA, Lem. 4.4.3] as $k \neq 0$.

(2) Note that the unit map $X \to \mathbf{R}i^*\mathbf{L}i_!X$ is an isomorphism, and f being final means that the induced map $\mathbf{L}i_!X \to Y$ is an isomorphism in $\mathrm{RFib}(\Delta^n)$, thus the unit is represented by $X \to \Lambda^n_k \times_{\Delta^n} Y$. Being an isomorphism in $\mathrm{RFib}(\Lambda^n_k)$ implies that this is a weak equivalence of right fibrant objects. □

Remark 1.1.54 Let us record two important simple facts used in the proof which will come up all the time. Let $f\colon A \to B$ is a morphism of simplicial sets:

1. $\mathbf{L}f_!\colon \mathrm{RFib}(A) \to \mathrm{RFib}(B)$ is fully faithful if and only if for every commutative diagram
$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$
in which p and q are right fibrations and $\overline{f}$ is final, the map $X \to A \times_B Y$ is final. *Proof* Full faithfulness is equivalent to the unit map $F \to \mathbf{R}f^*\mathbf{L}f_!F$ being an isomorphism in $\mathrm{RFib}(A)$ for all objects F. ($\Leftarrow$) Represent F by a right fibration $X \to A$, and choose a fibrant replacement $\overline{f}\colon \mathbf{L}f_!F = X \to Y$ for some right fibration $Y \to B$. Since Y is right fibrant over B, finality of $\overline{f}$ is equivalent to being an isomorphism in $\mathrm{RFib}(B)$, so
$$X \to \mathbf{R}f^*\mathbf{L}f_!X = f^*(Y) = A \times_B Y$$
represents the unit. Since $A \times_B Y \to A$ is a right fibration, this map being an isomorphism in $\mathrm{RFib}(X)$ is equivalent to finality. This also clearly demonstrates the ($\Rightarrow$) direction by treating the fibrant representatives as given.

2. If f is surjective on objects, then $\mathbf{R}f^*$ is conservative. *Proof* Let $F \to G$ be a morphism in $\mathrm{RFib}(A)$ such that $\mathbf{R}f^*F \to \mathbf{R}f^*G$ is an isomorphism. Then for each object a of A, applying $\mathbf{R}a^*$ to both sides shows that $\mathbf{R}f(a)^*F \to \mathbf{R}f(a)^*G$ is an isomorphism, i.e. $F_{f(a)/} \xrightarrow{\sim} G_{f(a)/}$ in $\mathcal{H}$ for all a. Then the assumption and Lemma 1.1.37 implies that $F \to G$ is an isomorphism.

Remark 1.1.55 In the following result, we will use a recollection about the small object argument, especially in the context of horn lifting $\{\Lambda^n_k \to \Delta^n\}$. For illustration, we'll write this in the context of left horn lifting. Recall how this works:

1. If $f\colon X \to B$ is a map, set $E_0 = X$ and $f_0 = f$, and assume inductively that $f_r\colon E_r \to B$ has been constructed.

2. At stage r, let S_r be the set of all lifting problems for all $\Lambda^n_k \hookrightarrow \Delta^n$ against f_r, and then define E_{r+1} as the pushout:
$$\begin{array}{ccc} \coprod_{s\in S_r} \Lambda^{m_s}_{j_s} & \longrightarrow & E_r \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{s\in S_r} \Delta^{m_s} & \longrightarrow & E_{r+1} \end{array}$$
For each lifting problem $s \in S_r$, the canonical map $\Delta^{m_s} \to B$ induces an extension $E_{r+1} \to B$.

3. Finally, define $E \coloneqq \varinjlim_r E_r$ with its canonical map $E \to B$, and consider the composite $X \hookrightarrow E \to B$, which is by construction a factorization of f. It follows from saturation that $E_r \to E_{r+1}$ is a left anodyne, hence $X \hookrightarrow E$ is left anodyne. The important point is that $E \to B$ is a left

fibration: given a lifting problem with map $\Lambda^n_k \to E$, it follows from the fact that Λ^n_k is a finite simplicial set (i.e. has finitely many non-degenerate sections) that this map factors through E_r for some r, by [HCHA, Cor. 1.3.12]. Thus the map exists at the rth stage and hence has a lift in E_{r+1}, completing the proof.

The point is that this construction can be chosen to stay in the fixed universe $\mathbf{U}$. Indeed, if E_r and B are $\mathbf{U}$-small sets, then S_r is $\mathbf{U}$-small set because it is a subset of

$$\coprod_{n\geq 1, 0\leq j\leq n} \mathrm{Hom}(\Lambda^n_j, E_r) \times \mathrm{Hom}(\Delta^m, B).$$

Therefore we can choose $\mathbf{U}$-small coproducts in step 2, and hence a $\mathbf{U}$-small E_{r+1}. Finally, by induction, we may choose a $\mathbf{U}$-small E to factor through, as long as X was $\mathbf{U}$-small to begin with. It is clear that this works for all of the lifting properties we encounter in practice, since they all consist of right lifting against countably many inclusions of finite simplicial sets.

Corollary 1.1.56 *Let $0 < k < n$. For any minimal left fibration $p\colon X \to \Lambda^n_k$ with $\mathbf{U}$-small fibers, there exists a minimal left fibration $q\colon Y \to \Delta^n$ with $\mathbf{U}$-small fibers and a cofinal map $f\colon X \to Y$ such that the square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle q} \\ \Lambda^n_k & \hookrightarrow & \Delta^n \end{array}$$

is cartesian.

Proof Note that for each $d \geq 0$, since $(\Lambda^n_k)_d$ is finite, it follows that X_d is isomorphic to a finite disjoint disjoint union of $\mathbf{U}$-small sets (namely the fibers over simplices), and thus that X is isomorphic to a $\mathbf{U}$-small simplicial set. Since the conclusions of the proof are invariant under isomorphism, we may assume without loss of generality that X is $\mathbf{U}$-small.

Now factor the composite $X \to \Lambda^n_k \to \Delta^n$ into $X \to X_0 \to \Delta^n$ where the first map is left anodyne and the second is a left fibration. By Remark 1.1.55 we may choose X_0 to be $\mathbf{U}$-small. Now factor $X_0 \to \Delta^n$ into $X_0 \to Y \to \Delta^n$, where $X_0 \to Y$ is a trivial fibration and $Y \to \Delta^n$ is a minimal left fibration, by [HCHA, Thm. 5.1.17]. Since a trivial fibration has a section, Y is isomorphic to a subobject of X_0, and hence we may assume that Y is $\mathbf{U}$-small. Thus $Y \to \Delta^n$ has $\mathbf{U}$-small fibers. This gives a diagram as in the statement, and it remains to show the cartesian property.

To conclude, note that f is a composition of a left anodyne map and a trivial fibration, hence is cofinal. Therefore the induced map

$$X \to \Lambda^n_k \times_{\Delta^n} Y$$

is a weak equivalence in $(\mathbf{sSet}/\Lambda^n_k)_{\mathrm{cov}}$, by Lemma 1.1.53. On the other hand, the structure maps to Λ^n_k are both minimal left fibrations by pullback stability,

so this is a weak equivalence of minimal complexes in $(\mathbf{sSet}/\Lambda^n_k)_{\mathrm{cov}}$, hence an isomorphism by [HCHA, Prop. 5.1.10]. □

Remark 1.1.57 In the following major result, we will need to apply [HCHA, Thm. 5.1.20]; it is important to know that we can do this while preserving **U**-small fibers. Namely, in the book's notation, we are given the left and bottom maps in:

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow & \lrcorner & \vdots \\ Y & \xrightarrow{v} & Y' \end{array}$$

Here, $X' \to Y'$ is constructed by applying v_* to $X \to Y$, where v_* is the right adjoint to the pullback functor $v^* : \mathbf{sSet}/Y' \to \mathbf{sSet}/Y$. We want to show that if X, Y, Y' are **U**-small, then X' will also be. Indeed, writing $p : X' \to Y'$ the map, we have that for each simplex $\sigma : \Delta^n \to Y'$,

$$p_n^{-1}(\sigma) = \mathrm{Hom}_{\mathbf{sSet}/Y'}(\Delta^n, p) \cong \mathrm{Hom}_{\mathbf{sSet}/Y}(\Delta^n \times_{Y'} Y, X).$$

Then the RHS is a subset of $\mathrm{Hom}(\Delta^n \times_{Y'} Y, X)$, and Y and X are small therefore this set is small. Moreover X' is a Y'-indexed disjoint union of such sets, and since Y' is **U**-small, we conclude X' is isomorphic to a **U**-small set.

Theorem 1.1.58 *The simplicial sets* $\mathcal{S}$ *and* $\mathcal{S}_\bullet$ *are* ∞*-categories.*

Proof Since $\mathcal{S}_\bullet \to \mathcal{S}$ is a left fibration, it suffices to prove that $\mathcal{S}$ is an ∞-category. Let $\Lambda^n_k \to \mathcal{S}$ be a map for $0 < k < n$, and choose a pullback:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \Lambda^n_k & \longrightarrow & \mathcal{S} \end{array}$$

Thus $X \to \Lambda^n_k$ is a left fibration with **U**-small fibers. Now construct two cartesian squares as follows:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \longrightarrow & Y_0 \\ \downarrow & \lrcorner & \downarrow \\ \Lambda^n_k & \longrightarrow & \Delta^n \end{array}$$

First, factor $X \to \Lambda^n_k$ through X_0, so that $X_0 \to \Lambda^n_k$ is minimal left fibration and $X \to X_0$ is a trivial fibration, by [HCHA, Thm. 5.1.17]. Since $X \to X_0$ has a section, it follows that $X_0 \to \Lambda^n_k$ has **U**-small fibers, so by Cor. 1.1.56 we may choose a minimal left fibration $Y_0 \to \Delta^n$ with $Y_0 \to \Delta^n$ having **U**-small fibers. Then $X_0 \to Y_0$ is a monomorphism, so we apply [HCHA, Lem. 5.1.20] to obtain a trivial fibration $Y \to Y_0$ making the upper square cartesian. We claim that this has **U**-small fibers: indeed, this is clear from the proof of [HCHA, Lem. 5.1.20], as the right adjoint construction in that proof, and the fact that

X is isomorphic to a **U**-small simplicial set, imply that Y is isomorphic to a **U**-small simplicial set by the description of [HCHA, Thm. 1.1.10].

To complete the proof, note that the outer square is cartesian, and by construction $\Lambda^n_k \to \mathcal{S}$ is a classifying map for $X \to \Lambda^n_k$. by Example 1.1.51, we can choose a classifying map $\Delta^n \to U$ for the composite $Y \to \Delta^n$ which extends $\Lambda^n_k \to U$. Since $Y \to \Delta^n$ is the composite of a trivial fibration and a left fibration, it is a left fibration, so by Cor. 1.1.52 its classifying map factors through $\mathcal{S} \to U$. Thus we have extended $\Lambda^n_k \to \mathcal{S}$ to Δ^n. □

Example 1.1.59 The proof demonstrates the rough correspondence between the ∞-category $\mathcal{S}$ having certain properties, and general lifting properties of left/right fibrations. For instance, since $\mathcal{S} \to \Delta^0$ is a Joyal fibration, we deduce that for any left fibration $p\colon X \to A$ with **U**-small fibers and any Joyal trivial cofibration $A \to B$, we can find a cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle q} \\ A & \longrightarrow & B \end{array}$$

where $Y \to B$ is a left fibration with **U**-small fibers. Indeed, choose a classifying map $F\colon A \to \mathcal{S}$ for p, then the assumption is that this has a lift to $G\colon B \to \mathcal{S}$ along i, and then this classifies a map $q\colon Y \to B$. Thus by pasting, there is a unique map $X \to Y$ making the square above cartesian.

The last major result we want to demonstrate is that the Kan complex $\mathcal{S}^{\simeq}$ with its natural map from $\mathcal{S}^{\simeq}_{\bullet}$ classifies Kan fibrations.

Lemma 1.1.60 *Let $i\colon A \to B$ be an anodyne extension, then for any Kan fibration $p\colon X \to A$ with* **U***-small fibers, there is a cartesian diagram of the form*

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle q} \\ A & \longrightarrow & B \end{array}$$

where $q\colon Y \to B$ is a Kan fibration with **U***-small fibers.*

Proof First apply the same trick as in the proof of Thm. 1.1.58, which is that by factoring p into a trivial fibration followed by a minimal fibration (and using the **U**-small fibers version of [HCHA, Lem. 5.1.20]), we may assume that p is a minimal Kan fibration. Now factor the composite $X \to B$ as follows:

$$X \to X_0 \to Y \to B$$

where $X \to X_0 \to B$ is an anodyne extension followed by a trivial fibration, and then $X_0 \to Y \to B$ is a trivial fibration followed by a minimal fibration q. Then $X \to Y$ is a weak equivalence, and as in Cor. 1.1.56, and due to Remark 1.1.55, we may assume that $Y \to B$ has **U**-small fibers.

To conclude, the map $X \to A \times_B Y$ is a morphism of minimal complexes in $\mathbf{sSet}_{\mathrm{KQ}}/A$, and it is also a weak homotopy equivalence because the composite

$$X \to A \times_B Y \to Y$$

is one, and $A \times_B Y \to Y$ is also one by [HCHA, Prop. 3.1.28] and the fact that $A \to B$ is one. Therefore, by [HCHA, Prop. 5.5.10] we deduce $X \xrightarrow{\sim} A \times_B Y$ is an isomorphism. □

Remark 1.1.61 There's an issue with the previous proof in the case that A isn't **U**-small, because then Remark 1.1.55 doesn't immediately apply. Instead, we can fix this as follows: the assertion we are trying to prove is equivalent to saying that the subcomplex of $\mathcal{S}$ consisting of simplices (X, p, C) where each p is a Kan fibration, is in fact a Kan complex. Therefore we can reduce to the case where $A \to B$ is a horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, and thus both A and B are actually **U**-small.

Theorem 1.1.62 *The canonical map $p^{\simeq}_{\mathrm{univ}} \colon \mathcal{S}^{\simeq}_{\bullet} \to \mathcal{S}^{\simeq}$ is a Kan fibration of ∞-groupoids. The following are equivalent for a morphism $f \colon X \to Y$ with* **U***-small fibers:*

(*i*) *f is a Kan fibration.*

(*ii*) *There exists a classifying map for f which factors through $\mathcal{S}^{\simeq} \to U$.*

(*iii*) *Every classifying map for f factors through $\mathcal{S}^{\simeq} \hookrightarrow U$.*

Proof Since $p_{\mathrm{univ}} \colon \mathcal{S}_{\bullet} \to \mathcal{S}$ is a left fibration between ∞-categories, it is in particular a conservative isofibration ([HCHA, Prop. 3.4.8]), so we obtain a canonical cartesian square:

$$\begin{array}{ccc} \mathcal{S}^{\simeq}_{\bullet} & \longrightarrow & \mathcal{S}_{\bullet} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{S}^{\simeq} & \longrightarrow & \mathcal{S} \end{array}$$

This implies that $\mathcal{S}^{\simeq}_{\bullet} \to \mathcal{S}^{\simeq}$ is a left fibration between ∞-groupoids, hence a Kan fibration (by [HCHA, Prop. 3.5.5]). Now (*iii*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*i*) are immediate because if a classifying map factors through $\mathcal{S}^{\simeq}$, then f is a pullback of the Kan fibration $\mathcal{S}^{\simeq}_{\bullet} \to \mathcal{S}^{\simeq}$.

Assume (*i*), and let F be any classifying map for f. By factoring the map $Y \to \Delta^0$, choose an anodyne extension $Y \to Y'$ where Y' is a Kan complex (e.g. Ex^{∞}). By the previous lemma, choose a cartesian square

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle f} & \lrcorner & \downarrow{\scriptstyle g} \\ Y & \longrightarrow & Y' \end{array}$$

in which g is a Kan fibration with **U**-small fibers. By Example 1.1.51, choose a classifying map G extending the one for F. Since g is in particular a left fibration, G factors through $\mathcal{S}$. Since Y' is a Kan complex, G factors through $\mathcal{S}^{\simeq}$, hence F does too. □

1.2 Presheaves and the Yoneda Lemma

Our goal in this section is to prove two important results: First, the correspondence between right fibrations and presheaves valued in $\mathcal{S}$, and secondly the ∞-categorical Yoneda lemma. The latter will provide the basis for our theory of ∞-categorical constructions, which is studied in the next section.

1.2.1 Homotopy theory of left fibrations

Before getting to the main result, we need to spend a some work building up theory about maps relevant to the Joyal model category structure which aren't necessarily between fibrant objects. Thus we will first generalize some results that we already basically know in the case of a functor between ∞-categories.

Theorem 1.2.1 *If $f\colon X \to Y$ is a weak categorical equivalence, then the functors*

$$\mathbf{L}f_!\colon \mathrm{RFib}(X) \to \mathrm{RFib}(Y) \qquad \mathbf{L}f_!\colon \mathrm{LFib}(X) \to \mathrm{LFib}(Y)$$

are equivalences of categories.

Proof We need only prove the RFib case by dualization. We are going to apply the criterion of [HCHA, Prop. 3.6.2] to the class of morphisms such that $\mathbf{L}f_!$ is an equivalence; it is clear that this class satisfies 2-of-3, and every trivial fibration between ∞-categories is in this class, because all equivalences of ∞-categories are by Cor. 1.1.44. Therefore it remains to demonstrate this for all inner anodyne extensions. Since an inner anodyne extension induces a bijection on objects, we have previously explained why this implies that $\mathbf{R}f^*$ is conservative. Thus it remains to show that $\mathbf{R}f_!$ is fully faithful, i.e. $F \to \mathbf{R}f^*\mathbf{L}f_!(F)$ is an isomorphism for all F.

Represent F by a right fibration $F \to X$, and choose a universe $\mathbf{U}$ in which F is $\mathbf{U}$-small, so that this map has $\mathbf{U}$-small fibers. By Example 1.1.59, we can choose a cartesian square

$$\begin{array}{ccc} F & \xrightarrow{\widetilde{f}} & G \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

with vertical maps being right fibrations, since $X \to Y$ is a Joyal trivial cofibration. The composite

$$F \to \mathbf{R}f^*\mathbf{L}f_!(F) \xrightarrow{\overline{f}} \mathbf{R}f^*G = X \times_Y G$$

is an isomorphism by definition, so full faithfulness (at F) would be equivalent to $\mathbf{L}f_!(F) \to G$ being an isomorphism as $\mathbf{R}f^*$ is conservative. Since G is right fibrant over Y, this is equivalent to $F \to G$ being final, and it is a monomorphism, so this is equivalent to $F \to G$ being right anodyne. Thus, we may consider the class of all morphisms $f\colon X \to Y$ such that for all right fibrations $G \to Y$, $X \times_Y G \to G$ is right anodyne. We prove that this class has the following closure properties:

1. *Inner horn inclusions*: If f is an inner horn inclusion $\Lambda^n_k \hookrightarrow \Delta^n$, then we already know that $\mathbf{L}f_!$ is an equivalence by Lemma 1.1.53, so all $F \to G$ as above are right anodyne.

2. *Retracts*: Suppose f is a retract of $g\colon X' \to Y'$ in the class. For any right fibration $G \to Y$, applying the functor $(-)_Y G$ to the corresponding retract diagram gives the diagram:

$$\begin{array}{ccccc} X \times_Y G & \longrightarrow & X' \times_Y G & \longrightarrow & X \times_Y G \\ \downarrow & & \downarrow & & \downarrow \\ G & \longrightarrow & Y' \times_Y G & \longrightarrow & G \end{array}$$

   Since $Y' \times_Y G \to Y'$ is a right fibration, the assmuption implies that

$$X' \times_{Y'} Y' \times_Y G = X' \times_Y G \to Y' \times_Y G \qquad (*)$$

   is a right anodyne extension, hence $X \times_Y G \to G$ is a right anodyne extension because it is a retract of one.

3. *Pushouts*: Suppose f is the pushout of $g\colon X' \to Y'$ in the class. Since $(-) \times_Y G$ preserves pushouts, we obtain a pushout square

$$\begin{array}{ccc} X' \times_Y G & \longrightarrow & Y' \times_Y G \\ \downarrow & & \downarrow \\ X \times_Y G & \longrightarrow & G \end{array}$$

   for each right fibration $G \to Y$. Once again $(*)$ gives the result since right anodyne maps are closed under pushout.

4. *Transfinite compositions:* Write $Y = \varinjlim_{\alpha<\lambda} Y_\alpha$ and $X = Y_0$, and suppose that $X \to Y_\alpha$ is in the class. If $G \to Y$ is a right fibration, so too is $G \times_Y Y_\alpha \to Y_\alpha$, and thus

$$X \times_Y Y_\alpha \times_Y G \to G$$

   is a right anodyne extension. Applying $\varinjlim_{\alpha<\lambda}$ shows that $X \times_Y Y \times_Y G = X \times_Y G \to G$ is a right anodyne extension.

This shows that the class of inner anodyne extension satisfies the desired property, which completes the proof. □

Corollary 1.2.2 *Let $f\colon X \to Y$ be a morphism of simplicial sets such that $\tau(f)\colon \tau(X) \to \tau(Y)$ is essentially surjective, and $\mathbf{L}f_!\colon \mathrm{RFib}(X) \to \mathrm{RFib}(Y)$ is fully faithful. Then f is a weak categorical equivalence.*

Proof We may choose a commutative diagram on the left below,

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow\simeq & & \simeq\downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{D} \end{array} \qquad\qquad \begin{array}{ccc} \tau(X) & \longrightarrow & \tau(Y) \\ \downarrow & & \downarrow \\ \mathrm{h}\mathcal{C} & \longrightarrow & \mathrm{h}\mathcal{D} \end{array}$$

with vertical maps both weak categorical equivalences to ∞-categories, which induces a commutative diagram on the right in **Cat**. Since τ is left Quillen ([HCHA, Prop. 3.3.14]), the vertical maps in the second diagram are equivalences of categories. Hence $\mathrm{h}\mathcal{C} \to \mathrm{h}\mathcal{D}$ is essentially surjective. By Thm. 1.2.1, we conclude that $\mathrm{RFib}(\mathcal{C}) \to \mathrm{RFib}(\mathcal{D})$ is also fully faithful, hence $\mathcal{C} \to \mathcal{D}$ is fully faithful. Thus by Thm. 1.1.43 we conclude $\mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories. □

Lemma 1.2.3 *Let $f : X \to \mathcal{D}$ be a morphism of simplicial sets and $\mathcal{D}$ be an ∞-category. There is a factorization of f into*

$$X \to \mathcal{C} \to \mathcal{D}$$

where $X \to \mathcal{C}$ is a right anodyne weak categorical equivalence, and $\mathcal{C} \to \mathcal{D}$ is a Joyal fibration between ∞-categories.

Proof Consider the morphisms $\Lambda^n_k \hookrightarrow \Delta^n$ for $0 < k < n$, and $\{\epsilon\} \hookrightarrow I_{\simeq}$ for $\epsilon = 0, 1$. These are all right anodyne extensions by [HCHA, Prop. 4.1.7]. They are also Joyal trivial cofibrations, because either they are inner anodyne, or they are sections of $I_{\simeq} \to \Delta^0$ which we have shown is a trivial cofibration.

Thus, the small object argument gives a factorization $X \to \mathcal{C} \to \mathcal{D}$ where $X \to \mathcal{C}$ is a right anodyne extension and a Joyal trivial cofibration. In particular it is a weak categorical equivalence. On the other hand $\mathcal{C} \to \mathcal{D}$ must be an inner fibration, so $\mathcal{C}$ is an ∞-category. The map $\mathcal{C} \to \mathcal{D}$ also has right lifting against $\{\epsilon\} \to I_{\simeq}$, which implies from the definition that it is an isofibration. Thus it is a Joyal fibration. □

Corollary 1.2.4 *A weak categorical equivalence is final and cofinal. In particular, a Joyal trivial cofibration is both left and right anodyne.*

Proof It suffices to show final. Given a weak equivalence $X \to Y$, this is equivalent to showing that it induces an isomorphism in $\mathrm{RFib}(Y)$. By choosing a Joyal fibrant resolution $Y \to \mathcal{D}$, and using that $\mathrm{RFib}(Y) \to \mathcal{D}$ is an equivalence by Thm. 1.2.1, we may assume that $Y = \mathcal{D}$ is an ∞-category. Using the factorization in the previous lemma, we obtain $X \to \mathcal{C} \to \mathcal{D}$ where the first map is final and a weak categorical equivalence. Thus by 2-of-3, $\mathcal{C} \to \mathcal{D}$ is a Joyal fibration and a weak categorical equivalence, hence a trivial fibration, which is final. The second statement follows because a final (resp. cofinal) monomorphism is a right (resp. left) anodyne extension. □

Example 1.2.5 Any left or right fibration is a Joyal fibration, because these have right lifting against Joyal trivial cofibrations by the above corollary.

Proposition 1.2.6 *Consider a cartesian square*

$$\begin{array}{ccc} X & \xrightarrow{\overline{f}} & Y \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array}$$

in which q is a right or left fibration. Then the square is homotopy cartesian in $\mathbf{sSet}_{\mathrm{Joy}}$. *In particular, if f is a weak categorical equivalence then $\overline{f}$ is one too.*

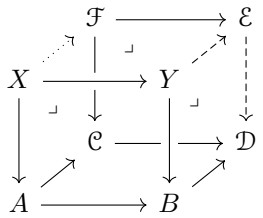
Proof It suffices to do the 'right' case. First we prove the consequence in the following special case:

Step 1: f is a weak equivalence and bijective on objects. Then $\overline{f}$ is also bijective on objects, so $\tau\overline{f}\colon \tau(X) \to \tau(Y)$ is also bijective on objects, and in particular essentially surjective. By Cor. 1.2.2, it remains to show that $\mathbf{L}\overline{f}_!\colon \mathrm{RFib}(X) \to \mathrm{RFib}(Y)$ is fully faithful. We do this using the criterion of Remark 1.1.54, so assume we have a commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & X & \xrightarrow{p} & A \\ \downarrow & & \overline{f}\downarrow & \lrcorner & f\downarrow \\ G & \longrightarrow & Y & \xrightarrow{q} & B \end{array}$$

on the left side, where the maps to X and Y are right fibrations, and $F \to G$ is final. We have to show that $F \to G \times_Y X = G \times_B A$ is final, but this is true because $G \to B$ and $F \to A$ are also right fibrations, and $A \to B$ induces an equivalence $\mathrm{RFib}(A) \to \mathrm{RFib}(B)$ by Thm. 1.2.1. Thus we conclude by Remark 1.1.54 again.

Step 2: In general. Take a fibrant resolution via an inner anodyne extension $B \to \mathcal{D}$, and factorize $A \to \mathcal{D}$ as an inner anodyne extension $A \to \mathcal{C}$ followed by an inner fibration $\mathcal{C} \to \mathcal{D}$ (so $\mathcal{C}$ is an ∞-category). Form a commutative cube as follows:



The bottom commutative square is the one we just made. Since $Y \to B$ is a right fibration and $B \to \mathcal{D}$ is inner anodyne, we can apply Example 1.1.59 to get a cartesian square as on the right, with $\mathcal{E} \to \mathcal{D}$ a right fibration (so $\mathcal{E}$ an ∞-category). Now pullback $\mathcal{C} \to \mathcal{D} \leftarrow \mathcal{E}$ to obtain the right fibration $\mathcal{F} \to \mathcal{C}$ and hence $\mathcal{F}$ is an ∞-category. By the pasting lemma, there's a unique map $X \to \mathcal{F}$ making the left face cartesian.

To conclude, note that $A \to \mathcal{C}$ and $B \to \mathcal{D}$ are inner anodyne, and hence weak equivalences that are bijective on objects. Thus by Step 1, $X \to \mathcal{F}$ and $Y \to \mathcal{E}$ are weak categorical equivalences. The back face is a cartesian square in which all objects are fibrant and $\mathcal{E} \to \mathcal{D}$ is a right fibration, hence a Joyal fibration, so it is homotopy cartesian by [HCHA, Cor. 2.3.28]. The front face is isomorphic to the back one in $\mathrm{Ho}(\mathbf{sSet}^{\square}_{\mathrm{Joy}})$, so we conclude that it is homotopy cartesian by [HCHA, Rmk. 2.3.24]. □

We will now begin working towards proving the correspondence between right fibrations and presheaves of spaces. If A is a simplicial set, we denote by $\underline{\mathrm{Hom}}_{\mathbf{sS}/A}(-,-)$ the internal Hom of $\mathbf{sSet}/A$, so that maps $A' \to \underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X,Y)$ correspond to maps $A' \times_A X \to Y$ over A, or equivalently maps $A' \times_A X \to A' \times_A Y$ over A'. We usually view this with its canonical map $\underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X,Y) \to A$, which is equal to $p_{X,*}p_X^*Y$.

For a fixed presheaf X over A, we have the defining adjunction

$$(-) \times_A X : \mathbf{sSet}/A \rightleftarrows \mathbf{sSet}/A : \underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X,-)$$

where the left adjoint always preserves monomorphisms. If X is left or right fibrant over A, then it also preserves weak categorical equivalences by Prop. 1.2.6, and hence this is a Quillen adjunction for the slice Joyal model structure on both sides. Thus the right adjoint preserves fibrant objects, meaning that if $Y \to A$ is a Joyal fibration, then the structure map

$$\underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X,Y) \to A$$

is a Joyal fibration. This applies e.g. when Y is also left fibrant over A.

Remark 1.2.7 Let $X \to Y$ be a morphism in $\mathbf{sSet}/A$. Then the following conditions are equivalent:

(*i*) $X \times_A B \to Y \times_A B$ is a weak categorical equivalence for all $B \to A$.

(*ii*) $X \times_A \sigma \to Y \times_A \sigma$ is a weak categorical equivalence for all simplices $\sigma\colon \Delta^n \to A$.

Indeed, this is by Remark 1.1.6 applied to the slice $\mathbf{\Delta}/A$: we have the morphism of colimit-preserving functors $X \times_A (-) \to Y \times_A (-)$, which also preserve monomorphisms. If either of these conditions holds, we say $X \to Y$ is *locally a weak categorical equivalence over* A. So, we may define the unique subobject $\underline{\mathrm{Eq}}_A(X,Y) \subset \underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X,Y)$ by the rule that $\mathrm{Hom}_{\mathbf{sSet}/A}(\Delta^n, \underline{\mathrm{Eq}}_A(X,Y))$ corresponds to the subset of $\mathrm{Hom}_{\mathbf{sSet}/\Delta^n}(X \times_A \Delta^n, Y \times_A \Delta^n)$ consisting of local weak categorical equivalences, for all $\Delta^n \to A$. This is well-defined because the condition is preserved under pullback, and we deduce from the equivalence that maps $A' \to \underline{\mathrm{Eq}}_A(X,Y)$ correspond to those maps $A' \times_A X \to A' \times_A Y$ over A' which are locally weak categorical equivalences (over A').

Lemma 1.2.8 *Let X and Y be left fibrant over A.*

1. *For all $A' \to A$, a map $A \to \underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X,Y)$ over A factors through $\underline{\mathrm{Eq}}_A(X,Y)$ iff $A' \times_A X \to A' \times_A Y$ is a weak categorical equivalence, iff it is a fiberwise equivalence over A'.*
2. *The map $\underline{\mathrm{Eq}}_A(X,Y) \to A$ is a Joyal fibration.*

Proof (1) By construction, it factors through iff $A' \times_A X \to A' \times_A Y$ is locally a weak categorical equivalence over A'. This means that for all simplices $\sigma\colon \Delta^n \to A'$, $\Delta^n \times_A X \to \Delta^n \times_A Y$ is a weak categorical equivalence.

This is a morphism of left fibrant objects over Δ^n, so by Example 1.1.2 this is equivalent to being a fiberwise equivalence over Δ^n. But the corresponding fiberwise map is just $\sigma(i) \times_A X \to \sigma(i) \times_A Y$ for all i, so this is the same as the map $A' \times_A X \to A' \times_A Y$ being a fiberwise equivalence over A'.

For the remaining assertion we prove that a weak categorical equivalence $A' \times_A X \to A' \times_A Y$ is already locally one over A'. Indeed, for all $A'' \to A'$ we have cartesian squares:

$$\begin{array}{ccccc} X \times_A A'' & \longrightarrow & Y \times_A A'' & \longrightarrow & A'' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X \times_A A' & \longrightarrow & Y \times_A A' & \longrightarrow & A' \end{array}$$

Since $Y \times_A A' \to A'$ and $X \times_A A' \to A'$ are left fibrations, the right-hand and outer square are homotopy cartesian by Prop. 1.2.6, so the left-hand square is as well by the pasting lemma. Therefore $X \times_A A'' \to Y \times_A A''$ is a weak equivalence.

(2) We already know that $\underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X, Y) \to A$ is a Joyal fibration, so it has right lifting against trivial cofibrations $K \to L$. The lifting problems and solutions are in 1-1 correspondence with fillers v in a diagram

$$\begin{array}{ccc} K \times_A X & \xrightarrow{u} & K \times_A Y \\ \downarrow & & \downarrow \\ L \times_A X & \overset{v}{\dashrightarrow} & L \times_A Y \end{array}$$

by adjunction. Now $K \to \underline{\mathrm{Hom}}_{\mathbf{sS}/A}(X, Y)$ factoring through $\underline{\mathrm{Eq}}_A(X, Y)$ means that u is a weak equivalence, by (1), and the existence of a solution in $\underline{\mathrm{Eq}}_A(X, Y)$ means that v is a weak equivalence. This is immediate because both vertical maps are weak categorical equivalences by Prop. 1.2.6, as $K \to L$ is a weak equivalence and X and Y are left fibrant over A. □

Remark 1.2.9 Let us briefly summarize all of the criterion we have discussed for equivalences of left fibrant objects. More precisely, the following are equivalent for a map $f\colon X \to Y$ of left fibrant objects over A:

1. f is a weak equivalence in $(\mathbf{sSet}/A)_{\mathrm{cov}}$.
2. f is locally a weak categorical equivalence.
3. f is a weak categorical equivalence.
4. f is fiberwise an equivalence of ∞-groupoids.
5. f is cofinal.

We will say that f is an equivalence of left fibrant objects over A if any of these equivalent conditions hold.

Now we will use $\mathcal{S}$ to classify other things about left fibrations, in particular morphisms between two left fibrations, and when they are isomorphisms. First,

the product $\mathcal{S} \times \mathcal{S}$ classifies pairs of left fibrations. Form the pullbacks

$$\begin{array}{ccc} \mathcal{S}_\bullet^{(i)} & \longrightarrow & \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{S} \times \mathcal{S} & \xrightarrow{\mathrm{pr}_i} & \mathcal{S} \end{array}$$

for $i = 0, 1$, giving us two left fibrant objects over $\mathcal{S} \times \mathcal{S}$, and hence Joyal fibrations:

$$\underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}\left(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}\right) \to \mathcal{S} \times \mathcal{S} \qquad \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}\left(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}\right) \to \mathcal{S} \times \mathcal{S}$$

The first map is the universal morphism of **U**-small left fibrations: Indeed, suppose we are given a pair of maps $(F, G) \colon A \to \mathcal{S} \times \mathcal{S}$ classifying two left fibrations $X \to A$ and $Y \to A$. Then a map to $\underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}\left(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}\right)$ over $\mathcal{S} \times \mathcal{S}$ is a morphism

$$A \underset{\mathcal{S}\times\mathcal{S}}{\times} \mathcal{S}_\bullet^{(0)} = X \to A \underset{\mathcal{S}\times\mathcal{S}}{\times} \mathcal{S}_\bullet^{(1)} = Y$$

over A, i.e. it is a morphism between the given left fibrations. Since $\mathcal{S}_\bullet^{(i)}$ are left fibrant over $\mathcal{S} \times \mathcal{S}$, such a map over the product factors through $\underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}\left(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}\right)$ iff $X \to Y$ is a fiberwise equivalence/weak categorical equivalence.

The universal left fibration is classified by the identity $\mathrm{Id}_\mathcal{S}$, and the identity morphism of the universal left fibrant object $\mathrm{id}_{\mathcal{S}_\bullet}$, which is a fiberwise equivalence, is classified by a map

$$\mathcal{S} \xrightarrow{\mathrm{id}_{\mathcal{S}_\bullet}} \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}) \xrightarrow{(s,t)} \mathcal{S} \times \mathcal{S} \tag{$*$}$$

on the left above. This gives a factorization of the diagonal, where we know that the second map is a Joyal fibration. We want to show that the first map is a weak equivalence, so that we can use this as a sort of 'universal' path object.

Lemma 1.2.10 *The map* $\mathrm{id}_{\mathcal{S}_\bullet} \colon \mathcal{S} \to \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})$ *is a Joyal trivial cofibration; thus* $(*)$ *is a path object.*

Proof It is a section of $t \colon \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}) \to \mathcal{S}$, so it suffices to prove that this map is a trivial fibration. Suppose we are given a lifting problem

$$\begin{array}{ccc} K & \longrightarrow & \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}) \\ \downarrow & \nearrow & \downarrow \\ L & \longrightarrow & \mathcal{S} \end{array}$$

against a monomorphism $K \hookrightarrow L$. This consists of the following data: $L \to \mathcal{S}$ classifies a left fibration $Y \to L$, and $F \colon K \to \mathcal{S}$ classifies one $X \to K$ which is its pullback. On the other hand, the upper map corresponds to another map

$F_0 \colon K \to \mathcal{S}$ classifying another left fibration $X' \to K$, and a weak categorical equivalence $X' \to X$ of left fibrant objects over K.

$$\begin{array}{ccccc} X' & \xrightarrow{\simeq} & X & \hookrightarrow & Y \\ & \searrow & \downarrow{\scriptstyle p} & \lrcorner & \downarrow{\scriptstyle q} \\ & & K & \hookrightarrow & L \end{array}$$

We are in exactly then position to apply [HCHA, Prop. 5.2.21] to the class of left fibrations and $\mathbf{sSet}_{\mathrm{Joy}}$. Indeed, every left fibration is a Joyal fibration, a trivial fibration is clearly a left fibration, and this class is closed under retracts and composition. Therefore, there exists a left fibration $X'' \to L$ and a weak equivalence $X'' \to Y$ producing a commutative diagram:

$$\begin{array}{ccccc} & & X & \longrightarrow & Y \\ & \overset{\simeq}{\nearrow} & & \overset{\simeq}{\nearrow} & \\ X' & \hookrightarrow & X'' & & \\ \downarrow & \lrcorner & \downarrow & \searrow^{q} & \\ K & \hookrightarrow & L & & \end{array}$$

F_0 is the classifying map for $X' \to K$, so by Example 1.1.51 we may choose $F_1 \colon L \to \mathcal{S}$ a classifying map for $X'' \to L$ such that $K \to L \to \mathcal{S}$ equals F_0. Then the weak equivalence $X'' \to Y$ corresponds to the requisite map $L \to \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})$ making both triangles commute. □

Remark 1.2.11 Technically, one should insert '**U**-small' in various parts of the proof above, but the only place this is possibly suspect is in the construction of the left fibration $X'' \to L$ from [HCHA, Prop. 5.1.21]. But this is not an issue, because this map is constructed by [HCHA, Lem. 5.1.20], and we already saw that we can choose it to have **U**-small fibers as long as $X' \to K$ has **U**-small fibers. TODO: Previously, I only showed a special case of this, so we should prove the whole thing.

Example 1.2.12 Let X and Y be left fibrant presheaves over A with **U**-small fibers, with chosen classifying maps $F, G \colon A \to \mathcal{S}$. There is a canonical cartesian square:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_A(X,Y) & \dashrightarrow & \underline{\mathrm{Hom}}\left(A, \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})\right) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s,t)} \\ \Delta^0 & \xrightarrow{(F,G)} & \underline{\mathrm{Hom}}(A,\mathcal{S}) \times \underline{\mathrm{Hom}}(A,\mathcal{S}) \end{array}$$

Indeed, a map from A' to the pullback of the given cospan corresponds to a map $A' \times A \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})$ over $\mathcal{S} \times \mathcal{S}$, where the structure map classifies the left fibrations $A' \times X$ and $A' \times Y$ over $A' \times A$. Therefore this is the same data as a morphism $A' \times X \to A' \times Y$ over $A' \times A$, which is also the same data as a map $A' \to \underline{\mathrm{Hom}}_A(X,Y)$ by definition. We also pullback along

the upper map here to obtain:

$$\begin{array}{ccc} \mathrm{Equiv}_A(X,Y) & \dashrightarrow & \underline{\mathrm{Hom}}\left(A, \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})\right) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}_A(X,Y) & \dashrightarrow & \underline{\mathrm{Hom}}\left(A, \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})\right) \end{array}$$

This functorially defines $\mathrm{Equiv}_A(-,-)$ as the subobject of $\underline{\mathrm{Hom}}_A(-,-)$ for which maps from $A' \to \mathrm{Equiv}_A(X,Y)$ correspond to those maps $A' \times X \to A' \times Y$ over $A' \times A$ which are equivalences of left fibrations, so this also shows that it is independent of $\mathbf{U}$ or any choice of classifying maps. Note that the following are equivalent for an n-simplex $\sigma\colon \Delta^n \times X \to \Delta^n \times Y$ over $\Delta^n \times A$ of $\underline{\mathrm{Hom}}_A(X,Y)$:

(i) σ is an equivalence of left fibrations over $\Delta^n \times A$.

(ii) $\sigma_{(i,a)}\colon X_a \to Y_a$ is an equivalence of ∞-groupoids for all (i,a).

(iii) $\sigma_i\colon X \to Y$ is an equivalence of left fibrations over A for all i.

This is simply because equivalences of left fibrations are checked fiberwise; it follows that $\mathrm{Equiv}_A(X,Y) \subset \underline{\mathrm{Hom}}_A(X,Y)$ forms a full subcategory, and is therefore an ∞-groupoid. In fact, the inclusion is actually a Kan fibration; indeed, given a lifting problem against $\Lambda^n_k \hookrightarrow \Delta^n$, there is already a solution in Equiv_A if $n \geq 2$ because then the vertices of the simplex lie in Equiv_A. If $n = 1$, this is implied by the fact that if we're given a map $h\colon \Delta^1 \times X \to \Delta^1 \times Y$ of left fibrations over $\Delta^1 \times A$ and one of the fibers $h_0\colon X \to Y$ (or h_1) is an equivalence, then the other is too. That's because we can check this on the fibers over all objects a, in which case it corresponds to a homotopy $\Delta^1 \times X_a \to Y_a$ from a map to an equivalence in $\mathbf{sSet}_{\mathrm{KQ}}$.

In our goal to view $\underline{\mathrm{Hom}}(A, \mathcal{S})$ as the space of left fibrations with $\mathbf{U}$-small fibers over A, the next step is to show that a morphism of classifying maps in the ∞-category $\underline{\mathrm{Hom}}(A, \mathcal{S})$ induces a morphism between the corresponding left fibrations, and that being invertible is equivalent to being an equivalence of left fibrations.

Construction 1.2.13 Let $p_X\colon X \to A$ and $p_Y\colon Y \to A$ be left fibrant objects (with $\mathbf{U}$-small fibers) over A, and suppose we are given classifying maps $F, G\colon A \to \mathcal{S}$ for them respectively.

1. A morphism f from F to G in the ∞-category $\underline{\mathrm{Hom}}(A, \mathcal{S})$ is a map $\Delta^1 \times A \to \mathcal{S}$ starting at F and ending at G; up to choice of pullback, this classifies a left fibration $\pi\colon W \to \Delta^1 \times A$ with $\mathbf{U}$-small fibers, equipped with isomorphisms $W_0 \cong X$ and $W_1 \cong Y$ over A, where $W_\epsilon = W \times_{\Delta^1 \times A} \{\epsilon\} \times A$ for $\epsilon \in \{0,1\}$.

2. Since $\{0\} \times X \to \Delta^1 \times X$ is left anodyne (by [HCHA, Prop. 3.4.3]) and π is a left fibration, the commutative diagram

$$\begin{array}{ccc} \{0\} \times X & \longrightarrow & W \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \Delta^1 \times X & \longrightarrow & \Delta^1 \times A \end{array}$$

admits a lift φ.

3. Since it is a map over $\Delta^1 \times A$, the induced map on fibers over $\{0\} \times A$ and $\{1\} \times A$ is $\varphi_0 \colon X \xrightarrow{\sim} W_0$, and $\varphi_1 \colon X \to W_1 \cong Y$. φ_1 is a morphism of left fibrations classified by F and G, hence induces a map

$$\widetilde{\varphi}_1 \colon A \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet)$$

over $\mathcal{S} \times \mathcal{S}$ classifying φ_1.

Lemma 1.2.14 *Assume we are in the situation of Constr. 1.2.13. The morphisms φ_1 and $\widetilde{\varphi}_1$ are independent of the choice of lift φ up to J-homotopy over A and $\mathcal{S} \times \mathcal{S}$ respectively. In particular, they define unique elements of the sets*

$$\mathrm{Hom}_{\mathrm{Ho}((\mathbf{sSet}/A)_{\mathrm{cov}})}(X, Y),\ \mathrm{Hom}_{\mathrm{Ho}(\mathbf{sSet}_{\mathrm{Joy}}/\mathcal{S}\times\mathcal{S})}\big(A, \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet)\big)$$

respectively.

Proof Consider the model category $(\mathbf{sSet}/\Delta^1 \times A)_{\mathrm{cov}}$; note that $J \times (-)$ gives model cylinders in this category too, because a weak categorical equivalence is cofinal by Cor. 1.2.4. A lift of the commutative square in Constr. 1.2.13(2) is necessarily a map over $\Delta^1 \times A$, but in $\mathrm{Ho}((\mathbf{sSet}/\Delta^1 \times A)_{\mathrm{cov}})$ such a map is unique, because $0 \times X \to \Delta^1 \times X$ is left anodyne, hence an isomorphism in this category. Since $\Delta^1 \times X$ is cofibrant and W is left fibrant over $\Delta^1 \times A$, any two lifts are J-homotopic rel $\Delta^1 \times A$.

Taking a J-homotopy $J \times \Delta^1 \times X \to W$ over $\Delta^1 \times A$, passing to $\{1\} \times A$ fibers gives a homotopy $h \colon J \times X \to W_1 \cong Y$ over A, between the two induced maps $X \to Y$. This proves the first claim. For the second, consider the induced map $(\mathrm{pr}_J, h) \colon J \times X \to J \times Y$ over $J \times A$, which is a morphism of left fibrations with $\mathbf{U}$-small fibers over $J \times A$, hence induces a map

$$J \times A \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet).$$

over $\mathcal{S} \times \mathcal{S}$. Restricting along $\{\epsilon\} \times A \to J \times A$ produces the two possible induced maps $\widetilde{\varphi}_1, \widetilde{\varphi}'_1$ induced by two choices $X \to Y$, so this provides J-homotopy invariance.

The last uniqueness statement follows because, as explained, $J \times (-)$ gives model cylinders in the covariant model structure over a presheaf, and Y is left fibrant over A. In the second case we can use the Joyal model structure because

$$\underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet) \to \mathcal{S} \times \mathcal{S}$$

is a Joyal fibration, hence the source is a fibrant object of the slice. □

Lemma 1.2.15 *Assume we are in the situation of Constr. 1.2.13. Then the following conditions are equivalent:*

(*i*) $\widetilde{\varphi}_1$ *factors through* $\underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet)$.

(*ii*) $f\colon F \to G$ *is invertible in* $\underline{\mathrm{Hom}}(A, \mathcal{S})$.

(*iii*) φ_1 *is a fiberwise equivalence over* A.

(*iv*) *For each object* a *of* A, *the induced map* $\pi_a\colon W_a \to \Delta^1$ *is a Kan fibration.*

Proof We already know that (*i*) = (*iii*). We know from [HCHA, Cor. 3.5.12] that (*ii*) is equivalent to $f_a\colon F_a \to G_a$ being invertible in $\mathcal{S}$ for each object a of A. This is equivalent to, for each a, $\Delta^1 \times \{a\} \to \mathcal{S}$ factoring through $\mathcal{S}^{\simeq}$, which itself is equivalent to maps that these classify, namely $\pi_a\colon W_a \to \Delta^1$, being Kan fibrations, by Thm. 1.1.62.

(*iv*) ⇒ (*iii*) Fix an object a of A. If $W_a \to \Delta^1$ is a Kan fibration, then consider the cartesian squares:

$$\begin{array}{ccc} W_{(\epsilon,a)} & \longrightarrow & \{\epsilon\} \\ \downarrow & \lrcorner & \downarrow \\ W_a & \longrightarrow & \Delta^1 \end{array}$$

for $\epsilon \in \{0,1\}$. By [HCHA, Cor. 3.1.28], pulling back along the anodyne extension $\{\epsilon\} \to \Delta^1$ produces weak homotopy equivalences $W_{(\epsilon,a)} \to W_a$. Taking the fiber of the commutative square in Constr. 1.2.13 at a gives the diagram:

$$\begin{array}{ccccc} W_{(0,a)} & = & \{0\} \times X_a & \longrightarrow & W_a \\ & & \downarrow & \nearrow & \downarrow \\ & & \Delta^1 \times X_a & \longrightarrow & \Delta^1 \end{array}$$

In partcular, the upper map is a weak homotopy equivalence and the left-hand map is an anodyne extension, so $\varphi_a\colon \Delta^1 \times X_a \to W_a$ is a weak homotopy equivalence. Now X_a is a Kan complex so the bottom map is a Kan fibration like the right-hand map; thus, a second application of right-properness of $\mathbf{sSet}_{\mathrm{KQ}}$ ([HCHA, Cor. 3.5.12]) implies that the induced map on pullbacks

$$\varphi_{(1,a)}\colon X_a = \{1\} \times X_a \to W_{(a,1)} = Y_a,$$

is a weak homotopy equivalence, hence an equivalence of ∞-groupoids.

(*iii*) ⇒ (*iv*) Conversely, suppose $\varphi_{(1,a)}\colon X_a \to W_{(1,a)} \cong Y_a$ is an equivalence of ∞-groupoids. The map induced by $\varphi_a\colon \Delta^1 \times X_a \to W_a$ on the fibers over 0 is already an isomorphism, so φ_a is a fiberwise equivalence of left fibrations over Δ^1. Now, in $(\mathbf{sSet}/\Delta^1)_{\mathrm{cov}}$, the class of presheaves which are Kan fibrant over Δ^1 is closed under retracts and precomposition by trivial fibrations, and consists of left fibrant objects. Thus by [HCHA, Prop. 5.1.13], the fact that

$\Delta^1 \times X_a \to W_a$ is a weak equivalence in the covariant model structure over Δ^1 implies that $W_a \to \Delta^1$ is a Kan fibration, because X_a is an ∞-groupoid. □

1.2.2 Right fibrations and presheaves of spaces

Let us combine all of these results together. Recall the object $h(-,-)$ defined in [HCHA, Sec. 3.5]. Using the evaluation map, form cartesian squares:

$$\begin{array}{ccccc} W^h & \hookrightarrow & W & \longrightarrow & \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ \Delta^1 \times h(\Delta^1, \mathcal{S}) & \hookrightarrow & \Delta^1 \times \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) & \xrightarrow{\mathrm{ev}} & \mathcal{S} \end{array}$$

The evluation map provides a morphism in the category $\underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}), \mathcal{S})$, classifying the left fibration $W \to \Delta^1 \times \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})$, which allows us to apply Constr. 1.2.13. In particular, we obtain a canonical map

$$\varphi_1 \colon \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet)$$

which is well-defined up to J-homotopy over $\mathcal{S} \times \mathcal{S}$. Precompose by the inclusion $h(\Delta^1, \mathcal{S})$, which now corresponds to applying the same construction to $W^h \to \Delta^1 \times h(\Delta^1, \mathcal{S})$. Note that this time, the morphism we started with corresponds to

$$\Delta^1 \to \underline{\mathrm{Hom}}(h(\Delta^1, \mathcal{S}), \mathcal{S}),$$

which is invertible, because that condition is equivalent to the adjunct map $h(\Delta^1, \mathcal{S}) \to \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})$ factoring through $h(\Delta^1, \mathcal{S})$, which of course it does. Therefore by Lemma 1.2.15, we get a commutative diagram:

$$\begin{array}{ccc} h(\Delta^1, \mathcal{S}) & \longrightarrow & \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet) \end{array} \qquad (**)$$

This is in fact cartesian. Indeed, note that the whole square is over $\mathcal{S} \times \mathcal{S}$; a map from A to the pullback of the cospan corresponds to a map $\Delta^1 \to \underline{\mathrm{Hom}}(A, \mathcal{S})$ for which the corresponding map between left fibrations is an equivalence, but this is equivalent to $\Delta^1 \to \underline{\mathrm{Hom}}(A, \mathcal{S})$ being an invertible morphism, which means precisely that it factors through $h(\Delta^1, \mathcal{S})$.

Finally, we also have a cartesian square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A, \mathcal{S})^{\simeq}(F, G) & \longrightarrow & \underline{\mathrm{Hom}}(A, h(\Delta^1, \mathcal{S})) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}(A, \mathcal{S})(F, G) & \longrightarrow & \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})) \end{array}$$

which is by definition of h: The bottom map is the restriction of the isomorphism $\underline{\mathrm{Hom}}(\Delta^1, \underline{\mathrm{Hom}}(A, \mathcal{S})) = \underline{\mathrm{Hom}}(A, \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}))$ to the mapping space. Since the right-hand map is a full subcategory inclusion, so too is the pullback, so we can check this on objects. Indeed, a morphism from $F \to G$, i.e. a

map $\Delta^1 \to \underline{\mathrm{Hom}}(A, \mathcal{S})$, factors through the mapping space of the core iff it is invertible, which is the case iff the transposed map $A \to \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})$ factors through $h(\Delta^1, \mathcal{S})$.

Apply $\mathrm{Hom}(A, -)$ to $(**)$, then by pasting this gives a cartesian square fitting in the left-hand map above and the right hand map of the second square in Example 1.2.12. It follows by a second application of pasting that we have another cartesian square

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A, \mathcal{S})^{\simeq}(F, G) & \longrightarrow & \mathrm{Equiv}_A(X, Y) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}(A, \mathcal{S})(F, G) & \longrightarrow & \underline{\mathrm{Hom}}_A(X, Y) \end{array}$$

where by Example 1.2.12 the right-hand map is a Kan fibration.

Remark 1.2.16 This is a very important cartesian square, and also encodes the lower map that we will eventually show is an equivalence of ∞-groupoids. Here are some immediate observations:

1. Note that the only choice made here took place before the introduction of A, X, Y, F, G, as we made a single universal choice when constructing the map $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet)$ and its pullback. Thus the square is functorial in all of these new objects.
2. The lower map in the square is well-defined up to J-homotopy, because a J-homotopy connecting two choices for the bottom map of $(**)$ (unique by Lemma 1.2.14) induces a J-homotopy connecting the two corresponding choices of the lower map. This is in fact relative to $\underline{\mathrm{Hom}}(A, \mathcal{S} \times \mathcal{S})$. Since these are ∞-groupoids, this is determined in both the Joyal and Kan-Quillen model structures.
3. Since the right-hand map is a Kan fibration, this square is homotopy cartesian in $\mathbf{sSet}_{\mathrm{KQ}}$ (and $\mathbf{sSet}_{\mathrm{Joy}}$, since it is in particular a left fibration). Thus the upper map is also uniquely determined up to homotopy in both homotopy categories.

Lemma 1.2.17 *Let X and Y be left fibrant objects over A with $\mathbf{U}$-small fibers, classified by maps $F, G \colon A \to \mathcal{S}$. The morphism*

$$\underline{\mathrm{Hom}}(A, \mathcal{S})^{\simeq}(F, G) \to \mathrm{Equiv}_A(X, Y),$$

canonical up to homotopy, is an equivalence of ∞-groupoids. In other words, the unique corresponding morphism in $\mathcal{H}$ is an isomorphism.

Proof By construction, once we have chosen the map φ_1 as above, the relevant morphism is the fiber of the morphism

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A, h(\Delta^1, \mathcal{S})) & \xrightarrow{\varphi_1} & \underline{\mathrm{Hom}}(A, \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})) \\ & \searrow & \downarrow \\ & & \underline{\mathrm{Hom}}(A, \mathcal{S}) \times \underline{\mathrm{Hom}}(A, \mathcal{S}) \end{array}$$

over $\underline{\mathrm{Hom}}(A, \mathcal{S}) \times \underline{\mathrm{Hom}}(A, \mathcal{S})$ at the object (F, G). The right hand map is a Joyal fibration by [HCHA, Cor. 3.6.4], and the left-hand map is one by [HCHA, Cor. 3.6.7]. We claim the upper map is a weak equivalence. Indeed, project down to the second factor $\mathrm{Hom}(A, \mathcal{S})$, then we claim the two maps are trivial fibrations. It suffices to show that the Joyal fibrations

$$h(\Delta^1, \mathcal{S}) \xrightarrow{\mathrm{ev}_1} \mathcal{S} \qquad \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}) \xrightarrow{t} \mathcal{S}$$

are weak categorical equivalences. The right-hand one is clear because it has a section which is a trivial cofibration, by Lemma 1.2.10. The left-hand one is known to us from [HCHA, Cor. 3.5.10], since $\mathcal{S}$ is an ∞-category. This proves that φ_1 is a weak equivalence. To conclude, apply [HCHA, Cor. 2.3.29] in $\mathbf{sSet}_{\mathrm{Joy}}$ and we deduce that the induced map on the fiber over (F, G) is a weak equivalence. □

We now begin to outline the main result of this section. For each A, we define an assignment

$$p^*_{\mathrm{univ}} \colon \mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S}) \to \mathrm{LFib}(A)$$

on objects and morphisms as follows:

1. For each object $F \colon A \to \mathcal{S}$ of $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S})$, we define $p^*_{\mathrm{univ}}(F)$ to be the left fibration $A \times_{\mathcal{S}} \mathcal{S}_\bullet \to A$ classified by F.
2. If F and G are objects $A \to \mathcal{S}$, then applying π_0 to the map $\underline{\mathrm{Hom}}(A, \mathcal{S})(F, G) \to \underline{\mathrm{Hom}}_A(X, Y)$ induces a map

   $$p^*_{\mathrm{univ}} \colon \mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(A,\mathcal{S})}(F, G) \to \mathrm{Hom}_{\mathrm{LFib}(A)}(p^*_{\mathrm{univ}} F, p^*_{\mathrm{univ}} G)$$

   providing the assignment on morhpisms. Since $\underline{\mathrm{Hom}}(A, \mathcal{S})(F, G) \to \underline{\mathrm{Hom}}_A(X, Y)$ is determined up to Δ^1- (and J-)homotopy, the induced map is independent of all choices.

Remark 1.2.18 Let us recall how the induced map on morphisms is computed concretely. We took a representative in a J-homotopy class

$$\varphi_1 \colon \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})$$

defined by Constr. 1.2.13, then applying $\underline{\mathrm{Hom}}(A, -)$ we view this as a map out of $\underline{\mathrm{Hom}}(\Delta^1, \underline{\mathrm{Hom}}(A, \mathcal{S}))$, so a morphism of Joyal fibrations over $\underline{\mathrm{Hom}}(A, \mathcal{S}) \times \underline{\mathrm{Hom}}(A, \mathcal{S})$. Then we take the induced map on fibers over (F, G). Writing $W \to \Delta^1 \times \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})$ for the left fibration classified by evaluation, we know that φ_1 above is induced by a morphism $W_0 \to W_1$ of left fibrant objects over

$\underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})$. Then, the map of (2) above works as follows: given a morphism $f\colon F \to G$ in $\underline{\mathrm{Hom}}(A, \mathcal{S})$, viewed as a map $f\colon A \to \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S})$, the induced map on fibers is

$$F \times_{\mathcal{S}} \mathcal{S}_\bullet = W_0 \underset{\underline{\mathrm{Hom}}(\Delta^1,\mathcal{S})}{\times} A \to W_1 \underset{\underline{\mathrm{Hom}}(\Delta^1,\mathcal{S})}{\times} A = G \times_{\mathcal{S}} \mathcal{S}_\bullet$$

over A, producing the corresponding object of $\underline{\mathrm{Hom}}_A(p^*_{\mathrm{univ}}F, p^*_{\mathrm{univ}}G)$. In light of this, consider the universal lift:

$$\begin{array}{ccc} 0 \times W_0 & \hookrightarrow & W \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \Delta^1 \times W_0 & \longrightarrow & \Delta^1 \times \underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) \end{array}$$

If we apply $(-) \times_{\underline{\mathrm{Hom}}(\Delta^1,\mathcal{S})} A$ to this, associated to a morphism $f\colon F \to G$ classifying X and Y over A, we produce a diagram:

$$\begin{array}{ccc} 0 \times X & \hookrightarrow & W_f \\ \downarrow & \overset{\varphi_f}{\nearrow} & \downarrow \\ \Delta^1 \times X & \longrightarrow & \Delta^1 \times A \end{array}$$

By Lemma 1.2.14, this shows that the assignment p^*_{univ} on morphisms can be computed either by using the universal lift φ above, or by choosing a lift as in Constr. 1.2.13 for a given pair of left fibrations.

Example 1.2.19 Here is an example of the above phenomenon. We'll check that $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S}) \to \mathrm{LFib}(A)$ preserves identity morphisms. Let $F\colon A \to \mathcal{S}$ classify $p\colon p^*_{\mathrm{univ}}(F) = X \to A$; the morphism $1_F\colon F \to F$ corresponds to $\Delta^1 \times A \to A \to \mathcal{S}$, which classifies $\Delta^1 \times X \to \Delta^1 \times A$. In particular, we have to choose a lift in:

$$\begin{array}{ccc} 0 \times X & \hookrightarrow & \Delta^1 \times X \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \Delta^1 \times X & \longrightarrow & \Delta^1 \times A \end{array}$$

Of course we can just take $\varphi = \mathrm{Id}$, and the induced map φ_1 is $1_X\colon X \to X$ over A, proving that $p^*_{\mathrm{univ}}(1_F) = 1_X$.

To demonstrate that the assignment is functorial, we need a lemma extending Constr. 1.2.13 to higher order simplices.

Lemma 1.2.20 *Let $p\colon W \to \Delta^n \times A$ be left fibration.*

1. *The fiber inclusion $W_n \hookrightarrow W$ is a weak equivalence in $(\mathbf{sSet}/A)_{\mathrm{cov}}$.*
2. *A lift φ in the left-hand diagram below*

$$\begin{array}{ccc} 0 \times W_0 & \hookrightarrow & W \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \Delta^n \times W_0 & \longrightarrow & \Delta^n \times A \end{array} \qquad \begin{array}{ccc} i \times W_0 & \xrightarrow{\varphi_n} & W_n \\ \downarrow & & \downarrow \\ \Delta^n \times W_0 & \xrightarrow{\varphi} & W \end{array}$$

always exists, and for any such lift the right-hand diagram commutes in $\mathrm{LFib}(A)$. *In particular, the triangle*

$$\begin{array}{ccc} W_0 & \xrightarrow{\varphi_n} & W_n \\ & \searrow & \downarrow \\ & & W \end{array}$$

commutes in $\mathrm{LFib}(A)$.

Proof (1) $W_n \to A$ is a left fibration, and $W \to \Delta^n \to A \to A$ is a left fibration composed with a projection, so $W_n \to W$ is a morphism of proper maps over A. By [HCHA, Cor. 4.4.28], it suffices to show that $W_n \to W$ is a fiberwise weak homotopy equivalence over A. But $\{n\} \hookrightarrow \Delta^n$ is a right anodyne extension, and for each a the maps $W_a \to \Delta^n$ are left fibrations, hence proper. Thus the pullback $W_{(n,a)} \hookrightarrow W_a$ is right anodyne, in particular a weak homotopy equivalence.

(2) Since 0 is an initial object of Δ^n, it follows that $0 \times W_0 \to \Delta^n \times W_0$ is left anodyne, thus a lift φ always exists. Moreover, this means that the projection $\Delta^n \times W_0 \to W_0$ is a covariant weak equivalence, and thus all sections of it (i.e. $i \times W_0 \hookrightarrow \Delta^n \times W_0$) are equal in $\mathrm{LFib}(A)$. We conclude because the right-hand diagram commutes by construction when $i = n$. The final commutative triangle is the case that $i = 0$. □

Corollary 1.2.21 *The assignment* $p^*_{\mathrm{univ}} \colon \mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S}) \to \mathrm{LFib}(A)$ *is a well-defined functor.*

Proof We have checked that it preserves identities, so it remains to check that it preserves commutative triangles in $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S})$, i.e. the boundary of 2-simplices. Consider a map $\Delta^2 \times A \to \mathcal{S}$ classifying a left fibration $W \to \Delta^2 \times A$, then pulling back along $\Delta^{\{i,j\}} \times A \to \Delta^2 \times A$ provides the restriction to a face, $W_{ij} \to \Delta^1 \times A$. This provides a homotopy class $f_{ij} \colon W_i \to W_j$ in $\mathrm{LFib}(A)$.

Applications of Lemma 1.2.20 to $W_{01} \to \Delta^{\{0,1\}} \times A$ and $W_{12} \to \Delta^{\{1,2\}} \times A$ produce, by construction, a commutative diagram

$$\begin{array}{ccccc} W_0 & \xrightarrow{f_{01}} & W_1 & \xrightarrow{f_{12}} & W_2 \\ & \searrow & \downarrow & \searrow & \downarrow \\ & & W_{01} & & W_{12} \\ & & & \searrow & \downarrow \\ & & & & W \end{array}$$

in $\mathrm{LFib}(A)$. Now apply Lemma 1.2.20 to $W \to \Delta^2 \times A$ and choose a lift φ as in (2). Applying $\Delta^{\{0,2\}} \times_{\Delta^2} (-)$ to the whole square shows that the image of φ provides a lift $\Delta^{\{0,2\}} \times W_0 \to W_{0,2}$, which implies that $\varphi_2 \colon W_0 \to W_2$ coincides with f_{02}. Thus the lemma and commutative diagram above show that f_{02} and $f_{12} f_{01}$ are equal in $\mathrm{LFib}(A)$ after composing with $W_2 \to W$. But this map is an isomorphism, so $f_{02} = f_{12} f_{01}$. □

The main work will be full faithfulness. We need some technical results for this, which will concern the following construction. First, let $p\colon \Delta^1 \times A \to A$ be the projection, which gives the adjunction $p^* \dashv p_*$ on the slice categories over A and $\Delta^1 \times A$. If X and Y are in $\mathbf{sSet}/\Delta^1 \times A$, we denote the corresponding fibers over $\{\epsilon\} \times A$ as X_ϵ, Y_ϵ. We know that there is a lifting correspondence of the following form:

$$\begin{array}{ccc} \Delta^1 \times K \underset{\{\epsilon\}\times K}{\amalg} \{\epsilon\} \times L & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 \times L & \longrightarrow & Y \end{array} \qquad \begin{array}{ccc} K & \longrightarrow & p_*X \\ \downarrow & \nearrow & \downarrow \\ L & \longrightarrow & X_\epsilon \times_{Y_\epsilon} p_*Y \end{array}$$

There are a number of consequences we can easily deduce from this:

1. Take Y to be the terminal object $\Delta^1 \times A$ and ignore the lifting map. Then we get a natural bijection between morphisms of arrows from $K \to L$ and $p_*X \to X_\epsilon$ and morphisms of arrows from

$$\Delta^1 \times K \underset{\{\epsilon\}\times K}{\amalg} \{\epsilon\} \times L \to \Delta^1 \times L$$

and $X \to \Delta^1 \times A$. If we assume that K and L start as objects in $\mathbf{sSet}/A$, then the lower map $\Delta^1 \times L \to \Delta^1 \times \Delta^1 \times A$ is the structure map, so this becomes two adjunctions

$$s_!, t_!\colon \operatorname{Arr}(\mathbf{sSet}/A) \rightleftarrows \mathbf{sSet}/(\Delta^1 \times A) : t^*, s^*,$$

where $s_! \dashv s^*$ is the case that $\epsilon = 0$ and $t_! \dashv t^*$ is $\epsilon = 1$.

2. If X is left fibrant over $\Delta^1 \times A$, then $s^*(X)\colon p_*(X) \to X_0$ is a trivial fibration: by the lifting correspondence, this corresponds to the fact that $\Delta^1 \times K \cup 0 \times L \hookrightarrow \Delta^1 \times L$ is left anodyne for any monomorphism $K \to L$ ([HCHA, Prop. 3.4.3]).

By the identification $\operatorname{Arr}(\mathbf{sSet}/A)_{\text{cov}} \equiv (\mathbf{sSet}/A)^{\bullet \to \bullet}_{\text{cov}}$, we can view the arrow category as equipped with the injective model structure on top of the covariant model structure as given by [HCHA, Prop. 2.3.11]. From that result, the cofibrations and weak equivalences are defined pointwise; a (trivial) fibration is a commutative square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

in which $X \to Y$ is a (trivial) left fibration and the natural map $X' \to X \times_Y Y'$ is a (trivial) left fibration.

3. Under this definition, we claim that $t_! \dashv t^*$ is a Quillen adjunction. Indeed, suppose that $X \to Y$ is a (trivial) left fibration over $\Delta^1 \times A$. To see that $t^*(X) \to t^*(Y)$ is a (trivial) fibration, we have to show that $X_1 \to Y_1$ and $p_*X \to X_1 \times_{Y_1} p_*Y$ are (trivial) fibrations. The first fact is because pulling back along the inclusion $1 \times A \hookrightarrow \Delta^1 \times A$ is right Quillen. The second is

because if, in the lifting correspondence above, $K \to L$ is left anodyne (a monomorphism), then the induced map $\Delta^1 \times K \cup 1 \times L \to \Delta^1 \times L$ is left anodyne (a monomorphism) [HCHA, Prop. 3.4.3].

4. The right derived adjoint $\mathbf{R}t^*$ is conservative. Indeed, let $X \to Y$ be a morphism of left fibrant presheaves over $\Delta^1 \times A$ for which $t^*X \to t^*Y$ is a weak equivalence. This means $p_*X \to p_*Y$ and $X_1 \to Y_1$ is an equivalence. It follows from (2) and the commutative diagram

$$\begin{array}{ccc} p_*X & \longrightarrow & p_*Y \\ {\scriptstyle s_*(X)}\downarrow & & \downarrow{\scriptstyle s_*(Y)} \\ X_0 & \longrightarrow & Y_0 \end{array}$$

that $X_0 \to Y_0$ is also a weak (covariant) equivalence. Thus both $X_0 \to Y_0$ and $X_1 \to Y_1$ are fiberwise equivalences, hence $X \to Y$ is a fiberwise equivalence.

Example 1.2.22 Assume the setup as in Lemma 1.2.20 or Constr. 1.2.13, i.e. we're given a left fibration $W \to \Delta^1 \times A$ and a chosen lift φ as in:

$$\begin{array}{ccc} 0 \times W_0 & \longrightarrow & W \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow \\ \Delta^1 \times W_0 & \longrightarrow & \Delta^1 \times A \end{array}$$

Define $i_W \colon W_0 \to p_*W$ to be the adjoint map to the lift φ. Corresponding to the two endpoints, we have the following two commutative squares:

$$\begin{array}{ccc} W_0 & \xrightarrow{i_W} & p_*W \\ \| & & {\scriptstyle s^*(W)}\downarrow \\ W_0 & = & W_0 \end{array} \qquad \begin{array}{ccc} W_0 & \xrightarrow{i_W} & p_*W \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle t^*(W)} \\ W_1 & = & W_1 \end{array}$$

The left-hand square and (2) above imply that i_W is a trivial cofibration, so on the right-hand side we see that $(i_W, 1_{W_1})$ gives a trivial cofibration between φ_1 and $t^*(W)$. Hence, even though φ_1 depends on a choice, we have a weak replacement $t^*(W)$ which is natural and functorial in W.

For the next result, let us establish the following notation: if W is a presheaf over $\Delta^1 \times A$, then for $\epsilon \in \{0, 1\}$ write W/ϵ for the pullback along $\Delta^1/\epsilon \times A \to \Delta^1 \times A$, viewed as an object over A. Of course, this is either $c_0 \colon 0 \times A \to \Delta^1 \times A$ or the identity map, so $W/0$ is W_0 over A, and $W/1$ is simply $p_!W$. This definition satisfies the following properties:

1. $W \mapsto W/\epsilon$ is left Quillen as a functor $(\mathbf{sSet}/A)_{\mathrm{cov}} \to (\mathbf{sSet}/\Delta^1 \times A)_{\mathrm{cov}}$. Indeed, if $\epsilon = 1$ then this is clear, and otherwise we know that $\Delta^1/0 \to \Delta^1$ is a right fibration, hence $0 \times A \to \Delta^1 \times A$ is smooth.

2. The natural map $W_\epsilon \to W/\epsilon$ is a weak equivalence if W is left fibrant over $\Delta^1 \times A$. Indeed, either this map is the identity or the weak equivalence of Lemma 1.2.20.

Proposition 1.2.23 *For any A, the adjunction $t_! \dashv t^*$ forms a Quillen equivalence.*

Proof We have checked that it forms a Quillen adjunction and that the right adjoint $\mathbf{R}t^*$ is conservative, so it remains to see that $\mathbf{L}t_!$ is fully faithful, i.e. that the unit $F \to \mathbf{R}t^*\mathbf{L}t_!F$ is an isomorphism. Write $F = (F^0 \to F^1)$, and consider a fibrant replacement $t_!F \to G$ in $(\mathbf{sSet}/\Delta^1 \times A)_{\text{cov}}$. This induces a commutative diagram for $\epsilon = 0, 1$:

$$\begin{array}{ccccc} F^\epsilon & \longrightarrow & t^*(t_!F)^\epsilon & \longrightarrow & (t_!F)_\epsilon & \longrightarrow & t_!F/\epsilon \\ & & \downarrow & & \downarrow & & \downarrow \simeq \\ & & t^*(G)^\epsilon & \longrightarrow & G_\epsilon & \xrightarrow{\simeq} & G/\epsilon \end{array}$$

Here, the right-most map is a weak equivalence because of (1) above, and $G_\epsilon \to G/\epsilon$ is one because of (2) above. Moreover, the map $t^*(G)^\epsilon \to G_\epsilon$ is a weak equivalence: for $\epsilon = 1$ it is the identity, and for $\epsilon = 0$ we showed it is a trivial fibration. The composite $F \to t^*(G)$ represents the adjunction unit, so to finish the proof we may show that $F^\epsilon \to (t_!F)/\epsilon$ is a weak equivalence for $\epsilon = 0, 1$.

This is a morphism of left Quillen functors on the category

$$\text{Arr}(\mathbf{sSet}/A)_{\text{cov}} = \text{PSh}(\mathbf{\Delta}/A)^{\bullet \to \bullet} = \text{PSh}((\bullet \to \bullet)^{\text{op}} \times \mathbf{\Delta}/A),$$

where $(\bullet \to \bullet)^{\text{op}} \times \mathbf{\Delta}/A$ is Eilenberg-Zilber. By Remark 1.1.6 we may prove that it is a weak equivalence for F a representable functor, of which there are two forms $(0, \sigma)$ or $(1, \sigma)$, where $\sigma \colon \Delta^n \to A$ is an n-simplex.

1. $(0, \sigma)$: There's exactly one map $0 \to 0$ and $1 \to 0$ in the opposite category of $\bullet \to \bullet$, so $F = (\Delta^n \xrightarrow{1} \Delta^n)$ with the structure maps given by σ. Thus $t_!(F) = \Delta^1 \times \Delta^n$, and the $\epsilon = 0, 1$ cases correspond to the two maps

$$\Delta^n \xrightarrow{1} \Delta^1 \times \Delta^n / 0 = \Delta^n \qquad \Delta^n \xrightarrow{c_0} \Delta^1 \times \Delta^n$$

which are weak equivalences: On the right, this is because c_0 is a section of the same map that c_1 is a section of, and c_1 is left anodyne.

2. $(1, \sigma)$: There's no map $0 \to 1$ and one map $1 \to 1$ in the opposite category, so $F = (\varnothing \to \Delta^n)$, meaning $t_!(F) = \Delta^n$, and the two maps are

$$\varnothing \to \varnothing \qquad \Delta^n \xrightarrow{1} \Delta^n,$$

both of which are weak equivalences.

This proves that the unit is a pointwise weak equivalence and therefore completes the proof. □

Remark 1.2.24 The next theorem is the main result of this section. There is a significant size issue in the original version of Cisinski in the proof of fullness, and in an updated version there is a somewhat unsatisfying fix to this issue. The proof presented below is different from both of these but is also a bit unsatisfying, and involves a direct construction of fullness.

We write $\mathrm{LFib}_{\mathbf{U}}(A)$ for the full subcategory of $\mathrm{LFib}(A)$ consisting of objects represented by left fibrations with **U**-small fibers.

Theorem 1.2.25 *The functor* $p^*_{\mathrm{univ}} \colon \mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S}) \to \mathrm{LFib}(A)$ *is fully faithful, and induces an equivalence of categories between* $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S})$ *and* $\mathrm{LFib}_{\mathbf{U}}(A)$.

Proof The second statement follows immediately, because any left fibration of the form $p^*_{\mathrm{univ}} F$ has **U**-small fibers, and conversely every left fibration with **U**-small fibers admits a classifying map factoring through $\mathcal{S}$.

Fullness: Let $F, G \colon A \to \mathcal{S}$ classify left fibrations $X \to A$ and $Y \to A$ with **U**-small fibers, and suppose that $\psi \colon X \to Y$ is a morphism over A. Let $c_0 \colon 0 \times A \hookrightarrow \Delta^1 \times A$, then we have a natural map $p^*Y \to c_{0,*}Y$ adjoint to the identity, so we may form a pullback

$$\begin{array}{ccc} W & \longrightarrow & c_{0,*}X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle c_{0,*}\psi} \\ p^*Y & \longrightarrow & c_{0,*}Y \end{array}$$

inside $\mathbf{sSet}/\Delta^1 \times A$. By adjunction, for any presheaf K over $\Delta^1 \times A$, a map $K \to W$ is a commutative diagram

$$\begin{array}{ccc} K_0 & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ K & \longrightarrow & Y \end{array}$$

over A, where K_0 is the fiber over $0 \times A$. In particular, an n-simplex of W lying over $(\alpha, \sigma) \colon \Delta^n \to \Delta^1 \times A$ is a map from $(\Delta^r, \Delta^n) \to \psi$ over A. This construction has the following properties:

1. For each such n-simplex of $\Delta^1 \times A$, there can be only **U**-many maps $\Delta^n \to Y$ over A, and only **U**-many maps $\Delta^r \to X$ over A, because of the assumptions that X and Y have **U**-small fibers over A. Therefore W has **U**-small fibers over $\Delta^1 \times A$.

2. $W \to \Delta^1 \times A$ is a left fibration. Suppose we're lifting against $\Lambda^n_k \hookrightarrow \Delta^n$ for $k < n$, and assume the n-simplex of (α, σ) is so that $\Delta^r \subset \Delta^n$ has $r < n$. Then $\Delta^r \subset \Lambda^n_k$, so the zero fiber is already $\Lambda^n_k \times_{\Delta^1} 0 = \Delta^r$, so the given horn provides a map $\Delta^r \to X$ over $\Lambda^n_k \to Y$. This lower map lifts over A, producing a morphsim from $(\Delta^r, \Delta^n) \to \psi$ as desired. The other case is $r = n$, so α is constantly 0, in which case $K_0 \to K$ becomes $\Lambda^n_k = \Lambda^n_k$, and we just extend the map to X over A.

3. When α is a constant map at 0 or 1, Δ^r is either Δ^n or $\varnothing$ respectively, so the simplex is either any simplex of X or any simplex of Y. Thus canonically $W_0 = X$ and $W_1 = Y$. By Example 1.1.51 applied to the inclusion $\partial\Delta^1 \times A \subset \Delta^1 \subset A$, we can choose a classifying map $\Delta^1 \times A \to \mathcal{S}$ for $W \to \Delta^1 \times A$ with endpoints F and G.

To conclude, take $K = \Delta^1 \times X$ and $K_0 = X$ with the obvious maps in the diagram above, producing a map $\Delta^1 \times X \to W$ over $\Delta^1 \times A$. This is the identity on the zero fiber and ψ on the one fiber, so by Lemma 1.2.14 in Constr. 1.2.13 we conclude that the associated map $F \to G$ maps to ψ.

Faithfulness: Let $F, G \colon A \to \mathcal{S}$ be maps classifying $X \to A$ and $Y \to A$, and $f, g \colon F \to G$ two morphisms in $\underline{\mathrm{Hom}}(A, \mathcal{S})$. Assume that $f, g \colon \Delta^1 \times A \to \mathcal{S}$ classify $W_f \to \Delta^1 \times A$ and $W_g \to \Delta^1 \times A$. Apply Lemma 1.2.17 with $\Delta^1 \times A$ to obtain bijections

$$\begin{aligned} \mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(\Delta^1 \times A, \mathcal{S})^{\simeq}}(f, g) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{LFib}(\Delta^1 \times A)^{\simeq}}(W_f, W_g) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Arr}(\mathbf{sSet}/A)_{\mathrm{cov}})^{\simeq}}(t^*(W_f), t^*(W_g)) \end{aligned}$$

because $\pi_0 \operatorname{Equiv}_{\Delta^1 \times A}(W_f, W_g) = \mathrm{Hom}_{\mathrm{LFib}(\Delta^1 \times A)}(W_f, W_g)$, along with the equivalence of Prop. 1.2.23. As f and g vary, this shows that

$$\mathrm{h}\,\underline{\mathrm{Hom}}(\Delta^1 \times A, \mathcal{S})^{\simeq} \to \mathrm{Ho}(\mathrm{Arr}(\mathbf{sSet}/A)^{\simeq}_{\mathrm{cov}})$$

is fully faithful. Now, keeping the same notation, assume that the two maps $\varphi_1, \psi_1 \colon X \to Y$ induced by f and g are equal in $\mathrm{LFib}(A)$. Then they are J-homotopic because X and Y are left fibrant over A, and we can construct a big diagram

$$\begin{array}{ccccccccc} p_*W_f & \xleftarrow{i_f} & X & \xrightarrow{c_0} & J \times X & \xleftarrow{c_1} & X & \xrightarrow{i_g} & p_*W_g \\ {\scriptstyle t^*(W_f)}\downarrow & & {\scriptstyle \varphi_1}\downarrow & & {\scriptstyle h}\downarrow & & {\scriptstyle \psi_1}\downarrow & & \downarrow{\scriptstyle t^*(W_g)} \\ Y & = & Y & = & Y & = & Y & = & Y \end{array}$$

using then J-homotopy h. Here the right-most squares are given by Example 1.2.22. Each horizontal map is a weak equivalence in $\mathrm{LFib}(A)$, and $c_0 = c_1$ in $\mathrm{LFib}(A)$. Thus viewing the diagram is a zig-zag in $\mathrm{Arr}(\mathbf{sSet}/A)_{\mathrm{cov}}$, it provides an isomorphism $t^*(W_f) \cong t^*(W_g)$, which by full faithfulness lifts to an invertible map $\alpha \colon f \simeq g$ in $\mathrm{Hom}(\Delta^1 \times A, \mathcal{S})$. In other words we have a diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha_0} & F \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ G & \xrightarrow{\alpha_1} & G \end{array}$$

commuting in $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S})$. Now apply Lemma 1.2.17 to obtain full faithfulness of $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S})^{\simeq} \to \mathrm{LFib}(A)^{\simeq}$. Transporting this diagram along the functor shows immediately that the image of α_1 is 1_Y, so $\alpha_1 = G$. Likewise, α_0 maps to the composite

$$X \xrightarrow{c_0} J \times X \xrightarrow{c_1^{-1}} X$$

in $\mathrm{LFib}(A)$, which is the identity because $c_0 = c_1$ in $\mathrm{LFib}(A)$. Thus $a_0 = 1_F$ in $\mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{S})$, so we conclude $f = g$ in that category too. □

Corollary 1.2.26 *Let A be a simplicial set.*

1. *For any choice of the map*

$$\underline{\mathrm{Hom}}(\Delta^1, \mathcal{S}) \to \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet),$$

   canonical up to J-homotopy, the induced map

$$\underline{\mathrm{Hom}}(\Delta^1, \underline{\mathrm{Hom}}(A, \mathcal{S})) \to \underline{\mathrm{Hom}}\left(A, \underline{\mathrm{Hom}}_{\mathbf{sS}/\mathcal{S}\times\mathcal{S}}(\mathcal{S}^{(0)}_\bullet, \mathcal{S}^{(1)}_\bullet)\right)$$

   is an equivalence of ∞-categories over $\underline{\mathrm{Hom}}(A, \mathcal{S}) \times \underline{\mathrm{Hom}}(A, \mathcal{S})$.

2. *For any $F, G\colon A \to \mathcal{S}$, the induced functor*

$$\underline{\mathrm{Hom}}(A, \mathcal{S})(F, G) \to \underline{\mathrm{Hom}}_A(p^*_{\mathrm{univ}} F, p^*_{\mathrm{univ}} G)$$

   on fibers is an equivalence of ∞-groupoids.

Proof We prove this in a roundabout way, namely starting with (2). For any simplicial set K, apply $\underline{\mathrm{Hom}}(K, -)$ to both sides of the fiberwise map. By the canonical identifications coming from Hom-ing to $\underline{\mathrm{Hom}}_A$, we obtain

$$\underline{\mathrm{Hom}}(K \times A, \mathcal{S})(F_K, G_K) \to \underline{\mathrm{Hom}}_{K\times A}(p^*_{\mathrm{univ}} F_K, K \times p^*_{\mathrm{univ}} G_K),$$

where F_K, G_K represent $K \times A \to A \to \mathcal{S}$. The functor p^*_{univ} on morphisms was defined by applying π_0 to a map of the form above, so applying π_0 gives a bijection by full faithfulness of Thm. 1.2.25. On the other hand, $\pi_0 \underline{\mathrm{Hom}}(K, -) = \mathrm{Hom}_{\mathcal{H}}(K, -)$ on ∞-groupoids, so by the Yoneda lemma in $\mathcal{H}$ we deduce that the functor is an equivalence of ∞-groupoids.

(1) The second map is, up to isomorphism, obtained form the first by applying $\underline{\mathrm{Hom}}(A, -)$. Thus it suffices to prove that the $A = \Delta^0$ case is an equivalence of ∞-categories. On the other hand, (2) is the statement that after applying $\underline{\mathrm{Hom}}(A, -)$ we obtain a fiberwise equivalence. Therefore the result can be deduced from the following claim:

Claim: Let $f\colon X \to Y$ be a morphism between Joyal fibrations $X \to B \leftarrow Y$ where B is an ∞-category. If $\underline{\mathrm{Hom}}(A, X) \to \underline{\mathrm{Hom}}(A, Y)$ is a fiberwise equivalence of ∞-categories over $\underline{\mathrm{Hom}}(A, B)$, then f is an equivalence of ∞-categories.

Proof: Since X and Y are Joyal fibrant over B, the induced diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A, X)^{\simeq} & \longrightarrow & \underline{\mathrm{Hom}}(A, Y)^{\simeq} \\ & \searrow & \downarrow \\ & & \underline{\mathrm{Hom}}(A, B)^{\simeq} \end{array}$$

has Kan fibrations as vertical maps by [HCHA, Thm. 3.5.11]. Since $(-)^{\simeq}$ preserves pullbacks and equivalences of ∞-categories, the horizontal map is

a fiberwise equivalence. Thus $\underline{\mathrm{Hom}}(A,X)^{\simeq} \to \underline{\mathrm{Hom}}(A,Y)^{\simeq}$ is an equivalence of ∞-groupoids (e.g. by Cor. 1.1.7 and Example 1.1.2), so $X \to Y$ is an equivalence of ∞-categories by [HCHA, Thm. 3.6.9]. □

Example 1.2.27 This is important to us especially in the case that $A = \Delta^0$. It first says that $\mathrm{h}\mathcal{S} \subset \mathcal{H}$ is equivalent to the full subcategory of $\mathcal{H}$ spanned by **U**-small ∞-groupoids. It also says that if $\mathcal{D}$ and $\mathcal{D}'$ are ∞-groupoids, then the mapping space $\mathcal{S}(\mathcal{D},\mathcal{D}')$ is equivalent to $\mathrm{Fun}(\mathcal{D},\mathcal{D}')$, the ∞-groupoid of functors between them. This is the full justification that $\mathcal{S}$ is the right ∞-category of ∞-groupoids. Repeating the same remarks using A, it is also the justification that $\underline{\mathrm{Hom}}(A,\mathcal{S})$ is the right ∞-category of left fibrations over A, and that $\underline{\mathrm{Hom}}(A^{\mathrm{op}},\mathcal{S})$ is the ∞-category of right fibrations over A.

Example 1.2.28 In this example, write $\mathcal{S}_{\mathbf{U}}$ and $\mathcal{S}_{\mathbf{V}}$ for the ∞-category of spaces $\mathcal{S}$ associated to the universes **U** and **V**. Assume that $\mathbf{U} \in \mathbf{V}$. Since a **U**-small simplicial set is **V**-small, there is a natural inclusion $U \subset V$, and $\mathcal{S}_{\mathbf{U}} \subset \mathcal{S}_{\mathbf{V}}$. It is clear that the inclusion classifies the left fibration $\mathcal{S}_{\mathbf{U},\bullet} \to \mathcal{S}_{\mathbf{U}}$ with **V**-small fibers. We deduce immediately from Cor. 1.2.26 that the induced map $\underline{\mathrm{Hom}}(A,\mathcal{S}_{\mathbf{U}}) \to \underline{\mathrm{Hom}}(A,\mathcal{S}_{\mathbf{V}})$ is fully faithful, because the left fibrations classified by two maps are the same regardless of the universe. Note that as a consequence, the natural map from $\mathcal{S}_U$ to the full subcategory of $\mathcal{S}_{\mathbf{V}}$ consisting of ∞-groupoids which are equivalent to **U**-small ∞-groupoids, is fully faithful and essentially surjective, by Lemma 1.2.17.

The next corollary uses the universe axiom in a somewhat essential way.

Corollary 1.2.29 *The following conditions are equivalent for a left fibrant presheaf X over A:*

(*i*) *There is an equivalence of left fibrant objects $X \simeq X'$ over A such that $X' \to A$ has* **U***-small fibers.*

(*ii*) *For each object a of A, the ∞-groupoid X_a is equivalent to a* **U***-small ∞-groupoid.*

(*iii*) *There exists a homotopy cartesian square of the form*

$$\begin{array}{ccc} X & \dashrightarrow & \mathcal{S}_\bullet \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ A & \dashrightarrow & \mathcal{S} \end{array}$$

in $\mathbf{sSet}_{\mathrm{Joy}}$.

Proof (*i*) ⇒ (*iii*) is immediate, because we can just replace the cartesian square classifying $X' \to A$ with $X \to A$, as it is homotopy cartesian by Prop. 1.2.6 (and the fact that this is equivalent to being a weak categorical equivalence). (*iii*) ⇒ (*ii*) By Prop. 1.2.6 and the pasting lemma, we deduce that for each a in A, $X_a \to \Delta^0$ is a homotopy pullback of $\mathcal{S}_\bullet \to \mathcal{S}$. The same result again implies that $X_a \to \{a\} \times_{\mathcal{S}} \mathcal{S}_\bullet$ is a weak categorical equivalence,

hence an equivalence of ∞-groupoids. But $\{a\} \times_{\mathcal{S}} \mathcal{S}_\bullet$ is isomorphic to a **U**-small simplicial set, so we take it to be **U**-small.

$(ii) \Rightarrow (i)$ Choose a universe $\mathbf{V} \ni \mathbf{U}$ such that $X \to A$ has **V**-small fibers. Let $\mathcal{S}'_{\mathbf{V}}$ be the full subcategory of $\mathcal{S}_{\mathbf{V}}$ of ∞-groupoids equivalent to **U**-small ∞-groupoids, then we saw in Example 1.2.28 that $\mathcal{S}_{\mathbf{U}} \to \mathcal{S}'_{\mathbf{V}}$ is an equivalence of ∞-categories. Thus

$$\underline{\mathrm{Hom}}(A, \mathcal{S}_{\mathbf{U}}) \to \underline{\mathrm{Hom}}(A, \mathcal{S}'_{\mathbf{V}})$$

is an equivalence of ∞-categories, hence essentially surjective. Since $X \to A$ is classified by a map in the RHS, we can choose a classifying map in the RHS equivalent to it. If this classifies $X' \to A$ with **U**-small fibers, then Lemma 1.2.17 shows that $X' \simeq X$ as $\mathrm{Equiv}_A(X', X) \simeq \underline{\mathrm{Hom}}(A, \mathcal{S}_{\mathbf{V}})(F_{X'}, F_X)$ is nonempty. □

We sometimes say that a left fibration $X \to A$ has *essentially* **U***-small fibers* if it satisfies one of the equivalent conditions of the previous result. As we will see, the Joyal homotopy pullback coincides with the ∞-categorical pullback in $\mathbf{Cat}_\infty$; thus, when working in a universe containing **U** as an element, the left fibrations with essentially **U**-small fibers are the pullbacks of the universal such fibration in $\mathbf{Cat}_\infty^{\mathbf{U}}$.

Some remarks on uniqueness: Let $p\colon X \to A$ be a map, and let us temporarily define the mapping set

$$\mathcal{M} \coloneqq \underline{\mathrm{Hom}}(X, \mathcal{S}^\bullet) \underset{\underline{\mathrm{Hom}}(X, \mathcal{S})}{\times} \underline{\mathrm{Hom}}(A, \mathcal{S}),$$

whose objects are morphisms $p \to p_{\mathrm{univ}}$ of arrows. Since $\underline{\mathrm{Hom}}(X, -)$ preserves left fibrations, $\mathcal{M} \to \underline{\mathrm{Hom}}(A, \mathcal{S})$ is a left fibration, and in particular $\mathcal{M}$ is an ∞-category. Now assume that p is itself a left fibration. For K a simplicial set, a map $K \to \mathcal{M}$ specifies a commutative diagram:

$$\begin{array}{ccc} K \times X & \dashrightarrow & \mathcal{S}_\bullet \\ \downarrow & & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ K \times A & \dashrightarrow & \mathcal{S} \end{array}$$

Then the following conditions are equivalent:

1. The diagram above is homotopy cartesian.
2. The map $K \times X \to (K \times A) \times_{\mathcal{S}} \mathcal{S}_\bullet$ is an equivalence of left fibrations over $K \times A$.
3. $K \to \mathcal{M}$ factors through the full subcategory of $\mathcal{M}$ spanned by arrows $(\alpha, \beta)\colon p \to p_{\mathrm{univ}}$ such that $X \to A \times_{\mathcal{S}} \mathcal{S}_\bullet$ is an equivalence of left fibrations over A.

(1) and (2) are equivalent because of Prop. 1.2.6; (2) and (3) are equivalent because both conditions are equivalent to the claim that, for every object k of

K, the map $\{k\} \times X \to (\{k\} \times A) \times_{\mathcal{S}} \mathcal{S}_\bullet$ is an equivalence of left fibrations, because everything is checked fiberwise. We therefore define

$$\mathrm{hClass}_{\mathbf{U}}(p) \subset \mathcal{M}$$

to be the full subcategory consisting of commuting maps $p \to p_{\mathrm{univ}}$ so that $X \to A \times_{\mathcal{S}} \mathcal{S}_\bullet$ is an equivalence of left fibrations, or equivalently that the obvious square is homotopy cartesian. In other words, this is the ∞-category of choices of classifying data for the left fibration p. Note that Cor. 1.2.29 says that $\mathrm{hClass}_{\mathbf{U}}(p)$ is nonempty iff p has essentially $\mathbf{U}$-small fibers.

Proposition 1.2.30 *Let $p\colon X \to A$ be a left fibration with essentially $\mathbf{U}$-small fibers. Then $\mathrm{hClass}_{\mathbf{U}}(p)$ is a contractible ∞-groupoid.*

Proof Then $\mathrm{hClass}_{\mathbf{U}}(p)$ is nonempty, so choose a classifying map $F\colon A \to \mathcal{S}$ and pullback $\widetilde{F}\colon X \to \mathcal{S}_\bullet$ such that $X \to A \times_{\mathcal{S}} \mathcal{S}_\bullet$ is an equivalence of left fibrations. Recall the source-target projection

$$(s,t)\colon \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)}) \to \mathcal{S} \times \mathcal{S}$$

which is a Joyal fibration. Hence s is one too, but it is a section of a trivial cofibration by Lemma 1.2.10, so s is a trivial fibration. We claim there is a pullback square:

$$\begin{array}{ccc} \mathrm{hClass}_{\mathbf{U}}(p) & \longrightarrow & \underline{\mathrm{Hom}}(A, \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \Delta^0 & \xrightarrow{F} & \underline{\mathrm{Hom}}(A, \mathcal{S}) \end{array}$$

Indeed, a map from K to the pullback is a map $K \times A \to \underline{\mathrm{Eq}}_{\mathcal{S}\times\mathcal{S}}(\mathcal{S}_\bullet^{(0)}, \mathcal{S}_\bullet^{(1)})$ over F, or in other words, a map $G\colon A \to \mathcal{S}$ and morphism $F \times_S \mathcal{S}_\bullet \to G \times_{\mathcal{S}} \mathcal{S}_\bullet$ of left fibrations which is an equivalence. Such a map is an equivalence iff the map $X \to G \times_{\mathcal{S}} \mathcal{S}_\bullet$ is an equivalence, so we conclude the cartesian property. This shows that $\mathrm{hClass}_{\mathbf{U}}(p) \to \Delta^0$ is a trivial fibration. □

Example 1.2.31 Therefore, for a given left fibration (with essentially small fibers), a choice of classifying map is essentially unique, i.e. unique up to a canonical contractible groupoid. We also see that in the case $A = \Delta^0$, this says that for a given ∞-groupoid $\mathcal{D}$, there is an essentially unique object of $\mathcal{S}$ classifying it. We already knew this on the level of the homotopy category from Thm. 1.2.25.

Remark 1.2.32 Since the projection $\mathcal{M} \to \underline{\mathrm{Hom}}(A, \mathcal{S})$ is a left fibration, we also know it is a conservative isofibration. This means that a morphism in $\mathcal{M}$ is invertible iff its image in $\underline{\mathrm{Hom}}(A, \mathcal{S})$ is invertible. One can use this to deduce quickly that $\mathrm{hClass}_{\mathbf{U}}(p) \hookrightarrow \mathcal{M}$ is an isofibration, because it is a full subcategory closed under equivalences, and hence so is $\mathrm{hClass}_{\mathbf{U}}(p) \to \underline{\mathrm{Hom}}(A, \mathcal{S})$. This also means that given two classifying data in $\mathrm{hClass}_{\mathbf{U}}(p)$, the fact that

they are equivalent means that the corresponding maps $A \to \mathcal{S}$ are equivalent in $\underline{\mathrm{Hom}}(A, \mathcal{S})$, i.e. for each a the map $F_a \to G_a$ of classifying maps is an equivalence of ∞-groupoids.

$\mathrm{hClass}_{\mathbf{U}}(p)$ is the space of homotopy classifying data; we could also consider the full subcategory of $\mathcal{M}$ spanned by those pairs for which the corresponding square is *literally* cartesian. This would then also be a full subcategory of $\mathrm{hClass}_{\mathbf{U}}(p)$, hence contractible if it is nonempty.

Remark 1.2.33 The full subcategory construction in the literally cartesian case described above is not the 'right' contractible ∞-groupoid to study. There is another non-full simplicial subset $\mathrm{class}_{\mathbf{U}}(p) \subset \mathcal{M}$ described as follows: We claim that there is a unique subcomplex such that the maps $K \to \mathcal{M}$ factoring through $\mathrm{class}_{\mathbf{U}}(p)$ are the morphisms $(K \times X \to K \times A) \to p_{\mathrm{univ}}$ for which

$$K \times X \to (K \times A) \times_{\mathcal{S}} \mathcal{S}_\bullet$$

is an isomorphism. Indeed, we can define it by taking this same condition for $K = \Delta^n$, and it is easy to check that the condition passes to all K. The objects of $\mathrm{class}_{\mathbf{U}}(p)$ are still literal classifying data for p, but the higher morphisms are also classifying data for $\Delta^n \times X \to \Delta^n \times A$. Then we claim that if p admits a classifying map, i.e. p has $\mathbf{U}$-small fibers, the $\mathrm{class}_{\mathbf{U}}(p)$ is a contractible groupoid. Indeed, suppose we have a lifting problem $K \hookrightarrow L$ and a map $K \to \mathrm{class}_{\mathbf{U}}(p)$. We have cartesian squares:

$$\begin{array}{ccc} K \times X & \xrightarrow{\widetilde{F}} & \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ K \times A & \xrightarrow{F} & \mathcal{S} \end{array} \qquad \begin{array}{ccc} K \times X & \longrightarrow & L \times X \\ \downarrow & \lrcorner & \downarrow \\ K \times A & \longrightarrow & L \times A \end{array}$$

If $K \neq \varnothing$, then we can simply apply Example 1.1.51 and extend F to a classifying map G for $L \times X \to L \times A$, which will provide a lift to $L \to \mathrm{class}_{\mathbf{U}}(p)$. If $K = \varnothing$, then we just need to find any classifying map for $L \times X \to L \times A$, which is easy because it is a pullback of $X \to A$, hence has $\mathbf{U}$-small fibers.

These results are important for conceptual understanding, especially since we do not yet have an instrinsically constructed ∞-category of left fibrations over A. Once we discuss localization, that will be another perspective. We will also use Prop. 1.2.30 to justify that mapping space functors $\mathrm{Map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ exist and are unique up to contractible ∞-groupoid.

Example 1.2.34 This is a continuation of Example 1.1.32 and Example 1.1.33. We consider the setup there, in which we had a right fibration $F \colon C \to D$ of small 1-categories induced by choices of groupoids F_d for each d, with pullback maps $u^* \colon F_{d'} \to F_d$ for each $u \colon d \to d'$ in D. Now, choose a universe $\mathbf{U}$ in which C is $\mathbf{U}$-small, so that $F^{\mathrm{op}} \colon C^{\mathrm{op}} \to D^{\mathrm{op}}$ is a left fibration with $\mathbf{U}$-small fibers. Hence there is an essentially unique functor

$$\widetilde{F} \colon D^{\mathrm{op}} \to \mathcal{S}$$

classifying F^{op}, so we can identify $\widetilde{F}(d) = F_d$ for all $d \in D$. We want to see that under the equivalence Cor. 1.2.26, the map $\widetilde{F}(d') \to \widetilde{F}(d)$ in $\mathcal{S}$ associated to $u\colon d \to d'$ is given by the pullback functor u^*. To see this, return to Constr. 1.2.13: if C_u^{op} is the fiber $\Delta^1 \times_{D^{\mathrm{op}}} C^{\mathrm{op}}$ associated to u, we need to come up with a lift in the diagram:

$$\begin{array}{ccc} 0 \times F_{d'} & \longrightarrow & C_u^{\mathrm{op}} \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 \times F_{d'} & \longrightarrow & \Delta^1 \end{array}$$

To do this, we have the two functors $F_{d'} \hookrightarrow C_u^{\mathrm{op}}$ and $F_{d'} \xrightarrow{u^*} F_d \hookrightarrow C_u^{\mathrm{op}}$, and we want to exhibit a natural transformation between them. Indeed there is the canonical one which is the map $u\colon (d, u^*x) \to (d', x)$ on components $x \in F_{d'}$, providing the relevant homotopy. This ends at the map $u^*\colon F_{d'} \to F_d$ by definition, so the corresponding map $\widetilde{F}(d') \to \widetilde{F}(d)$ is precisely given by pullback.

For example, we consider the case of Example 1.1.33. Using a universe/cardinality bound, we can restrict to small enough schemes **Sch** and small enough quasicoherent sheaves so that C and **Sch** are small categories, and the fiber $F_X = \mathrm{QCoh}(X)^{\simeq}$ is a small category. Then we write the induced classifying map as

$$\mathrm{QCoh}^{\simeq}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathcal{S} \qquad X \mapsto \mathrm{QCoh}(X)^{\simeq}.$$

By our computation above, a morphism $f\colon X \to Y$ of schemes induces the pullback map

$$\mathrm{QCoh}^{\simeq}(f) \cong f^*\colon \mathrm{QCoh}(Y)^{\simeq} \to \mathrm{QCoh}(X)^{\simeq},$$

as we would expect, at least up to natural isomorphism. This is especially nice because it records various natural coherences that don't need to be checked independently, especially associativity-compatibility of the natural isomorphisms $f^*g^* \cong (gf)^*$.

1.2.3 Locally U-small ∞-categories

In this section we mainly prove two size results on **U**-smallness and locally **U**-smallness, and setup the theory of the twisted diagonal and fat twisted diagonal of an ∞-category. Both of these are in preparation to state and prove the Yoneda lemma. In the latter half of this section, we will freely use and cite to the theory of bivariant model structures, as discussed in [HCHA, Ch. 5.5].

Convention In this section, we will use the temporary phrase 'cardinally **U**-small' or of **U**-small cardinality if an object is isomorphic to a **U**-small object. So, a cardinally **U**-small set is one bijective to an element of **U**, cardinally **U**-small simplicial set is one isomorphic to a **U**-small simplicial set, etc. Later, when we are not concerned with set theory anymore and just have a fixed background universe, we will say small to mean cardinally **U**-small.

Definition 1.2.35 An ∞-category $\mathcal{C}$ is *locally* $\mathbf{U}$*-small* if for all objects x, y of $\mathcal{C}$, the ∞-groupoid $\mathcal{C}(x, y)$ is equivalent to a $\mathbf{U}$-small ∞-groupoid.

Example 1.2.36 If A is a $\mathbf{U}$-small simplicial set, then $\underline{\mathrm{Hom}}(A, \mathcal{S})$ is locally $\mathbf{U}$-small. Indeed, its mapping spaces are equivalent to $\underline{\mathrm{Hom}}_A(X, Y)$ for left fibrations $X \to A$ and $Y \to A$ with $\mathbf{U}$-small fibers (Cor. 1.2.26). But then X and Y are isomorphic to $\mathbf{U}$-small simplicial sets, so $\underline{\mathrm{Hom}}(X, Y)$ is isomorphic to a $\mathbf{U}$-small simplicial set, so $\underline{\mathrm{Hom}}_A(X, Y)$ is too. In particular $\mathcal{S}$ is locally $\mathbf{U}$-small. We'll generalize this below.

Example 1.2.37 By definition, if $F\colon \mathcal{C} \to \mathcal{D}$ is a fully faithful functor of ∞-categories, then if $\mathcal{D}$ is locally $\mathbf{U}$-small we see that $\mathcal{C}$ is also locally $\mathbf{U}$-small. In particular the notion is invariant under equivalences of ∞-categories.

Example 1.2.38 If $\mathcal{C}$ is locally $\mathbf{U}$-small, then for any tuple $(x_0, \ldots, x_n)$ of objects the n-fold mapping space $\mathcal{C}(x_0, \ldots, x_n)$ is also equivalent to a $\mathbf{U}$-small ∞-groupoid. Indeed, by [HCHA, 3.7.7] there is a canonical trivial fibration

$$\mathcal{C}(x_0, \ldots, x_n) \to \mathcal{C}(x_0, x_1) \times \cdots \times \mathcal{C}(x_{n-1}, x_n),$$

and the RHS is equivalent to a $\mathbf{U}$-small ∞-groupoid.

Example 1.2.39 Suppose $\mathcal{C}$ is a (cardinally) $\mathbf{U}$-small ∞-category. Then $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})$ is cardinally $\mathbf{U}$-small, so $\mathcal{C}(x, y)$ is cardinally $\mathbf{U}$-small for all x and y. Hence $\mathcal{C}$ is locally $\mathbf{U}$-small.

Remark 1.2.40 We recall that an ordinary locally small category C is locally $\mathbf{U}$-small if for each x, y, the set $\mathrm{Hom}_C(x, y)$ is cardinally $\mathbf{U}$-small. An *essentially* $\mathbf{U}$*-small ordinary category* C is one which is locally $\mathbf{U}$-small, and the isomorphism classes of objects form a cardinally $\mathbf{U}$-small set. We recall that this is equivalent to C being equivalent to a $\mathbf{U}$-small category.

Example 1.2.41 If $\mathcal{C}$ is cardinally $\mathbf{U}$-small, then $\mathrm{h}\mathcal{C}$ is obviously also cardinally $\mathbf{U}$-small. Also clearly (or consequently), C is cardinally $\mathbf{U}$-small iff $N(C)$ is cardinally $\mathbf{U}$-small. Suppose $\mathcal{C}$ is locally $\mathbf{U}$-small. Then $\mathcal{C}(x, y)$ is equivalent to a $\mathbf{U}$-small ∞-groupoid $\mathcal{D}$, so $\pi_0\mathcal{C}(x, y) \cong \pi_0(\mathcal{D})$ is cardinally $\mathbf{U}$-small. In other words $\mathrm{h}\mathcal{C}$ is locally $\mathbf{U}$-small. Conversely, if $\mathcal{C}$ is a 1-category in the sense of Remark 1.0.3 and $\mathrm{h}\mathcal{C}$ is locally $\mathbf{U}$-small, then $\mathcal{C}$ is locally $\mathbf{U}$-small. Indeed, its mapping spaces are equivalent to mapping spaces of the form $N(\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y))$ where $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y)$ is viewed as a discrete groupoid, which is cardinally $\mathbf{U}$-small.

Proposition 1.2.42 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between locally $\mathbf{U}$-small ∞-categories. The following conditions are equivalent:*

(*i*) *F is fully faithful.*

(*ii*) *For any $\mathbf{U}$-small simplicial set A, the induced map*

$$F_*\colon \underline{\mathrm{Hom}}(A, \mathcal{C}) \to \underline{\mathrm{Hom}}(A, \mathcal{D})$$

is a fully faithful functor, with essential image consisting of the maps $\varphi\colon A \to \mathcal{D}$ *such that for each* a *in* A, $\varphi(a)$ *is in the essential image of* F.

(*iii*) *For any* $\mathbf{U}$*-small simplicial set* A, *the induced map*

$$F_*\colon \mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{C}) \to \mathrm{h}\,\underline{\mathrm{Hom}}(A, \mathcal{D})$$

is a fully faithful functor.

(*iv*) *There is a factorization of* F *into* $\mathcal{C} \to \mathcal{C}' \to \mathcal{D}$ *where* $\mathcal{C} \to \mathcal{C}'$ *is an equivalence of* ∞*-categories, and* $\mathcal{C}' \to \mathcal{D}$ *has the right lifting property against* $\partial\Delta^n \to \Delta^n$ *for* $n > 0$.

Remark 1.2.43 If X is a simplicial set and $S \subset X_0$ is a subset, we write $X_S \subset X$ for the full simplicial subset spanned by S. Then we recall that $X_S \to X$ has the RLP against all maps which are bijections on objects, in particular $\partial\Delta^n \to \Delta^n$ for $n > 0$. If $f\colon X \to Y$ is a morphism, then it has the RLP against such inclusions iff the map $X \to Y_{\mathrm{im}\, f}$ is a trivial fibration, where $Y_{\mathrm{im}\, f}$ is the full subcomplex spanned by $f_0(X_0) \subset Y_0$.

Proof (of Prop. 1.2.42) $(i) = (iv)$ $(\Rightarrow)$ Let $\mathcal{D}' \subset \mathcal{D}$ be the full subcategory spanned by the image of $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$, so that F factors as $\mathcal{C} \to \mathcal{D}' \to \mathcal{D}$. Then the first map is fully faithful and essentially surjective, and $\mathcal{D}' \hookrightarrow \mathcal{D}$ is a full subcategory inclusion so we conclude by the remark above. $(\Leftarrow)$ Given such a factorization, we may assume $\mathcal{C} = \mathcal{C}'$ and that F has this property. Then again repeat the same factorization $\mathcal{C} \to \mathcal{D}' \to \mathcal{D}$ as above. By the remark, $\mathcal{C} \to \mathcal{D}'$ has to be a trivial fibration, hence an equivalence, so $\mathcal{C} \to \mathcal{D}$ is fully faithful.

$(i) = (ii)$ One direction is clear by taking $A = \Delta^0$. Conversely, let $\mathcal{D}' \subset \mathcal{D}$ be the essential image of F, so that $\mathcal{C} \to \mathcal{D}' \hookrightarrow \mathcal{D}$ where the first functor is an equivalence. Then F_* factors into

$$\underline{\mathrm{Hom}}(A, \mathcal{C}) \to \underline{\mathrm{Hom}}(A, \mathcal{D}') \hookrightarrow \underline{\mathrm{Hom}}(A, \mathcal{D})$$

where the first map is an equivalence, and we claim that the second inclusion is of the full subcategory spanned by maps $\varphi\colon A \to \mathcal{D}$ such that $\varphi(a) \in \mathcal{D}'$ for all a in A. Indeed, an n-simplex $\psi\colon \Delta^n \times A \to \mathcal{D}$ lies in $\underline{\mathrm{Hom}}(A, \mathcal{D}')$ iff the map factors through $\mathcal{D}'$, i.e. $\psi(i, a) \subset \mathcal{D}'$, iff $\varphi(a)\colon \Delta^n \to \mathcal{D}$ lies in $\mathcal{D}'$. This proves the claim.

$(ii) = (iii)$ The $(\Rightarrow)$ direction is already known. If (iii) holds, we use the normal trick: starting with the map $\mathcal{C}(x, y) \to \mathcal{D}(Fx, Fy)$, we first observe that it suffices to show that applying $\pi_0\,\underline{\mathrm{Hom}}(A, -)$ for all $\mathbf{U}$-small A gives an equivalence. Indeed, this is because if we choose $\mathbf{U}$-small equivalent representatives $\mathcal{E}_1, \mathcal{E}_2$ for each mapping space, then this will imply that

$$\mathrm{Hom}_{\mathcal{H}_\mathbf{U}}(A, \mathcal{E}_1) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{H}_\mathbf{U}}(A, \mathcal{E}_2)$$

is an isomorphism for all $A \in \mathcal{H}_\mathbf{U}$, so $\mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ is an equivalence of ∞-groupoids, hence the original map is one too.

To prove this claim, observe that we get

$$\pi_0 \underline{\mathrm{Hom}}(A, \mathcal{C})(x, y) \to \pi_0 \underline{\mathrm{Hom}}(A, \mathcal{D})(Fx, Fy)$$

where x, y, Fx, Gy denote the corresponding constant-in-A maps. This is isomorphic to the map

$$\mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(A,\mathcal{C})}(x, y) \to \mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(A,\mathcal{D})}(Fx, Fy),$$

which we have assumed is a bijection. □

The next result will rely on a more in-depth computation regarding minimal complexes as in [HCHA, Ch. 5.1]. We will use the following lemma, which is also interesting in its own right. Assume $n \geq 1$; if $\mathcal{C}$ is an ∞-category and $b\colon \partial\Delta^n \to \mathcal{C}$ is a boundary, we construct a cartesian diagram:

$$\begin{array}{ccccc}
\mathrm{Fill}^n_{\mathcal{C}}(b) & \hookrightarrow & \underline{\mathrm{Hom}}(\Delta^n, \mathcal{C})^{\simeq} & \hookrightarrow & \underline{\mathrm{Hom}}(\Delta^n, \mathcal{C}) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\{b\} & \hookrightarrow & \underline{\mathrm{Hom}}(\partial\Delta^n, \mathcal{C})^{\simeq} & \hookrightarrow & \underline{\mathrm{Hom}}(\partial\Delta^n, \mathcal{C})
\end{array}$$

Since $\partial\Delta^n \to \Delta^n$ is surjective on vertices, it follows that the right-hand map is conservative, so the right-hand square is cartesian. Since $\mathcal{C}$ is an ∞-category, the right-hand map is a Joyal fibration ([HCHA, Cor. 3.6.4]), so the middle vertical map is a Kan fibration ([HCHA, Thm. 3.5.11]), so $\mathrm{Fill}^n_{\mathcal{C}}(b)$ is an ∞-groupoid. If b is instead a map $\Lambda^n_k \to \Delta^n$, we use the same notation to denote the corresponding construction; in this case if $n \geq 2$, then we again have bijectivity on vertices so the result is an ∞-groupoid.

Lemma 1.2.44 *Let $\mathcal{C}$ be an ∞-category, $n > 0$, and $b\colon \partial\Delta^n \to \mathcal{C}$ be a boundary. If b has a filler $f\colon \Delta^n \to \mathcal{C}$, there is an equivalence*

$$\mathrm{Fill}^n_{\mathcal{C}}(b) \simeq \Omega^{n-1}(\mathcal{C}(x_0, x_n), f_{0,n})$$

of ∞-groupoids, where $f_{0,n}\colon x_0 \to x_n$ is the long edge of f.

Proof For $n = 1$, $\mathrm{Fill}^1_{\mathcal{C}}(b) = \mathcal{C}(x_0, x_1)$ by definition. Assuming $n \geq 2$, observe that

$$\Lambda^n_1 \cap \Delta^{[n]\setminus 1} = \partial\Delta^{[n]\setminus 1} \subset \Delta^n,$$

and the union of these two is $\partial\Delta^n$, giving a bicartesian square. Applying $\underline{\mathrm{Hom}}(-, \mathcal{C})$, we obtain the bottom right square in the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Fill}^n_{\mathcal{C}}(b) & \longrightarrow & \mathrm{Fill}^n_{\mathcal{C}}(b|_{\Lambda^n_1}) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^n, \mathcal{C}) & & \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow & & \\
\{b\} & \longrightarrow & \mathrm{Fill}^{n-1}_{\mathcal{C}}(\partial\tau) & \longrightarrow & \underline{\mathrm{Hom}}(\partial\Delta^n, \mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^{n-1}, \mathcal{C}) \\
 & & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
 & & \{b|_{\Lambda^n_1}\} & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda^n_1, \mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(\partial\Delta^{n-1}, \mathcal{C})
\end{array}$$

Here, we have let $\tau = b|_{\Delta^{[n]\setminus 1}}$. Then the outer rectangle on the bottom is cartesian by definition, so the left-bottom one is; then the middle outer rectangle

is cartesian, so the middle one is. The outer-upper one is cartesian, so the left one is.

Since $\mathcal{C}$ is an ∞-category, restriction along $\Lambda^n_1 \to \Delta^n$ produces a trivial fibration ([HCHA, Cor. 3.2.8]). Thus $\mathrm{Fill}^n_{\mathcal{C}}(b|_{\Lambda^n_1})$ is a contractible ∞-groupoid. On the other hand, the map

$$\mathrm{Fill}^n_{\mathcal{C}}(b|_{\Lambda^n_1}) \to \mathrm{Fill}^{n-1}_{\mathcal{C}}(\partial\tau)$$

is a Joyal fibration of Kan complexes, hence a Kan fibration. Therefore the upper right square is homotopy cartesian. Since b has a filler f, it maps to a filler of $b|_{\Lambda^n_1}$ which maps to the filler τ of $\partial\tau$. Therefore,

$$\begin{aligned}\mathrm{Fill}^n_{\mathcal{C}}(b) &\simeq \{\tau\} \times^{\mathbf{R}}_{\mathrm{Fill}^{n-1}_{\mathcal{C}}(\partial\tau)} \mathrm{Fill}^n_{\mathcal{C}}(b|_{\Lambda^n_1}) \\ &\simeq \{\tau\} \times^{\mathbf{R}}_{\mathrm{Fill}^{n-1}_{\mathcal{C}}(\partial\tau)} \{\tau\} \simeq \Omega\big(\mathrm{Fill}^{n-1}_{\mathcal{C}}(\partial\tau), \tau\big)\end{aligned}$$

where the last step is by Example 1.1.25. On the other hand, by definition $\tau = d_1(f)$ has vertices $x_0, \ldots, x_n$ with long edge $f_{0n} \colon x_0 \to x_n$. By induction, we conclude

$$\big(\mathrm{Fill}^{n-1}_{\mathcal{C}}(\partial\tau), \tau\big) \simeq \Omega^{n-2}(\mathcal{C}(x_0, x_n), f_{0n})$$

as a pair, so combining with the above observation completes the proof. □

We observe that if X is a locally $\mathbf{U}$-small ∞-groupoid, then $\Omega(X, x_0) = X(x, x_0)$ is equivalent to a $\mathbf{U}$-small ∞-groupoid, which is in particular locally $\mathbf{U}$-small, and we may repeat. In particular, if X is equivalent to a $\mathbf{U}$-small ∞-groupoid then $\Omega^n(X, x_0)$ is as well for all $n \geq 0$.

Corollary 1.2.45 *Let $\mathcal{C}$ be an ∞-category. The following conditions are equivalent:*

(*i*) *$\mathcal{C}$ is equivalent to a $\mathbf{U}$-small ∞-category.*

(*ii*) *There exists a $\mathbf{U}$-small simplicial set A and a weak categorical equivalence $\mathcal{C} \to A$.*

(*iii*) *$\mathcal{C}$ admits a $\mathbf{U}$-small Joyal minimal model.*

(*iv*) *$\mathcal{C}$ is locally $\mathbf{U}$-small and $\mathrm{h}\mathcal{C}$ is essentially $\mathbf{U}$-small.*

In this case, we say $\mathcal{C}$ is essentially $\mathbf{U}$-small.

Proof (*i*) = (*ii*) One direction is clear, and if $\mathcal{C} \to A$ is a weak equivalence, then by Remark 1.1.55 we may factor $A \to \Delta^0$ through an inner fibration $A \to \mathcal{D} \to \Delta^0$ with $A \to \mathcal{D}$ inner anodyne and $\mathcal{D}$ $\mathbf{U}$-small, proving (*i*).

(*i*) = (*iii*) One direction is clear because $\mathcal{C}$ is equivalent to its minimal model. Conversely, if $\mathcal{C} \simeq \mathcal{D}$ where $\mathcal{D}$ is $\mathbf{U}$-small, and $S \hookrightarrow \mathcal{D}$ is a minimal model, then S is minimal and $\mathbf{U}$-small; so if $S' \hookrightarrow \mathcal{C}$ is a minimal model, then using a retract of $S \hookrightarrow \mathcal{D}$ we can produce a weak equivalence $S' \to S$. This is an isomorphism by [HCHA, Prop. 5.1.10], proving the claim.

$(i) \Rightarrow (iv)$ If $\mathcal{C} \simeq \mathcal{C}'$ with $\mathcal{C}'$ **U**-small, then we know this implies $\mathcal{C}'$ and hence $\mathcal{C}$ are locally **U**-small. Moreover $\mathrm{h}\mathcal{C} \simeq \mathrm{h}\mathcal{C}'$, and we have seen that $\mathrm{h}\mathcal{C}'$ is cardinally **U**-small, so $\mathrm{h}\mathcal{C}$ is essentially **U**-small.

$(iv) \Rightarrow (iii)$ Since a minimal model is equivalent to $\mathcal{C}$, the hypotheses transfer onto any choice of minimal model, so we may assume that $\mathcal{C}$ is a minimal complex. The chosen cylinder is J, and two objects of $\mathcal{C}$ are equivalent iff there is a map $J \to \mathcal{C}$ connecting them, iff they are equal in $\mathcal{C}$ by minimality. Therefore there are no non-identity isomorphisms in $\mathrm{h}\mathcal{C}$, so we conclude that $\mathcal{C}_0$ is cardinally **U**-small by essential smallness.

We claim by induction that $\mathcal{C}_n$ is cardinally **U**-small. Indeed, assume $\mathcal{C}_{n-1}$ is cardinally **U**-small. Then $\mathrm{Hom}(\partial\Delta^n, \mathcal{C})$ is cardinally **U**-small, so it suffices to show that for each $b\colon \partial\Delta^n \to \mathcal{C}$, the set $\{\sigma \in \mathcal{C}_n : \partial\sigma = b\}$ is cardinally **U**-small. Indeed, we have the canonical surjection

$$\mathrm{Obj}\left(\mathrm{Fill}^n_{\mathcal{C}}(b)\right) \to \{\sigma \in \mathcal{C}_n : \partial\sigma = b\}.$$

If there is a morphism $\sigma_1 \to \sigma_2$ in $\mathrm{Fill}^n_{\mathcal{C}}(b) \subset \underline{\mathrm{Hom}}(\Delta^n, \mathcal{C})^{\simeq}$, it corresponds to an invertible map $J \to \underline{\mathrm{Hom}}(\Delta^n, \mathcal{C})^{\simeq}$, i.e. $J \times \Delta^n \to \mathcal{C}$, whose restriction to $J \times \partial\Delta^n$ is the constant map at b. By minimality, $\sigma_1 = \sigma_2$, so the map factors through a surjection

$$\pi_0 \mathrm{Fill}^n_{\mathcal{C}}(b) \to \{\sigma \in \mathcal{C}_n : \partial\sigma = b\}.$$

By Lemma 1.2.44, if the LHS is nonempty, then it isomorphic to

$$\pi_0\big(\Omega^{n-1}(\mathcal{C}(x_0, x_n), f_{0n})\big)$$

which is cardinally **U**-small because $\mathcal{C}$ is locally **U**-small. Thus we are done by choosing a **U**-small simplicial set isomorphic to $\mathcal{C}$. □

Example 1.2.46 If $\mathcal{D}$ is an ∞-groupoid, we can extract the following argument from the proof. If $\mathcal{D}$ is essentially **U**-small, then so is $\Omega^n(\mathcal{D}, x_0)$ for all x_0, thus $\pi_n(\mathcal{D}, x_0)$ is cardinally **U**-small. Conversely, that $\pi_n(\mathcal{D}, x_0)$ is cardinally **U**-small for all n and x_0, then we'll show $\mathcal{D}$ is essentially **U**-small. We may assume $\mathcal{D}$ is a Joyal minimal complex, in which case we may proceed by induction in the proof. The point is that whenever there is a filler f of b with long map $f_{0n}\colon x_0 \to x_n$, then the assumption implies

$$\pi_n(\mathcal{D}, x_0) \cong \pi_{n-1}(\mathcal{D}(x_0, x_n), f_{0n}) = \pi_0\Omega^{n-1}(\mathcal{D}(x_0, x_n), f_{0n})$$

(by Example 1.1.11), and we know that the RHS surjects onto the $\{\sigma : \partial\sigma = b\}$, completing the proof. (Note that this statement cannot hold for ∞-categories: if C is an arbitrarily large (small) 1-category with a terminal object, then it is weakly contractible.)

In light of the above result, we will also say that a simplicial set X is *essentially* **U**-*small* if there is a **U**-small simplicial set A and a weak categorical equivalence $X \to A$. Then (1) by (ii) above this coincides with the original definition for the ∞-category case, and (2) it is clearly equivalent to saying

that X has a **U**-small Joyal fibrant replacement $X \to \mathcal{C}$. However, the notion is not stable under weak equivalences in both directions (i.e. it only goes from target to source).

Corollary 1.2.47 *Let $\mathcal{C}$ be a locally* **U***-small ∞-category.*

1. *For each essentially* **U***-small simplicial set A, $\underline{\mathrm{Hom}}(A, \mathcal{C})$ is a locally* **U***-small ∞-category.*
2. *For each object x of $\mathcal{C}$, the (fat) slice and coslices*
$$\mathcal{C}/x \qquad x\backslash\mathcal{C} \qquad \mathcal{C}//x \qquad x\backslash\backslash\mathcal{C}$$
are locally **U***-small ∞-categories.*
3. *Any ∞-subcategory $\mathcal{B} \subset \mathcal{C}$ is locally* **U***-small.*

Proof (1) A weak categorical equivalence $A \to A'$ induces an equivalence of ∞-categories $\underline{\mathrm{Hom}}(A', \mathcal{C}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(A, \mathcal{C})$, so we may assume that A is **U**-small. Let $f, g\colon A \to \mathcal{C}$ be maps; the union $f(A_0) \cup g(A_0) \subset \mathrm{Ob}(\mathcal{C})$ is cardinally **U**-small, so let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory spanned by these objects. Then $\mathcal{C}'$ is locally small and has a cardinally **U**-small set of objects, so by Cor. 1.2.45 it is essentially small, so choose an equivalence $F\colon \mathcal{C}' \simeq \mathcal{D}$ where $\mathcal{D}$ is **U**-small. $\underline{\mathrm{Hom}}(A, \mathcal{C}') \to \underline{\mathrm{Hom}}(A, \mathcal{C})$ is fully faithful by Prop. 1.2.42, so
$$\underline{\mathrm{Hom}}(A, \mathcal{D})(Ff, Fg) \simeq \underline{\mathrm{Hom}}(A, \mathcal{C}')(f, g) \to \underline{\mathrm{Hom}}(A, \mathcal{C})(f, g)$$
is an equivalence. Since $\underline{\mathrm{Hom}}(A, \mathcal{D})(Ff, Fg)$ is **U**-small we're done.

(2) Let us do $\mathcal{C}/x$, as the others will follow either by duality or from being equivalent to this one. An n-simplex of $\mathcal{C}/x$ is an $(n+1)$-simplex of $\mathcal{C}$ with last vertex x; therefore if $\mathcal{C}' \subset \mathcal{C}$ is a full subcategory containing x, then $\mathcal{C}'/x \subset \mathcal{C}/x$ is a full subcategory. In light of this, choose objects α, β of $\mathcal{C}/x$, then we can as in (1) choose a full subcategory $\mathcal{C}' \subset \mathcal{C}$ containing α, β, x with a **U**-small object set. Thus we may assume that $\mathcal{C}$ has a **U**-small object set. By Cor. 1.2.45, $\mathcal{C}$ is essentially **U**-small, hence choose $F\colon \mathcal{C} \simeq \mathcal{D}$ where $\mathcal{D}$ is **U**-small. By Remark 1.1.17, we deduce that $\mathcal{C}/x \simeq \mathcal{D}/F(x)$, and the latter is **U**-small.

(3) For x, y in $\mathcal{B}$, choose $\mathcal{D} \simeq \mathcal{C}(x, y)$ a **U**-small ∞-groupoid. Form the cartesian square:
$$\begin{array}{ccc} \mathcal{D}' & \hookrightarrow & \mathcal{D} \\ \simeq\downarrow & \lrcorner & \simeq\downarrow \\ \mathcal{B}(x, y) & \hookrightarrow & \mathcal{C}(x, y) \end{array}$$
We know from Remark 1.0.8 that the lower map is an monic Kan fibration, the left-hand map is an equivalence of ∞-groupoids while $\mathcal{D}'$ is cardinally **U**-small, as a subcomplex of $\mathcal{D}$. □

From what we have said, we already have one-sided functorial mapping spaces. Indeed, by Cor. 1.2.29, it is clear that $\mathcal{C}$ is locally **U**-small if and only if for each

object x, the left fibration $x\backslash\mathcal{C} \to \mathcal{C}$ has essentially **U**-small fibers. Thus by choosing a homotopy classifying map $\mathrm{Map}_{\mathcal{C}}(x,-)\colon \mathcal{C} \to \mathcal{S}$ for this left fibration, the homotopy fiber over y is $\mathcal{C}(x,y)$. We can even use $x\backslash\backslash\mathcal{C}$ to make it more natural.

In the last part of this section, we will develop the theory of the twisted diagonal, which is an ∞-category $\mathrm{S}(\mathcal{C})$ equipped with a canonical left fibration $\mathrm{S}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Its homotopy fibers are the mapping spaces $\mathcal{C}(x,y)$, and a choice of classifying data (which is unique up to contractible ∞-groupoid) produces a mapping space functor

$$\mathrm{Map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$$

which is (almost) equipped with canonical equivalences $\mathrm{Map}_{\mathcal{C}}(x,y) \simeq \mathcal{C}(x,y)$. This will be essential to the Yoneda lemma and the next section.

Definition 1.2.48 Let A be a simplicial set. We define two bisimplicial sets

$$\mathbf{S}(A) \coloneqq \big((m,n) \mapsto \mathrm{Hom}((\Delta^m)^{\mathrm{op}} * \Delta^n, A)\big)$$
$$\mathbf{S}_{\diamond}(A) \coloneqq \big((m,n) \mapsto \mathrm{Hom}((\Delta^m)^{\mathrm{op}} \diamond D^n, A)\big)$$

which are equipped with their natural maps

$$\mathbf{S}(A) \to A^{\mathrm{op}} \boxtimes A \qquad \mathbf{S}_{\diamond}(A) \to A^{\mathrm{op}} \boxtimes A.$$

We write $\mathrm{S}(A) = \delta^*(\mathbf{S}(S))$ and $\mathrm{S}_{\diamond}(A) = \delta^*(\mathbf{S}_{\diamond}(S))$, equipped with

$$\mathrm{S}(A) \to A^{\mathrm{op}} \times A \qquad \mathrm{S}_{\diamond}(A) \to A^{\mathrm{op}} \times A$$

for their diagonals. In the case that $A = \mathcal{C}$ is an ∞-category, we call $\mathrm{S}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ the *twisted diagonal* of $\mathcal{C}$, and $\mathrm{S}_{\diamond}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ the *fat twisted diagonal*.

We observe that since the natural map $(\Delta^m)^{\mathrm{op}} * (\Delta^n) \to (\Delta^m)^{\mathrm{op}} \diamond (\Delta^n)$ (Prop. 1.1.12) is a morphism under the component inclusions, the induced map $\mathrm{S}(A) \to \mathrm{S}_{\diamond}(A)$ is a morphism over $A^{\mathrm{op}} \times A$.

Lemma 1.2.49 *Let $\mathcal{C}$ be an ∞-category.*

1. *The natural maps*

$$\mathbf{S}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \qquad \mathbf{S}_{\diamond}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C}$$

   are left bifibrations.

2. *The natural maps*

$$\mathrm{S}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \qquad \mathrm{S}_{\diamond}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$$

   are left fibrations.

In particular, $\mathrm{S}(\mathcal{C})$ *and* $\mathrm{S}_{\diamond}(\mathcal{C})$ *are ∞-categories.*

Proof The left adjoint $\delta_!$ sends left bi-anodyne extensions to left anodyne extensions by [HCHA, Cor. 5.5.25], so δ^* sends left bifibrations to left fibrations by [HCHA, Prop. 2.1.12]. Thus we may prove (1). There are two classes to check right lifting against, and the proof in each case is similar so we do the first one. For $m \geq 1, n \geq 0$, and $0 \leq k < m$, we have the the following lifting correspondence:

$$\begin{array}{ccc} (\Delta^m)^{\mathrm{op}} * \partial\Delta^n \cup (\Lambda^m_k)^{\mathrm{op}} * \Delta^n & \to & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ (\Delta^m)^{\mathrm{op}} * \Delta^n & \longrightarrow & \Delta^0 \end{array} \qquad \begin{array}{ccc} \Delta^m \boxtimes \partial\Delta^n \cup \Lambda^m_k \boxtimes \Delta^n & \longrightarrow & \mathbf{S}(\mathcal{C}) \\ \downarrow & \nearrow & \downarrow \\ \Delta^m \boxtimes \Delta^n & \longrightarrow & \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \end{array}$$

Since applying $(-)^{\mathrm{op}}$ to $\Lambda^m_k \to \Delta^m$ gives $\Lambda^m_{m-k} \to \Delta^m$, the left hand inclusion is isomorphic to

$$\left(\Delta^m * \partial\Delta^n \cup \Lambda^m_{m-k} * \Delta^n \hookrightarrow \Delta^m * \Delta^n\right) = \left(\Lambda^{m+n+1}_{m-k} \hookrightarrow \Delta^{m+n+1}\right),$$

where the equality is due to [HCHA, Prop. 3.4.17]. Now $m - k < m + n + 1$ and $m - k > 0$ by our assumptions, so this is an inner horn inclusion. Thus the lift exists since $\mathcal{C}$ is an ∞-category. For the case of $\mathbf{S}_\diamond(\mathcal{C})$, all $*$'s are replaced with $\diamond$'s, so the strict equality above is false. However, since $\Lambda^m_{m-k} \hookrightarrow \Delta^m$ is a right anodyne extension, the map above is still a Joyal trivial cofibration by [HCHA, Lemma 5.6.11], hence $\mathcal{C} \to \Delta^0$ still lifts against it. □

If A is a simplicial set and a is an object of A, we define

$$_a\,\mathrm{S}(A) = (\{a\} \times A) \times_{A^{\mathrm{op}} \times A} \mathrm{S}(A) \qquad \mathrm{S}(A)_a = (A^{\mathrm{op}} \times \{a\}) \times_{A^{\mathrm{op}} \times A} \mathrm{S}(A)$$

and likewise for $_a\,\mathrm{S}_\diamond(A)$ and $\mathrm{S}_\diamond(A)_a$.

Proposition 1.2.50 *Let $\mathcal{C}$ be an ∞-category. For each object x of $\mathcal{C}$, there are canonical equivalences of left fibrations*

$$x\backslash\mathcal{C} \xrightarrow{\sim} {}_x\,\mathrm{S}(\mathcal{C}) \qquad x\backslash\backslash\mathcal{C} \xrightarrow{\sim} {}_x\,\mathrm{S}_\diamond(\mathcal{C})$$

over $\mathcal{C}$, and

$$(\mathcal{C}/x)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{S}(\mathcal{C})_x \qquad (\mathcal{C}//x)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{S}_\diamond(\mathcal{C})_x$$

over $\mathcal{C}^{\mathrm{op}}$. Consequently, the canonical map $\mathrm{S}(\mathcal{C}) \to \mathrm{S}_\diamond(\mathcal{C})$ is an equivalence of left fibrations over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$.

Proof By definition ${}_x\,\mathrm{S}(\mathcal{C})$ is defined by applying δ^* to the fiber product on the left below:

$$\begin{array}{ccc} {}_x\mathbf{S}(\mathcal{C}) & \longrightarrow & \mathbf{S}(\mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \Delta^0 \boxtimes \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathrm{op}} \boxtimes \mathcal{C} \end{array} \qquad \begin{array}{ccc} {}^{\Delta^0}({}_x\mathbf{S}(\mathcal{C})) & \longrightarrow & {}^{\Delta^0}(\mathbf{S}(\mathcal{C})) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C} & \xrightarrow{(x,\mathrm{id}_\mathcal{C})} & \mathrm{Ob}(\mathcal{C}) \times \mathcal{C} \end{array}$$

Therefore, we have the one on the right as well. The n-simplices of the upper-right simplicial set are $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^0 * \Delta^n, \mathcal{C})$, and the square asserts precisely

that the n-simplices of the pullback will be

$$\mathrm{Hom}_{\Delta^0\backslash\mathbf{sSet}}(\Delta^0 * \Delta^n, \mathcal{C}) = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, x\backslash\mathcal{C}) = (x\backslash\mathcal{C})_n.$$

Thus ${}^{\Delta^0}({}_x\mathbf{S}(\mathcal{C})) = x\backslash\mathcal{C}$. The same logic works for $x\backslash\backslash\mathcal{C}$ given its adjunction definition in terms of $\diamond$. Thus we obtain the asserted functorial equivalences by [HCHA, Lemma 5.5.20]. By functoriality, we have for each x a commutative diagram

$$\begin{array}{ccc} x\backslash\mathcal{C} & \xrightarrow{\simeq} & x\backslash\backslash\mathcal{C} \\ \downarrow\simeq & & \simeq\downarrow \\ {}_x\,\mathrm{S}(\mathcal{C}) & \longrightarrow & {}_x\,\mathrm{S}_\diamond(\mathcal{C}) \end{array}$$

where the upper map is an equivalence of ∞-categories over $\mathcal{C}$, and hence of left fibrations, by Cor. 1.1.16. Therefore passing to all fibers over y in $\mathcal{C}$ produces equivalences, in the lower map, so $\mathrm{S}(\mathcal{C}) \to \mathrm{S}_\diamond(\mathcal{C})$ is a fiberwise equivalence over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, hence an equivalence of left fibrations. □

According to the previous proposition, if $\mathcal{C}$ is an ∞-category, we have the canonical equivalence of left fibrations $x\backslash\mathcal{C} \simeq {}_x\,\mathrm{S}(\mathcal{C})$, and thus an equivalence $(x\backslash\mathcal{C})_y \simeq \mathrm{S}(\mathcal{C})_{(x,y)}$, whilst on the other hand $(x\backslash\mathcal{C})_y \simeq \mathcal{C}(x,y)$, so there is a canonical zig-zag of equivalences

$$\mathcal{C}(x,y) \leftarrow (x\backslash\mathcal{C})_y \to \mathrm{S}(\mathcal{C})_{(x,y)}. \qquad (*)$$

The situation is even better if we use $\mathrm{S}_\diamond(\mathcal{C})$, since in this case we just have the literal identification $\mathcal{C}(x,y) = (x\backslash\backslash\mathcal{C})_y$, and hence a canonical equivalence $\mathcal{C}(x,y) \to \mathrm{S}_\diamond(\mathcal{C})_{(x,y)}$. This is nice because by choosing one inverse equivalence to $\mathrm{S}(\mathcal{C}) \to \mathrm{S}_\diamond(\mathcal{C})$ over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, we can get an equivalence of both sides of $(*)$ which at least doesn't depend on x and y.

Example 1.2.51 Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. Then this induces a canonical morhpism

$$\mathrm{S}(\mathcal{C}) \to \mathrm{S}(\mathcal{D}) \underset{\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}{\times} (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \qquad (*)$$

of left fibrations over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, which is canonically equivalent to the corresponding version using $\mathcal{C}_\diamond$. Thus we may form a commutative diagram:

$$\begin{array}{ccccc} \mathcal{C}(x,y) & \longrightarrow & \mathrm{S}_\diamond(\mathcal{C}) & & \\ \downarrow & h & \downarrow & & \\ \mathcal{D}(Fx,Fy) & \longrightarrow & \mathrm{S}_\diamond(\mathcal{D}) \underset{\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}{\times} (\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) & \longrightarrow & \mathrm{S}_\diamond(\mathcal{D}) \\ \downarrow & h & \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow[(x,y)]{} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{F^{\mathrm{op}}\times F} & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \end{array}$$

Thus the natural map $\mathcal{C}(x,y) \to \mathcal{D}(Fx,Fy)$ is a homotopy pullback of the one above. In particular, we observe that this morphism detects all ∞-categorical

properties of this induced map; for instance, the functor F is fully faithful iff $(*)$ is an equivalence of left fibrations.

From what we have said, it is clear that the following conditions are equivalent for an ∞-category $\mathcal{C}$:

(i) $\mathcal{C}$ is locally $\mathbf{U}$-small.

(ii) The left fibration $\mathrm{S}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ has essentially $\mathbf{U}$-small fibers.

(iii) The left fibration $\mathrm{S}_{\diamond}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ has essentially $\mathbf{U}$-small fibers.

Therefore, if $\mathcal{C}$ is a locally $\mathbf{U}$-small ∞-category, the space $\mathrm{hClass}_{\mathbf{U}}(\mathrm{S}(\mathcal{C}))$ is a contractible ∞-groupoid by Prop. 1.2.30.

Definition 1.2.52 Let $\mathcal{C}$ be a locally $\mathbf{U}$-small ∞-category. A *choice of mapping space* or a *mapping space functor* for $\mathcal{C}$ is an object of the contractible ∞-groupoid $\mathrm{hClass}_{\mathbf{U}}\big(\mathrm{S}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}\big)$. In particular, it is the choice of functors

$$\mathrm{Map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S} \qquad \sigma_{\mathcal{C}} \colon \mathrm{S}(\mathcal{C}) \to \mathcal{S}_{\bullet}$$

making the diagram

$$\begin{array}{ccc} \mathrm{S}(\mathcal{C}) & \xrightarrow{\sigma_{\mathcal{C}}} & \mathcal{S}_{\bullet} \\ \downarrow & & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\mathrm{Map}_{\mathcal{C}}} & \mathcal{S} \end{array}$$

homotopy cartesian.

Remark 1.2.53 Here are the initial immediate remarks about this definition.

1. We saw previously that contractibility of $\mathrm{hClass}(\mathrm{S}(\mathcal{C}))$ has the following consequence: Given any two choices of mapping space functors for $\mathcal{C}$, the homotopy cartesian property induces a homotopy between the two choices of $\sigma_{\mathcal{C}}$ and $\mathrm{Map}_{\mathcal{C}}$. We also know separately that this map is unique in the homotopy category from 1.2.25.

2. Suppose that we instead make a choice of mapping space for the fat twisted diagonal first. Then the diagram

$$\begin{array}{ccccc} \mathrm{S}(\mathcal{C}) & \longrightarrow & \mathrm{S}_{\diamond}(\mathcal{C}) & \xrightarrow{\sigma_{\mathcal{C}}} & \mathcal{S}_{\bullet} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & = & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\mathrm{Map}_{\mathcal{C}}} & \mathcal{S} \end{array}$$

   is homotopy cartesian by the previous result, so we can always choose our mapping space to factor through $\mathrm{S}_{\diamond}(\mathcal{C})$. The advantage of doing this is that in this case we have a canonical homotopy cartesian diagram

$$\begin{array}{ccccc} \mathcal{C}(x,y) & \longrightarrow & \mathrm{S}_{\diamond}(\mathcal{C}) & \xrightarrow{\sigma_{\mathcal{C}}} & \mathcal{S}_{\bullet} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ \Delta^0 & \xrightarrow{(x,y)} & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{\mathrm{Map}_{\mathcal{C}}} & \mathcal{S} \end{array}$$

and hence a canonical equivalence $\mathcal{C}(x,y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(x,y)$. If we don't choose to factor through the fat twisted diagonal, then we instead have a canonical zig-zag

$$\mathcal{C}(x,y) \leftarrow (x\backslash\mathcal{C})_y \rightarrow \mathrm{Map}_{\mathcal{C}}(x,y)$$

of equivalences. In particular, if we apply the functor π_0, we do get a uniquely determined isomorphism $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(x,y)$.

3. Of course, the advantage of this construction is that $\mathrm{Map}_{\mathcal{C}}$ forms a (bi)functor to $\mathcal{S}$. In particular, any morphism $(x,y) \to (x',y')$ gives a morphism $\mathrm{Map}_{\mathcal{C}}(x',y) \to \mathrm{Map}_{\mathcal{C}}(x,y')$ in $\mathcal{S}$, and under the equivalence

$$\mathcal{S}\big(\mathrm{Map}_{\mathcal{C}}(x',y), \mathrm{Map}_{\mathcal{C}}(x,y')\big) \simeq \underline{\mathrm{Hom}}\big(\mathrm{Map}_{\mathcal{C}}(x',y), \mathrm{Map}_{\mathcal{C}}(x,y')\big)$$

corresponds to a functor $\mathrm{Map}_{\mathcal{C}}(x',y) \to \mathrm{Map}_{\mathcal{C}}(x,y')$ of ∞-groupoids. Recall that since we have fixed a universal choice of the equivalences above (depending only on the universe $\mathbf{U}$), the map actually gives such a functor. It also gives a map $\mathcal{C}(x',y) \to \mathcal{C}(x,y')$ which is well-defined up to homotopy. For instance, since we know that equivalences correspond to equivalences, we deduce that if $(x,y) \to (x',y')$ is an equivalence of pairs in $\mathcal{C} \times \mathcal{C}$, then the induced map

$$\mathrm{Map}_{\mathcal{C}}(x',y) \to \mathrm{Map}_{\mathcal{C}}(x,y')$$

is an equivalence in $\mathcal{S}$, or an equivalence of ∞-groupoids when taking the corresponding functor. It is clear from (2) that applying π_0 to this produces the induced map $\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x',y) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y')$. This is an expression of the Yoneda lemma.

4. This is a continuation of Example 1.2.51. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and suppose we have chosen mapping spaces $\mathrm{Map}_{\mathcal{C}}$ and $\mathrm{Map}_{\mathcal{D}}$. Then consider the pullback $(F^{\mathrm{op}} \times F)^* \mathrm{Map}_{\mathcal{D}}$ which on objects is $\mathrm{Map}_{\mathcal{D}}(Fx, Fy)$. Then we have a chain of equivalences

$$\begin{array}{c} \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})(\mathrm{Map}_{\mathcal{C}}, (F^{\mathrm{op}} \times F)^* \mathrm{Map}_{\mathcal{D}}) \\ \downarrow \simeq \\ \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}(p^*_{\mathrm{univ}} \mathrm{Map}_{\mathcal{C}}, p^*_{\mathrm{univ}} (F^{\mathrm{op}} \times F)^* \mathrm{Map}_{\mathcal{D}}) \\ \downarrow \simeq \\ \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}(\mathrm{S}(\mathcal{C}), p^*_{\mathrm{univ}} (F^{\mathrm{op}} \times F)^* \mathrm{Map}_{\mathcal{D}}) \end{array}$$

by Cor. 1.2.26 and then the structural equivalence $\mathrm{S}(\mathcal{C}) \to p^*_{\mathrm{univ}} \mathrm{Map}_{\mathcal{C}}$. On the other hand, we have the canonical map

$$\mathrm{S}(\mathcal{C}) \to (F^{\mathrm{op}} \times F)^* \mathrm{S}(\mathcal{D}) \to p^*_{\mathrm{univ}} (F^{\mathrm{op}} \times F)^* \mathrm{Map}_{\mathcal{D}}$$

from Example 1.2.51 followed by the structural map. The homotopy fiber of this object under the above equivalence gives a unique morphism

$$\mathrm{Map}_{\mathcal{C}} \to (F^{\mathrm{op}} \times F)^* \mathrm{Map}_{\mathcal{D}} \qquad \text{in } \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$$

up to contractible ∞-groupoid. For instance, we see that this is invertible iff $\mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{D}}(Fx,Fy)$ is invertible for all x,y, iff the map $\mathrm{S}(\mathcal{C}) \to (F^{\mathrm{op}} \times F^*)\,\mathrm{S}(\mathcal{D})$ is an equivalence, iff F is fully faithful. More specifically, using the canoncial zig-zag identifying the fibers of $\mathrm{Map}_{\mathcal{C}}$ with $\mathcal{C}(x,y)$, the same zig-zag for $\mathcal{D}$ provides equivalences between the fiberwise map on $\mathcal{C}(x,y)$. In other words, we have a strictly commutative diagram

$$\begin{array}{ccc} \mathcal{C}(x,y) & \xrightarrow{F} & \mathcal{D}(Fx,Fy) \\ \simeq\uparrow & & \uparrow\simeq \\ (x\backslash\mathcal{C})_y & \xrightarrow{F} & (Fx\backslash\mathcal{D})_{Fy} \\ \simeq\downarrow & & \downarrow\simeq \\ \mathrm{Map}_{\mathcal{C}}(x,y) & \dashrightarrow & \mathrm{Map}_{\mathcal{D}}(Fx,Fy) \end{array}$$

in which all maps are canonical once the mapping spaces are chosen except for the lower map, which is induced by the choice of natural transformation above. However, the morphism in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{S})$ is independent of the choice of object. As previously explained, we can remove the middle step by using $\mathrm{S}_\diamond$, but this doesn't change the literal non-uniqueness of the corresponding lower map of ∞-groupoids.

The composite functor $\mathrm{h}\mathcal{S} \xrightarrow{\sim} \mathcal{H}_{\mathbf{U}} \xrightarrow{\pi_0} \mathbf{Set}_{\mathbf{U}}$ induces a functor $\pi_0 \colon \mathcal{S} \to \mathbf{Set}_{\mathbf{U}}$, where we identify $\mathbf{Set}_{\mathbf{U}} \equiv N(\mathbf{Set}_{\mathbf{U}})$. We explained the canonical isomorphism $\mathrm{Hom}_{\mathrm{h}\mathbb{C}}(x,y) \xrightarrow{\sim} \pi_0(\mathrm{Map}_{\mathcal{C}}(x,y))$.

In the remainder of this section, we will prove one more result in preparation for the Yoneda lemma.

Lemma 1.2.54 *Let $\mathcal{C},\mathcal{D},\mathcal{E}$ be ∞-categories, and consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathcal{C}\times\mathcal{D} & \xrightarrow[1\times g]{} & \mathcal{C}\times\mathcal{E} \end{array}$$

in which p and q are left fibrations. If for each $a \in \mathcal{C}$, the fiberwise map $X_a \to Y_a$ is cofinal, then $X \to Y$ is cofinal.

Proof Since q is a left fibration, we may show that f is a weak equivalence in $(\mathbf{sSet}/\mathcal{C}\times\mathcal{E})_{\mathrm{cov}}$, which is equivalent to showing that

$$\mathbf{R}(c,e)^*X \to \mathbf{R}(c,e)^*Y \qquad \in \mathrm{LFib}(\mathcal{C}\times\mathcal{E})$$

is an isomorphism for all c,e, by Lemma 1.1.37. This map is $\mathbf{R}c^*$ applied to $\mathbf{R}(\mathcal{C}\times e)^*X \to \mathbf{R}(\mathcal{C}\times e)^*Y$, and moreover since $e \to \mathcal{E}/e$ is a (right anodyne) fibrant replacement for e over in $(\mathbf{sSet}/\mathcal{E})_{\mathrm{con}}$, it follows that $\mathcal{C}\times e \to \mathcal{C}\times\mathcal{E}/e$ is one for $\mathcal{C}\times e$ over $\mathcal{C}\times\mathcal{E}$. Therefore we have to show that the top map in the

diagram

$$\begin{array}{ccc} X \underset{\mathcal{C}\times\mathcal{E}}{\times} (\mathcal{C}\times\mathcal{E}/e) & \xrightarrow{f} & Y \underset{\mathcal{C}\times\mathcal{E}}{\times} (\mathcal{C}\times\mathcal{E}/e) \\ \| & & \| \\ X/e & \xrightarrow{f/e} & Y/e \\ \downarrow & & \downarrow \\ \mathcal{C}\times\mathcal{D}/e & \longrightarrow & \mathcal{C}\times\mathcal{E}/e \end{array}$$

is a weak equivalence in $(\mathbf{sSet}/\mathcal{C})_{\mathrm{cov}}$ for all e. The fibers over c are $X_c/e \to Y_c/e$, and the functor $(-)/e = (-)\times_{\mathcal{E}} \mathcal{E}/e$ preserves cofinality since $\mathcal{E}/e \to \mathcal{E}$ is a right fibration, hence smooth. Thus the upper map is still fiberwise cofinal over c, and the vertical maps are still left fibrations, hence proper. Therefore by composing with the projections to $\mathcal{C}$, we see that f/e is a morphism of proper maps over $\mathcal{C}$ which is fiberwise cofinal, hence a fiberwise weak homotopy equivalence. This proves that f/e is a covariant weak equivalence over $\mathcal{C}$ by [HCHA, Cor. 4.4.28]. □

If $\mathcal{C}$ is an ∞-category, then we can factor the morphism of Example 1.2.51 through intermediate steps, either via $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ or $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$. In particular we have maps as in the following proposition:

Proposition 1.2.55 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories. The maps*

$$\mathrm{S}(\mathcal{C}) \to (\mathcal{C}^{\mathrm{op}} \times \mathcal{D}) \underset{\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}{\times} \mathrm{S}(\mathcal{D}) \qquad \mathrm{S}(\mathcal{C}) \to (\mathcal{D}^{\mathrm{op}} \times \mathcal{C}) \underset{\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}{\times} \mathrm{S}(\mathcal{D})$$

are both cofinal and fiberwise cofinal over $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{C}$ respectively.

Proof Here is the proof of the first one. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{S}(\mathcal{C}) & \longrightarrow & (\mathcal{C}^{\mathrm{op}} \times \mathcal{D}) \underset{\mathcal{D}^{\mathrm{op}}\times\mathcal{D}}{\times} \mathrm{S}(\mathcal{D}) & \longrightarrow & \mathrm{S}(\mathcal{D}) \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \longrightarrow & \mathcal{C}^{\mathrm{op}} \times \mathcal{D} & \longrightarrow & \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \end{array}$$

where the right-hand square is cartesian, so the middle vertical map is a left fibration. By the previous lemma, it suffices to show that for all x, the lower horizontal map in the square

$$\begin{array}{ccc} x\backslash\mathcal{C} & \longrightarrow & F(x)\backslash\mathcal{D} \\ \downarrow\simeq & & \simeq\downarrow \\ {}_{x}\mathrm{S}(\mathcal{C}) & \longrightarrow & {}_{F(x)}\mathrm{S}(\mathcal{D}) \end{array}$$

is cofinal. Indeed, the vertical maps are both equivalences of left fibrations over the respective ∞-categories by Prop. 1.2.50, hence cofinal. The upper horizontal map is cofinal because both source and target have initial objects 1_x and $1_{F(x)}$, and the map sends one to the other. Thus the remaining map is cofinal by the cancellation property of cofinality (Remark 1.1.19). □

1.2.4 The Yoneda lemma

In line with thinking of ∞-categories as akin to categories enriched in spaces (i.e. $\mathcal{S}$), we should expect them to embed into $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and admit a Yoneda lemma. If $\mathcal{C}$ is a locally $\mathbf{U}$-small ∞-category with a chosen mapping space functor $\mathrm{Map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$, then by adjunction we obtain an induced functor

$$\text{よ}_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}),$$

called the *Yoneda embedding*. By definition, there are canonical equivalences

$$x\backslash\mathcal{C} \xrightarrow{\sim}{}_{x}\,\mathrm{S}(\mathcal{C}) \xrightarrow{\sim} \text{よ}(x),$$

or more precisely $\text{よ}(x) \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ classifies up to equivalence the left fibrations above. In other words, we have a canonical homotopy cartesian square

$$\begin{array}{ccc} (\mathcal{C}/x)^{\mathrm{op}} & \longrightarrow & \mathcal{S}_{\bullet} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\mathrm{op}} & \xrightarrow{\text{よ}(x)} & \mathcal{S} \end{array}$$

classifying the right fibration $\mathcal{C}/x \to \mathcal{C}$. For example, we should already expect this functor to be fully faithful, because we have abstract equivalences

$$\mathcal{C}(x, y) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}/x, \mathcal{C}/x) \simeq \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})(\text{よ}(x), \text{よ}(y)).$$

However, both intermediate steps here were defined somewhat abstractly, and it is not clear that this map is also the induced map by $\text{よ}_{\mathcal{C}}$. Thus, much of this section will be about checking functoriality/naturality of the constructions above. As an example, we will explicitly check such a compatibility from the previous section.

Lemma 1.2.56 *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories with mapping spaces $\mathrm{Map}_{\mathcal{C}}$ and $\mathrm{Map}_{\mathcal{D}}$.*

1. *For all objects x and morphisms $u \colon y \to y'$ in $\mathcal{C}$, the induced diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y) & \xrightarrow{u_*} & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y') \\ \cong\downarrow & & \downarrow\cong \\ \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y) & \xrightarrow{\pi_0(u_*)} & \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y') \end{array}$$

   is commutative.

2. *For all objects x, y of $\mathcal{C}$ and functors $F \colon \mathcal{C} \to \mathcal{D}$ of ∞-categories, the induced diagram*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x, y) & \xrightarrow{F} & \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(Fx, Fy) \\ \cong\downarrow & & \downarrow\cong \\ \pi_0 \mathrm{Map}_{\mathcal{C}}(x, y) & \xrightarrow{\pi_0 F} & \pi_0 \mathrm{Map}_{\mathcal{D}}(Fx, Fy) \end{array}$$

   is commutative.

Proof (2) follows from part (4) of Remark 1.2.53, so let's show (1). Let $E \to \mathcal{C}$ be the left fibration classified by $\mathrm{Map}_{\mathcal{C}}(x, -) \colon \mathcal{C} \to \mathcal{S}$, then we know that there is a canonical equivalence $\psi \colon x\backslash\mathcal{C} \to E$ of left fibrations over $\mathcal{C}$. Form W as the successive pullback as in the diagram below:

$$\begin{array}{ccccccc} 0 \times E_y & \longrightarrow & W & \longrightarrow & E & \longrightarrow & \mathcal{S}_\bullet \\ \downarrow & \overset{\varphi}{\nearrow} & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^1 \times E_y & \longrightarrow & \Delta^1 & \xrightarrow[u]{} & \mathcal{C} & \xrightarrow[\mathrm{Map}_{\mathcal{C}}(x,-)]{} & \mathcal{S} \end{array}$$

Then to compute the induced map, we choose a lift φ and take the 1-fiber $\varphi_1 \colon E_y \to W_1 = E_{y'}$; what we have to show is that in the following diagram

$$\begin{array}{ccc} (x\backslash\mathcal{C})_y & \xrightarrow{\simeq} & E_y \\ \downarrow & & \downarrow{\scriptstyle \varphi_1} \\ (x\backslash\mathcal{C})_{y'} & \xrightarrow{\simeq} & E_{y'} \end{array}$$

a morphism $f \colon x \to y$ in $\mathcal{C}$ corresponds after applying φ_1 to a choice of composite uf. Indeed, choose a 2-simplex $\sigma \colon \Delta^2 \to \mathcal{C}$ witnessing uf as the composite, which is equivalently a morphism σ in $x\backslash\mathcal{C}$ from f to uf. Its projection to $\mathcal{C}$ is u, hence $\psi(\sigma)$ is a morphism in E lying over u, so this defines a morphism $\gamma \colon \Delta^1 \to W$ over Δ^1 from $\psi(f) \in E_y$ to $\psi(uf) \in E_{y'}$.

In particular, this defines a morphism $0 \times E_y \cup \Delta^1 \times \{\psi(f)\} \to W$, and since

$$0 \times E_y \cup \Delta^1 \times \{\psi(f)\} \hookrightarrow \Delta^1 \times E_y$$

is left anodyne, we may choose a lift $\varphi \colon \Delta^1 \times E_y$ such that $\varphi|_{\Delta^1 \times \{\psi(f)\}} = \gamma$. In particular, $\varphi_1(\psi(f)) = \gamma(1) = \psi(uf)$, which is what we wanted to show. □

In particular, given $\mathrm{Map}_{\mathcal{C}}$, the diagram

$$\begin{array}{ccc} \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{C} & \xrightarrow{\mathrm{Map}_{\mathcal{C}}} & \mathrm{h}\mathcal{S} \\ & \searrow^{\mathrm{Hom}_{\mathrm{h}\mathcal{C}}} & \downarrow{\scriptstyle \pi_0} \\ & & \mathbf{Set} \end{array}$$

commutes up to a specified natural isomorphism.

Lemma 1.2.57 *Let A be a simplicial set and $\Phi \colon A^{\mathrm{op}} \to \mathcal{S}$ be a map classifying a right fibration $q \colon V \to A$ with $\mathbf{U}$-small fibers. If $i \colon U \to V$ is a right anodyne extension, then the ∞-category*

$$W \coloneqq \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \underset{\underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S})}{\times} \underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}_\bullet)$$

has as an initial object $(\Phi, U^{\mathrm{op}} \to V^{\mathrm{op}} \to \mathcal{S}_\bullet)$.

The simplicial set W in the lemma above is an ∞-category of mapping spaces out of U^{op} over A^{op}, parameterized by classifying maps. To see this, consider

the projection $W \to \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$, which is a left fibration as it is the pullback of $\underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}) \to \underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}_\bullet)$. If $F\colon A^{\mathrm{op}} \to \mathcal{S}$ is a map classifying a left fibration $X \to A^{\mathrm{op}}$, then we compute the fiber W_F as follows:

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X) & \longrightarrow & \underline{\mathrm{Hom}}(U^{\mathrm{op}}, X) & \longrightarrow & \underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}_\bullet) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^0 & \xrightarrow{qi} & \underline{\mathrm{Hom}}(U^{\mathrm{op}}, A^{\mathrm{op}}) & \xrightarrow{F_*} & \underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}) \end{array}$$

The left square is the definition of $\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)$, and the right square is the definition of X. On the other hand the outer square defines the fiber W_F, showing $W_F = \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)$.

Proof (of Lemma 1.2.57) Under the identification $W_F = \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)$ explained above, the object described in the statement is $i^{\mathrm{op}} \in W_\Phi$, which we write as $w_0 \in W$. We will show this is initial by giving two alternative constructions of a certain morphism and proving that they agree. First of all, fix $F\colon A^{\mathrm{op}} \to \mathcal{S}$ classifying $X \to A^{\mathrm{op}}$. Then we have a composite

$$\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})(\Phi, F) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(V^{\mathrm{op}}, X) \xrightarrow{i^*} \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X) \qquad (*)$$

where the first map is the equivalence of Cor. 1.2.26. On the other hand, since $W \to \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ is a left fibration, the induced map $w_0 \backslash W \to \Phi \backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ is a trivial fibration (by Lemma 1.1.22), so we may choose section and compose to obtain

$$\Phi \backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \dashrightarrow w_0 \backslash W \to W,$$

viewed as a morphism of left fibrations over $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$. Pulling back (i.e. taking the fiberwise map over F) provides a diagram

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})(\Phi, F) & \xleftarrow{\simeq} & \left(\Phi \backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})\right)_F & \longrightarrow & \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X) \\ & & \downarrow & \lrcorner & \downarrow \\ & & \Phi \backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) & \longrightarrow & W \end{array}$$

and by choosing an inverse we obtain a composite $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})(\Phi, F) \to \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)$, called $(**)$.

Claim: The maps $(*)$ and $(**)$ are equal in $\mathcal{H}_{\mathrm{Joy}}$.

Assuming this claim, here's why it completes the proof: It implies that the homotopy fibers of both maps over an object $s\colon U^{\mathrm{op}} \to X$ are weakly eqiuvalent in $\mathbf{sSet}_{\mathrm{Joy}}$. Using $(*)$, we obtain

$$\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})(\Phi, F) \underset{\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)}{\times^{\mathbf{R}}} \{s\} \xrightarrow{\sim} \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(V^{\mathrm{op}}, X) \underset{\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)}{\times^{\mathbf{R}}} \{s\}.$$

But note that $\underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(V^{\mathrm{op}}, X) \to \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X)$ is a trivial fibration by Lemma 1.1.3(b): Therefore we have both that the RHS is computed by the underived pullback, and that it is contractible.

On the other hand, consider the homotopy fiber at s of $(**)$; by definition, we may instead work with the homotopy fiber of the upper composite in the cartesian diagram:

$$\begin{array}{ccccc} \left(\Phi\backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})\right)_F & \longrightarrow & (w_0\backslash W)_F & \longrightarrow & \underline{\mathrm{Hom}}_{A^{\mathrm{op}}}(U^{\mathrm{op}}, X) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Phi\backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) & \overset{\simeq}{\dashrightarrow} & w_0\backslash W & \longrightarrow & W \end{array}$$

Since $w_0\backslash W \to W \to \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ is a composite of left fibrations, the dashed arow is an equivalence of left fibrations over $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$, hence a fiberwise equivalence. In particular the left-hand upper horizontal map is an equivalence, so we may compute the homotopy fiber of the upper map in the right-hand square over s. But this map is a left fibration, so this coincides with its usual pullback, namely $(w_0\backslash W)_{(F,s)}$, and we conclude because we know this space is equivalent to $W(w_0, (F, s))$. The previous paragraph shows that this is contractible, and since (F, s) was arbitrary this means w_0 is initial.

Proof (of claim): TODO: Fill in this identification using Constr. 1.2.13. □

Corollary 1.2.58 *Let $f\colon U \to A$ factor as a right anodyne extension $U \to V$ followed by a right fibration $q\colon V \to A$ with* **U***-small fibers, and let $\Phi\colon A^{\mathrm{op}} \to \mathcal{S}$ classify q. There is a homotopy cartesian square*

$$\begin{array}{ccc} \Phi\backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) & \dashrightarrow & \underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}_\bullet) \\ \downarrow & & \downarrow{\scriptstyle p_{\mathrm{univ},*}} \\ \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) & \xrightarrow{f^*} & \underline{\mathrm{Hom}}(U^{\mathrm{op}}, \mathcal{S}) \end{array}$$

where the upper horizontal map is canonical up to a contractible ∞-groupoid.

Proof Let W be as in Lemma 1.2.57 and w the initial object mapping to Φ. This time, both maps in

$$\Phi\backslash \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}) \leftarrow w\backslash W \to W$$

are trivial fibrations, by Thm. 1.1.27, so by choosing a section of the first map and pasting it with the square defining W, we obtain the claimed homotopy cartesian square. The final statement is simply because the space of sections of a trivial fibration is contractible. □

Example 1.2.59 Consider the case where $U = V = A = \Delta^0$. Choose an object e of $\mathcal{S}$ classifying Δ^0, which is clearly a terminal object by Cor. 1.2.26. Then the corollary above supplies an equivalence of ∞-categories $e\backslash\mathcal{S} \to \mathcal{S}_\bullet$, which makes sense as $\mathcal{S}_\bullet$ is supposed to be the pointed version of $\mathcal{S}$. If we choose $\mathrm{Map}_{\mathcal{S}}$, we saw that $e\backslash\mathcal{S}$ is (homotopy) classified by $\mathrm{Map}_{\mathcal{S}}(e, -)$, and since $\mathcal{S}_\bullet \to \mathcal{S}$ is classified by the identity map, this lifts to an equivalence $\mathrm{Map}_{\mathcal{S}}(e, -) \xrightarrow{\sim} \mathrm{id}_{\mathcal{S}}$ in $\underline{\mathrm{Hom}}(\mathcal{S}, \mathcal{S})$. We can upgrade this as follows: if A is a **U**-small simplicial set,

then we obtain an equivalence

$$e\backslash \underline{\mathrm{Hom}}(A, \mathcal{S}) \leftarrow \underline{\mathrm{Hom}}(A, e\backslash \mathcal{S}) \xrightarrow{\sim} \underline{\mathrm{Hom}}(A, \mathcal{S}_\bullet)$$

where the left-hand map is from Prop. 1.1.18.

Our next result is an intermediate version of the Yoneda lemma, which will be functorial in the presheaf but not in the object. The statement will look very similar to the classical Yoneda lemma.

Construction 1.2.60 Let $\mathcal{C}$ be an essentially **U**-small ∞-category with an object a. Fix mapping space functors for $\mathcal{C}, \mathcal{S}$, and $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, which are all locally small by assumption. The evaluation functor $a^*\colon \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ induces a (canonical up to contractible space) map on Hom-spaces, i.e. a morphism in the ∞-category $\mathcal{S}$

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(F', F) \to \mathrm{Map}_{\mathcal{S}}(F'(a), F(a))$$

which is functorial in F' and F, but possibly not in a. On the other hand, the map $a\colon \Delta^0 \to \mathcal{C}$ induces a morphism

$$e = \mathrm{Map}_{\Delta^0}(0, 0) \to \mathrm{Map}_{\mathcal{C}}(a, a)$$

in $\mathcal{S}$, where the LHS is a terminal object of $\mathcal{S}$. Now taking $F' = \text{よ}_{\mathcal{C}}$, and combining the induced map above with the natural equivalence of Example 1.2.59 we obtain a map

$$\begin{aligned}\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{よ}_{\mathcal{C}}(a), -) &\to \mathrm{Map}_{\mathcal{S}}(\mathrm{Map}_{\mathcal{C}}(a, a), a^*(-)) \\ &\to \mathrm{Map}_{\mathcal{S}}(e, a^*(-)) \xrightarrow{\sim} a^*\end{aligned}$$

in $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. In particular, for each $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ we obtain a Yoneda map

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{よ}_{\mathcal{C}}(a), F) \to F(a)$$

in $\mathcal{S}$ which is natural in F.

Theorem 1.2.61 *Let $\mathcal{C}$ be an essentially* **U***-small ∞-category with an object a. The morphism of Constr. 1.2.60*

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{よ}_{\mathcal{C}}(a), F) \to F(a)$$

in $\mathcal{S}$ is an equivalence for each $F\colon A^{\mathrm{op}} \to \mathcal{S}$.

Proof Let the two maps

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{よ}_{\mathcal{C}}(a), -), \quad a^*\colon \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$$

classify left fibrations V and $W \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ respectively. The Yoneda morphism (before choosing F) induces a morphism $V \to W$ of left fibrations, canonical up to homotopy, and it suffices to show that this is an equivalence (by Thm. 1.2.25). The choice of mapping space for $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ comes with an equivalence $\text{よ}_{\mathcal{C}}(a)\backslash \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\sim} V$, so we may prove that

$$\text{よ}_{\mathcal{C}}(a)\backslash \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to V \to W \qquad (*)$$

is an equivalence of left fibrations.

Now write $E \to \mathcal{C}^{\mathrm{op}}$ for the left fibration classified by $\text{ɟ}(a)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, which comes with an equivalence $\psi\colon (\mathcal{C}/a)^{\mathrm{op}} \xrightarrow{\sim} E$ over $\mathcal{C}^{\mathrm{op}}$. Thus $\psi(1_a) \in E^{\mathrm{op}}$ is a terminal object, so in the factorization

$$\left(\Delta^0 \xrightarrow{1_a} \mathcal{C}/a \xrightarrow{\psi} E^{\mathrm{op}}\right) \to \mathcal{C},$$

the first map is a right anodyne extension, and $E^{\mathrm{op}} \to \mathcal{C}$ is a right fibration. Thus we may apply Lemma 1.2.57: the composite above is $a\colon \Delta^0 \to \mathcal{C}$, so the W from that lemma conicides with the W defined here. The lemma says that W has the initial object $(\text{ɟ}(a), \psi(1_a))$, or more precisely the fiber $W_{\text{ɟ}(a)} = \text{ɟ}(a)(a) = \mathrm{Map}_{\mathcal{C}}(a,a)$ contains the initial object corresponding to the chosen $e \to \mathrm{Map}_{\mathcal{C}}(a,a)$.

Therefore, to show that $(*)$ is cofinal, we may show that it maps $1_{\text{ɟ}(a)}$ to the initial object of W described above. These both lie in the fiber over $\text{ɟ}(a)$ in $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, so we have to consider the induced map

$$\left(\text{ɟ}_{\mathcal{C}}(a) \backslash \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})\right)_{\text{ɟ}(a)} \to \mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɟ}(a), \text{ɟ}(a)) \to \mathrm{Map}_{\mathcal{C}}(a,a).$$

This is an exercise in unwinding definitions and applying Lemma 1.2.56; namely, apply π_0 to both sides, then the first step of the Yoneda construction is applying a^* to the mapping spaces:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{h}\,\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɟ}(a), \text{ɟ}(a)) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{S}}(\mathrm{Map}_{\mathcal{C}}(a,a), \mathrm{Map}_{\mathcal{C}}(a,a)) \\
\downarrow\wr & & \downarrow\wr \\
\pi_0\,\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɟ}(a), \text{ɟ}(a)) & \longrightarrow & \pi_0\,\mathrm{Map}_{\mathcal{S}}(\mathrm{Map}_{\mathcal{C}}(a,a), \mathrm{Map}_{\mathcal{C}}(a,a))
\end{array}$$

On the upper horizontal, $1_{\text{ɟ}(a)}$ maps to $1_{\mathrm{Map}_{\mathcal{C}}(a,a)}$ because a^* is a functor. The next step is to use the morphism $e \to \mathrm{Map}_{\mathcal{C}}(a,a)$, producing

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{S}}(\mathrm{Map}_{\mathcal{C}}(a,a), \mathrm{Map}_{\mathcal{C}}(a,a)) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{S}}(e, \mathrm{Map}_{\mathcal{C}}(a,a)) \\
\downarrow\simeq & & \downarrow\simeq \\
\pi_0\,\mathrm{Map}_{\mathcal{S}}(\mathrm{Map}_{\mathcal{C}}(a,a), \mathrm{Map}_{\mathcal{C}}(a,a)) & \longrightarrow & \pi_0\,\mathrm{Map}_{\mathcal{S}}(e, \mathrm{Map}_{\mathcal{C}}(a,a))
\end{array}$$

which on top is precomposition by $e \to \mathrm{Map}_{\mathcal{C}}(a,a)$. So the identity is sent precisely to this morphism. The last step is to use $\mathrm{Map}_{\mathcal{S}}(e, -) \xrightarrow{\sim} \mathrm{id}_{\mathcal{S}}$, producing:

$$\begin{array}{ccc}
\underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{S}}(e, \mathrm{Map}_{\mathcal{C}}(a,a)) & \dashrightarrow & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(a,a) \\
\downarrow\simeq & & \downarrow\simeq \\
\pi_0\,\mathrm{Map}_{\mathcal{S}}(e, \mathrm{Map}_{\mathcal{C}}(a,a)) & \xrightarrow{\simeq} & \pi_0\,\mathrm{Map}_{\mathcal{C}}(a,a)
\end{array}$$

Regardless of the upper map, by definition the initial object of W is the one corresponding to $e \to \mathrm{Map}_{\mathcal{C}}(a,a)$, so this morphism viewed in the upper left corner maps to the component of the initial object in $\pi_0\,\mathrm{Map}_{\mathcal{C}}(a,a)$ (this is also shows it maps to $1_a \in \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(a,a)$). This proves that the composite $(*)$

maps the initial object $1_{\gamma(a)}$ to an object equivalent to an initial object of W, hence an initial object. Thus $(*)$ is an equivalence of left fibrations. □

The last goal will be to prove the version of the Yoneda lemma which is functorial in both variables. Until the end of the proof, we establish the following notation: Let $\mathcal{C}$ be a locally **U**-small ∞-category, write $W \to \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ for the left fibration classified by

$$\mathrm{ev}\colon \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}.$$

Therefore we have a diagram

$$\begin{array}{ccccc} \mathrm{S}(\mathcal{C}) & \xrightarrow{w} & W & \longrightarrow & \mathcal{S}_{\bullet} \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{1 \times \text{ɟ}_{\mathcal{C}}} & \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) & \longrightarrow & \mathcal{S} \end{array}$$

factoring $\mathrm{Map}_{\mathcal{C}}$ through evaluation via the Yoneda embedding. On the other hand, we define $V \to \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ by the pullback:

$$\begin{array}{ccccc} \mathrm{S}(\mathcal{C}) & \xrightarrow{v} & V & \longrightarrow & \mathrm{S}(\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} & \xrightarrow{1 \times \text{ɟ}_{\mathcal{C}}} & \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \end{array}$$

In particular, for any choice of mapping space $\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}$ of the presheaf category, the left fibrant object V is a homotopy classifier for the functor $\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɟ}_{\mathcal{C}}(-), -)$. Therefore, the functorial Yoneda lemma consists of finding an equivalence $V \simeq W$ of left fibrations over $\mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

Lemma 1.2.62 *Let $\mathcal{C}$ be a locally* **U***-small ∞-category. The maps*

$$w\colon \mathrm{S}(\mathcal{C}) \to W \qquad v\colon \mathrm{S}(\mathcal{C}) \to V$$

constructed above are both cofinal.

Proof The case of v is Prop. 1.2.55 applied to the Yoneda embedding $\text{ɟ}\colon \mathcal{C} \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, which also says that v is fiberwise cofinal over $\mathcal{C}^{\mathrm{op}}$. This means that for each a, the map ${}_a\mathrm{S}(\mathcal{C}) \to V_a = {}_{\text{ɟ}(a)}\mathrm{S}(\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}))$, is cofinal, or by naturality the map $a\backslash\mathcal{C} \to \text{ɟ}(a)\backslash\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is cofinal. On the other hand, we know from Cor. 1.2.58 (or the previous proof) that there's an equivalence

$$\text{ɟ}(a)\backslash\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to W_a \tag{$*$}$$

of left fibrations over $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ sending $1_{\text{ɟ}(a)}$ to the initial object $(\text{ɟ}(a), 1_a)$ of W_a. We claim that the diagram

$$\begin{array}{ccc} a\backslash\mathcal{C} & \longrightarrow & \text{ɟ}(a)\backslash\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \\ \downarrow\simeq & & \downarrow\simeq \\ {}_a\mathrm{S}(\mathcal{C}) & \xrightarrow{w_a} & W_a \end{array}$$

commutes in $\mathrm{LFib}(\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}))$. This will complete the proof: the upper map is cofinal, and the vertical maps are both equivalences of left fibrations (over something, hence cofinal). Thus this will show that the lower map is an isomorphism, hence cofinal because W_a is left fibrant over $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. By Lemma 1.2.54 we will deduce that w is cofinal.

To show homotopy commutativity, we observe that both composites $a\backslash\mathcal{C} \to W_a$ are possible lifts in the commutative square below:

$$\begin{array}{ccc} \{1_a\} & \longrightarrow & W_a \\ \downarrow & \nearrow & \downarrow \\ a\backslash\mathcal{C} & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \end{array}$$

Here the upper map has image $\{(\text{ɬ}(a), 1_a)\}$; the lower map is defined using $a\backslash\mathcal{C} \to \mathrm{S}(\mathcal{C}) \to W_a \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$, and it's commutative because both directions map to $\text{ɬ}(a)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. Thus one composite is tautological, while the other one holds because the chosen equivalence $(*)$ sends $1_{\text{ɬ}(a)} \mapsto (\text{ɬ}(a), 1_a)$, and $1_a \mapsto \text{ɬ}(a)$. Since $1_a \in a\backslash\mathcal{C}$ is initial, the left-hand map in the square is left anodyne, so in $\mathrm{LFib}(\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}))$ it becomes an isomorphisms. Thus the two composite directions must be equal in that category. □

Theorem 1.2.63 *Let $\mathcal{C}$ be a locally $\mathbf{U}$-small category, $\mathbf{V} \supset \mathbf{U}$ a universe such that $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is locally $\mathbf{V}$-small where $\mathcal{S} = \mathcal{S}_{\mathbf{U}}$. Let*

$$\mathrm{Map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}, \mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}\colon \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{\mathbf{V}}$$

be choices of mapping spaces for $\mathcal{C}$ and $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

1. *The Yoneda embedding $\text{ɬ}_{\mathcal{C}}\colon \mathcal{C} \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is fully faithful, so in particular any choice of induced map*

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɬ}_{\mathcal{C}}(x), \text{ɬ}_{\mathcal{C}}(y)) \qquad (*)$$

   is an equivalence of ∞-groupoids for all objects x and y of $\mathcal{C}$.

2. *There is a natural equivalence between the functors*

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɬ}_{\mathcal{C}}(-), -) \xrightarrow{\sim} \mathrm{ev}\colon \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}_{\mathbf{V}},$$

   i.e. for each x in $\mathcal{C}$ and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, there is an equivalence

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɬ}_{\mathcal{C}}(x), F) \xrightarrow{\sim} F(x)$$

   in $\mathcal{S}_{\mathbf{V}}$ natural in F and x. This equivalence is characterized up to a canonical contractible ∞-groupoid by the property that for $F = \text{ɬ}_{\mathcal{C}}(y)$, it provides an inverse to $(*)$.

Proof Let $\mathcal{D} = \mathcal{C}^{\mathrm{op}} \times \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. We know that $W \to \mathcal{D}$ is a left fibration with $\mathbf{U}$-small fibers, but $V \to \mathcal{D}$ has only essentially $\mathbf{V}$-small fibers, so we let $V' \to \mathcal{D}$ be the left fibration classified by

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɬ}_{\mathcal{C}}(-), -)\colon \mathcal{D} \to \mathcal{S}_{\mathbf{V}},$$

then we have an equivalence $\psi\colon V \to V'$ of left fibrations over $\mathcal{D}$. As the map v of Lemma 1.2.62 is cofinal and V is left fibrant over $\mathcal{D}$, it is a weak equivalence of $(\mathbf{sSet}/\mathcal{D})_{\text{cov}}$. By Lemma 1.1.3, the induced map

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(V', W) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{D}}(V, W) \xrightarrow{v^*} \underline{\mathrm{Hom}}_{\mathcal{D}}(\mathrm{S}(\mathcal{C}), W) \qquad (**)$$

is an equivalence of ∞-groupoids. Therefore the homotopy fiber over the functor $w\colon \mathrm{S}(\mathcal{C}) \to W$ is a contractible ∞-groupoid; choosing an object in it provides a map $\gamma\colon V' \to W$ and a J-homotopy $h\colon \gamma\psi v \simeq w$ over $\mathcal{D}$. Moreover, since $\pi_0 \underline{\mathrm{Hom}}_{\mathcal{D}}(-, W) = \mathrm{Hom}_{\mathrm{LFib}(\mathcal{D})}(-, W)$, the diagram

$$\begin{array}{ccccc} \mathrm{S}(\mathcal{C}) & \xrightarrow{v} & V & \xrightarrow{\psi} & V' \\ & {\scriptstyle w}\searrow & & \swarrow{\scriptstyle \gamma} & \\ & & W & & \end{array}$$

commutes in $\mathrm{LFib}(\mathcal{D})$ (via the specified homotopy h). By Lemma 1.2.62, v, w, ψ are all weak equivalences, so γ is a weak equivalence, hence an equivalence $V' \to W$ of left fibrations over $\mathcal{D}$. Therefore it corresponds to an equivalence

$$\theta\colon \mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɬ}_{\mathcal{C}}(-), -) \xrightarrow{\sim} \mathrm{ev}$$

of functors $\mathcal{D} \to \mathcal{S}_{\mathbf{V}}$, by Lemma 1.2.17. The homotopy fiber of $(**)$ is the canonical contractible ∞-groupoid parameterizing choices of the isomorphism above, at least after applying Cor. 1.2.26.

It remains to prove full faithfulness and the characteristc essential uniqueness. Restrict all the data above along $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{1 \times \text{ɬ}} \mathcal{D}$. Then v restricts to the functoriality map of Remark 1.2.53(4), i.e. restricting h provides a homotopy commutative square

$$\begin{array}{ccccc} \mathrm{S}(\mathcal{C}) & \xrightarrow{\overline{v}} & V_{1 \times \text{ɬ}} & \xrightarrow{\psi} & V'_{1 \times \text{ɬ}} \\ & {\scriptstyle \overline{w}}\searrow & & \swarrow{\scriptstyle \gamma} & \\ & & W_{1 \times \text{ɬ}} & & \end{array}$$

which in particular commutes in $\mathrm{LFib}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C})$. The restrictions of ψ and γ over $1 \times \text{ɬ}$ are still equivalences between left fibrations because they are fiberwise eqiuvalences. The map $\overline{w}$ becomes the structural equivalence of $\mathrm{S}(\mathcal{C})$ with the left fibration classifying $\mathrm{Map}_{\mathcal{C}}$, by definition of W. Therefore $\overline{v}$ is an equivalence, meaning $\text{ɬ}_{\mathcal{C}}$ is fully faithful by Example 1.2.51. Hence any choice of induced map in (1) is an equivalence, by Remark 1.2.53.

Finally, the characterization statement: Most importantly, the homotopy commutative diagram shows that under the equivalence $\overline{w}$, the restriction of γ indeed gives a homotopy inverse to $(*)$. Moreover, we have the following alternative description of the map $(**)$ given by

$$\underline{\mathrm{Hom}}_{\mathcal{D}}(V', W) \to \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}(V'_{1 \times \text{ɬ}}, W_{1 \times \text{ɬ}}) \xrightarrow{(\psi\overline{v})^*} \underline{\mathrm{Hom}}_{\mathcal{C}^{\mathrm{op}} \times \mathcal{C}}(\mathrm{S}(\mathcal{C}), W_{1 \times \text{ɬ}})$$

where the right-hand space is equal to $\underline{\mathrm{Hom}}_{\mathcal{D}}(\mathrm{S}(\mathcal{C}), W)$ by adjunction. In particular, this is an equivalence, and on the other hand the second map is also an equivalence because we established that $\mathrm{S}(\mathcal{C}) \to V'_{1\times \curlywedge}$ is an equivalence. Therefore the restriction along $1 \times \curlywedge$ map induces an equivalence of ∞-groupoids.

Hence there is a canonical equivalence between the homotopy fiber over w described before, and the homotopy fiber of the equivalence $(\psi\overline{v})^*$ above at $\overline{w}$. Since $\overline{w}$ is the structural equivalence to $\mathrm{Map}_{\mathcal{C}}(-,-)$, its homotopy fiber corresponds to those maps inverting $\overline{v}$, i.e. providing homotopy inverses to $(*)$. This provides the desired characterization. □

It is easy to check in this proof that, if we restrict to a specific object $a \in \mathcal{C}^{\mathrm{op}}$ in the first factor, then we will obtain a case of Constr. 1.2.60, so the equivalence of Thm. 1.2.61 coincides up to homotopy with the one constructed in this theorem.

1.3 ∞-category theory

This section is all about carrying out the standard constructions of category theory, such as adjunctions, limits and colimits, and Kan extensions, in the ∞-categorical setting. The development of these concepts in [HCHA] differs from that in [HTT] in that, roughly, the standard category-theoretic definitions are used but **Set** is replaced with $\mathcal{S}$.

We will present this material as follows: The first will be an overview of the easy material: we define, the ∞-categorical concepts corresponding to the classical ones, and prove the most immediate properties from the definition. The second section proves the major technical results which allow us to extend the theory further. Its consequences are derived in the third section. In general, however, this will mostly be easier and more conceptual than the previous section.

We also make the following conventions: We work within a fixed universe **U**, and thus we will use the adjectives small, locally small, essentially small, or no qualifer to mean cardinally **U**-small, locally **U**-small, essentially **U**-small, or arbitrary. $\mathcal{S}$ will always mean $\mathcal{S}_{\mathbf{U}}$, the ∞-category of **U**-small ∞-groupoids. If $\mathcal{C}$ is an ∞-category, a *presheaf over* $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. If $\mathcal{C}$ is a locally small ∞-category, then it comes equipped with a choice of mapping space functor

$$\mathrm{Map}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S},$$

(and its associated data, namely a map $\mathrm{S}(\mathcal{C}) \to \mathcal{S}_\bullet$ making the canonical square homotopy cartesian), which is unique up to contractible choice. In particular it comes with a Yoneda embedding $\curlywedge_{\mathcal{C}} \colon \mathcal{C} \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. If $F \colon \mathcal{C} \to \mathcal{D}$ is a functor of ∞-categories, then we will assume there is a chosen 'canonical' map

$$\mathrm{Map}_{\mathcal{C}}(x, y) \to \mathrm{Map}_{\mathcal{D}}(Fx, Fy),$$

which is indeed functorial in x and y, but only canonical up to contractible choice. We will sometimes use the notation $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ if we want to emphasize

the objects as functors, instead of $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{D})$ emphasizing the objects as maps of simplicial sets.

1.3.1 Adjunctions and (co)limits

Definition 1.3.1 Let $\mathcal{C}$ and $\mathcal{D}$ be locally small ∞-categories. An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a pair of functors $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ equipped with an equivalence

$$\alpha\colon \mathrm{Map}_{\mathcal{C}}(-, G(-)) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(F(-), -)$$

in the ∞-category of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})$. Given F and G, we say that F is left adjoint to G, or G is right adjoint to F, if there exists an adjunction from $\mathcal{C}$ to $\mathcal{D}$ involving these functors; then α is called an adjunction formula. This relation between F and G is expressed as $F \dashv G$.

This definition is technically asymmetric, but of course an inverse to α in the functor category is unique up to contractible choice. In any case, the equivalence

$$\alpha_{c,d}\colon \mathrm{Map}_{\mathcal{C}}(c, Gd) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

does *not* canonically associate a morphism $c \to Gd$ with a morphism $Fc \to d$. Indeed, even if we assume that the equivalence of Cor. 1.2.26 is 'canonical', a map $f\colon c \to Gd$ is associated to some object of $\mathrm{Map}_{\mathcal{D}}(Fc, d)$, and we would need to choose an object of the homotopy fiber of the equivalence $(Fc\backslash\mathcal{D})_c \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Fc, d)$. In other words, the space of corresonding maps is contractible.

Example 1.3.2 Suppose we have an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ with an adjunction formula α as above, and transport α along the canoncial map

$$\mathrm{hFun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S}) \to \mathrm{Fun}(\mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{D}, \mathrm{h}\mathcal{S}) \xrightarrow{\pi_{0,*}} \mathrm{Fun}(\mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{h}\mathcal{D}, \mathbf{Set}).$$

It follows from Lemma 1.2.56 and the natural isomorphism directly following it that the image of α produces a canonical natural isomorphism

$$\alpha\colon \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c, Gd) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(Fc, f),$$

so in particular an adjunction $F\colon \mathrm{h}\mathcal{C} \rightleftarrows \mathrm{h}\mathcal{D} : G$.

Let X be a presheaf in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. The *$\infty$-category of elements of X*, denoted $\mathcal{C}/X$ or $\int_{\mathcal{C}} X$, is defined by the pullback:

$$\begin{array}{ccc} \mathcal{C}/X & \longrightarrow & \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})/X \\ \downarrow{\scriptstyle p_X} & \lrcorner & \downarrow \\ \mathcal{C} & \xrightarrow{\text{ɣ}_{\mathcal{C}}} & \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \end{array}$$

$\mathcal{C}/X \to \mathcal{C}$ is a right fibration, so $\mathcal{C}/X$ is an ∞-category. Let $\mathbf{V}$ be a universe containing $\mathbf{U}$ in which $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is locally $\mathbf{V}$-small; then the right-hand map is homotopy-classified by $\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(-, X)$, so $\mathcal{C}/X \to \mathcal{C}$ is homotopy-classified by

$$\mathrm{Map}_{\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})}(\text{ɣ}_{\mathcal{C}}(-), X) \simeq X,$$

by the Yoneda lemma (Thm. 1.2.63). Therefore $\mathcal{C}/X$ is canonically equivalent (over $\mathcal{C}$) to the right fibration classified by X, and we have a homotopy cartesian square:

$$\begin{array}{ccc} (\mathcal{C}/X)^{\mathrm{op}} & \longrightarrow & \mathcal{S}_\bullet \\ \downarrow & & \downarrow{\scriptstyle p_{\mathrm{univ}}} \\ \mathcal{C}^{\mathrm{op}} & \xrightarrow{X} & \mathcal{S} \end{array}$$

Consider the representable case $X = \text{ɣ}_{\mathcal{C}}(a)$. Then the Yoneda embedding induces the canonical map $\mathcal{C}/a \to \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})/\text{ɣ}(a)$, and hence a canonical map $\mathcal{C}/a \to \mathcal{C}/\text{ɣ}_{\mathcal{C}}(a)$ over $\mathcal{C}$. On the fiber over x, this is equivalent to

$$\mathcal{C}(x, a) \xrightarrow{\text{ɣ}_{\mathcal{C}}} \underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})(\text{ɣ}(x), \text{ɣ}(a)),$$

which is an equivalence because the Yoneda embedding is fully faithful, and hence $\mathcal{C}/a \xrightarrow{\sim} \mathcal{C}/\text{ɣ}_{\mathcal{C}}(a)$ is an equivalence of right fibrations. We will use these notions to make some elementary observations about adjunctions and universality in the next few lemmas.

Lemma 1.3.3 *Let $\mathcal{C}$ be a locally small ∞-category and $X\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ a presheaf. The following conditions are equivalent:*

(*i*) *X is equivalent in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ to $\text{ɣ}_{\mathcal{C}}(a)$ for some a in $\mathcal{C}$.*

(*ii*) *The ∞-category $\mathcal{C}/X$ has a terminal object.*

In this case, we say that X is representable. If e is a terminal object of $\mathcal{C}/X$, then X is representable by $p_X(e) \in \mathcal{C}$.

Proof Since $\mathcal{C}/X$ classifies X, having $\mathcal{C}/X \simeq \mathcal{C}/\text{ɣ}_{\mathcal{C}}(a)$ over $\mathcal{C}$ is equivalent to $X \simeq \text{ɣ}_{\mathcal{C}}(a)$ in the functor category, which is therefore further equivalent to $\mathcal{C}/X \simeq \mathcal{C}/a$ by above. Thus $(i) \Rightarrow (ii)$, because $\mathcal{C}/a$ has terminal object 1_a.

If $\mathcal{C}/X$ has terminal object e, then we know that $(\mathcal{C}/X)/e \to \mathcal{C}/X$ is a trivial fibration, and since p_X is a right fibration we know $(\mathcal{C}/X)/e \to \mathcal{C}/p_X(e)$ is a trivial fibration (Lemma 1.1.22). Both maps are over $\mathcal{C}$, so $\mathcal{C}/X \simeq \mathcal{C}/p_X(e)$, and thus $X \simeq \text{ɣ}_{\mathcal{C}}(p_X(e))$. □

Lemma 1.3.4 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between locally small ∞-categories. The following conditions are equivalent for an object $d \in \mathcal{D}$:*

(*i*) *There exists an object x of $\mathcal{C}$ and an equivalence*

$$\mathrm{Map}_{\mathcal{C}}(c, x) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

in $\mathcal{S}$ which is natural in c.

(*ii*) *$\mathcal{C}/d$ has a terminal object.*

(*iii*) *The presheaf $F^* \text{ɣ}_{\mathcal{D}}(d)$ over $\mathcal{C}$ is representable.*

More specifically, $f\colon F(x) \to d$ represents a terminal object of $\mathcal{C}/d$ if and only if any composite

$$\mathrm{Map}_{\mathcal{C}}(c,x) \xrightarrow{F_*} \mathrm{Map}_{\mathcal{D}}(Fc,Fx) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{D}}(Fc,d)$$

gives a natural equivalence for all c as in (i).

Proof If $\text{ɬ}_{\mathcal{C}}(x) \simeq F^*\text{ɬ}_{\mathcal{D}}(d)$ for some x, then evaluated pointwise this is the definition of (i); thus $(i) = (iii)$. There is an immediate proof if we ignore the last statement: Namely, since $\mathcal{D}/d \to \mathcal{D}$ is a right fibration, the defining square of $\mathcal{C}/d$ is homotopy cartesian, so it follows from the pasting lemma that $\mathcal{C}/d \to \mathcal{C}$ is homotopy classified by the composite $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$, i.e. exactly $F^*\text{ɬ}(d)$. Therefore

$$(\mathcal{C}/d)^{\mathrm{op}} \simeq F^*\text{ɬ}(d) \times^{\mathbf{R}}_{\mathcal{S}} \mathcal{S}_\bullet \simeq (\mathcal{C}/F^*\text{ɬ}(d))^{\mathrm{op}}.$$

Thus by the previous lemma we conclude $(ii) = (iii)$. The more specific statement follows immediately from the diagram of Lemma 1.A.4. □

Conceptually, the more specific statement in the proof is important for the general notion of ∞-categorical universal properties, and generalizes the comparatively obvious statement for ordinary categories.

Lemma 1.3.5 *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $f\colon \mathcal{B} \to \mathcal{D}$ be functors between ∞-categories, and define*

$$\mathcal{E} \subset (\mathcal{C} \times \mathcal{B}) \underset{\mathcal{D}\times\mathcal{D}}{\times} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D})$$

to be the full subcategory spanned by objects $(c, b, u\colon Fc \to f(b))$ such that (c,u) is terminal in $\mathcal{C}/f(b)$. Then the projection $\mathcal{E} \to \mathcal{B}$ is a trivial fibration, provided that $\mathcal{C}/f(b)$ has a terminal object for each $b \in \mathcal{B}$.

Proof Let us write $\mathcal{A} = (\mathcal{C}\times\mathcal{B})\times_{\mathcal{D}\times\mathcal{D}}\underline{\mathrm{Hom}}(\Delta^1,\mathcal{D})$, so that $\mathcal{A} \to \mathcal{B}$ is a pullback of evaluation-at-1 $\mathcal{C} \times_{\mathcal{D},\mathrm{ev}_0} \underline{\mathrm{Hom}}(\Delta^1,\mathcal{D}) \to \mathcal{D}$, which is a Joyal fibration by Example 1.A.6. We now make the following observations:

1. We first claim that $\mathcal{E} \subset \mathcal{A}$ is closed under equivalences, which will mean that the inclusion is an isofibration, and thus $\mathcal{E} \to \mathcal{B}$ is a Joyal fibration. Indeed, given an equivalence $(x,b,u) \simeq (x',b',u')$ in $\mathcal{A}$, it follows from the specific statement in Lemma 1.3.4 and functoriality of mapping spaces that one is terminal in $\mathcal{C}/f(b)$ iff the other is terminal in $\mathcal{C}/f(b')$.
2. We claim that $\mathcal{E} \to \mathcal{B}$ is fully faithful. Indeed, consider the cartesian diagram:

$$\begin{array}{ccc} \mathcal{A}\big((x,b,u),(x',b',u')\big) & \longrightarrow & \big(\mathcal{C} \times_{\mathcal{D}} \mathrm{Arr}(\mathcal{D})\big)\big((x,u),(x',u')\big) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{B}(b,b') & \longrightarrow & \mathcal{D}\big(f(b),f(b')\big) \end{array}$$

Since $\mathcal{C} \times_{\mathcal{D}} \mathrm{Arr}(\mathcal{D}) \to \mathcal{D}$ is a Joyal fibration, it follows from Lemma 1.0.7 that the right-hand map is a Kan fibration. Thus it suffices to show that the right-hand map is an equivalence as long as $(x, b, u), (x', b', u') \in \mathcal{E}$. We showed in Example 1.A.6 that this map is a homotopy pullback of

$$\mathcal{C}(x, x') \xrightarrow{F_*} \mathcal{D}(Fx, Fx') \xrightarrow{u'_*} \mathcal{D}(Fx, f(b')),$$

where the right-hand arrow is a choice of composition. But Lemma 1.3.4 shows that this being an equivalence is precisely the statement that (x', b', u') lies in $\mathcal{E}$.

To conclude, note that we have explicitly assumed that $\mathcal{E} \to \mathcal{B}$ is surjective on objects, so we have shown that it is an equivalence of ∞-categories as well as a Joyal fibration. □

Proposition 1.3.6 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between locally small ∞-categories. The following conditions are equivalent:*

(*i*) *F has a right adjoint.*

(*ii*) *For each $d \in \mathcal{D}$, there exists an object x_d of $\mathcal{C}$ and an equivalence*

$$\mathrm{Map}_{\mathcal{C}}(c, x) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

in $\mathcal{S}$ which is natural in c.

(*iii*) *For each $d \in \mathcal{D}$, $\mathcal{C}/d$ has a terminal object.*

Moreover, suppose that $\{(x_d, \varepsilon_d\colon F(x_d) \to d)\}$ is a choice of terminal object of $\mathcal{C}/d$ for each $d \in \mathcal{D}$. Then there is a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ such that $G(d) = x_d$, equipped with a natural transformation $\varepsilon\colon FG \to \mathrm{Id}_{\mathcal{D}}$ extending $\{\varepsilon_d\}$, and such that the composite

$$\mathrm{Map}_{\mathcal{C}}(c, x_d) \xrightarrow{F_*} \mathrm{Map}_{\mathcal{D}}(Fc, Fx_d) \xrightarrow{\varepsilon_d} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

provides an adjunction formula for all c and d. The choice of (G, ϵ) satisfying these conditions is essentially unique.

Proof We already know that (*ii*) = (*iii*) and that these are equivalent to $F^* \not\perp_{\mathcal{D}}(d)$ being representable for all d, by Lemma 1.3.4. By definition (*i*) $\Rightarrow$ (*ii*). Assume (*iii*), and choose the data $\{(x_d, \varepsilon_d)\}$ as in the more specific statement. Let $\mathcal{E} \subset \mathcal{C} \times_{\mathcal{D}, \mathrm{ev}_0} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D})$ denote the full subcategory spanned by the pairs $(c, u\colon Fc \to d)$ which represent terminal objects of $\mathcal{C}/d$. The composite $p\colon \mathcal{E} \to \mathcal{D}$ is a trivial fibration by Lemma 1.3.5 (taking $f = \mathrm{id}_{\mathcal{D}}$).

To conclude, we have a map $\mathrm{Ob}(\mathcal{D}) \to \mathcal{E}$ given by $d \mapsto (x_d, \varepsilon_d)$. Choose a lift in the commutative diagram:

$$\begin{array}{ccc} \mathrm{Ob}(\mathcal{D}) & \longrightarrow & \mathcal{E} \\ \downarrow & \nearrow & \downarrow{\scriptstyle p} \\ \mathcal{D} & = & \mathcal{D} \end{array}$$

The space of lifts forms a contractible ∞-groupoid, and a lift $\mathcal{D} \to \mathcal{E}$ consists precisely of the following data: a functor $G\colon \mathcal{D} \to \mathcal{C}$ with $Gd = x_d$, together with a natural transformation $FG \to \mathrm{id}_{\mathcal{D}}$ extending the given data ε_d. Then Lemma 1.3.4 shows that it induced the required equivalence on mapping spaces. □

Remark 1.3.7 The natural transformation $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ is called the *counit* of the adjunction $F \dashv G$. It follows from the Yoneda lemma that any natural transformation $\mathrm{Map}_{\mathcal{C}}(-, x) \to F^* \mathrm{Map}_{\mathcal{D}}(-, d)$ is induced, up to homotopy (i.e. in $\mathrm{h}\mathcal{S}$), by an object of $F^* \mathrm{Map}_{\mathcal{D}}(-, d)(x) = \mathrm{Map}_{\mathcal{D}}(Fx, d) \simeq \mathcal{C}(Fx, d)$. Unwinding the definitions shows that such a map $u\colon Fx \to d$ induces

$$\mathrm{Map}_{\mathcal{C}}(c, x_d) \xrightarrow{F_*} \mathrm{Map}_{\mathcal{D}}(Fc, Fx_d) \xrightarrow{\varepsilon_d} \mathrm{Map}_{\mathcal{D}}(Fc, d)$$

up to homotopy. Lemma 1.3.4 shows that this map being an equivalence is equivalent to (x, u) being terminal in $\mathcal{C}/d$. In particular, *every* adjunction formula is, up to homotopy, induced by an essentially unique counit map $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ which chooses such a terminal object of $\mathcal{C}/d$ at each d. Likewise, by considering the dual situation via a homotopy inverse of the adjunction formula, we can construct a unit.

Remark 1.3.8 In light of the previous two results, we see firstly that the existence of a right adjoint is independent of the ambient universe. For a functor $F\colon \mathcal{C} \to \mathcal{D}$ of arbitrary ∞-categories, we sometimes say that F has a partial right adjoint at d if $\mathcal{C}/d$ has a terminal object. Limits, colimits, and Kan extensions will all be defined by this condition. Thus F has a right adjoint in any universe iff it has a right adjoint at every d. This condition in ordinary categories is sometimes called the existence of a universal morphism from F to d, and the terminal pair $(c, u\colon Fc \to d)$ is the universal morphism.

Remark 1.3.9 Here are some immediate properties of adjunctions which we can deduce from the definitions or the previous result.

1. If $F\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint, then it is cofinal by Quillen's Theorem A (Cor. 1.1.38): the slices are weakly contractible because they have terminal objects.
2. If $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ is an adjunction between locally small categories, then for any naturally equivalent functor $G \simeq G'$ we canonically obtain an adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G'$, simply by choosing a composition of the adjunction formula with the induced equivalence in $\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})$.
3. If $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ and $H\colon \mathcal{D} \rightleftarrows \mathcal{E} : K$ are adjunctions between locally small ∞-categories, then there is a canonical adjunction

$$HF\colon \mathcal{C} \rightleftarrows \mathcal{E} : GK,$$

again by restricting the adjunction formulae and choosing a composition.

The next two properties require slightly more sophistication, but are basic applications of the Yoneda lemma.

4. If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor, there is a canonical space of right adjoints $\mathrm{RAdj}(F)$, which is given by the homotopy fiber of
$$\mathrm{Fun}(\mathcal{D},\mathcal{C})^{\simeq} \to \mathrm{Fun}(\mathcal{D},\underline{\mathrm{Hom}}(\mathcal{C}^{\mathrm{op}},\mathcal{S}))^{\simeq} = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}\times\mathcal{D},\mathcal{S})^{\simeq}$$
at $F^* \text{ Yo}_{\mathcal{D}}$. Indeed, an object is precisely a functor $G\colon \mathcal{D}\to\mathcal{C}$ equipped with a natural isomorphism $\text{Yo}_{\mathcal{C}}G \xrightarrow{\sim} F^*\text{Yo}_{\mathcal{D}}$. If F has no right adjoint then it is empty. Otherwise, we note that the functor above is fully faithful by the Yoneda lemma (and Prop. 1.2.42), and the mapping spaces of the homotopy fibers are the homotopy fibers of the mapping spaces, which are contractible ∞-groupoids. Therefore $\mathrm{RAdj}(F)$ is a contractible ∞-groupoid if F admits a right adjoint. This shows not only that any two right adjoints of F are equivalent, but that the space of equivalences between them is also contractible.

5. Suppose $F\colon \mathcal{C}\to\mathcal{D}$ is an equivalence of locally small ∞-categories. Then $G\colon \mathcal{D}\to\mathcal{C}$ is a right adjoint of F iff it is a homotopy inverse to F. (TODO: Fill in the proof.)

Our next topic is defining limits and colimits. We will primary state the case for limits, and those for colimits will follow dually.

Definition 1.3.10 Let $\mathcal{C}$ be an ∞-category and I be a simplicial set.

1. An *I-shaped diagram* in $\mathcal{C}$ is a morphism $D\colon I\to\mathcal{C}$ of simplicial sets.
2. Write $c_I\colon \mathcal{C}\to\underline{\mathrm{Hom}}(I,\mathcal{C})$ for the constant diagram functor. The ∞-category of cones over a diagram D is
$$\mathrm{Cone}_{\mathcal{C}}(D) \coloneqq \mathcal{C}\underset{c_I,\underline{\mathrm{Hom}}(I,\mathcal{C})}{\times}\underline{\mathrm{Hom}}(I,\mathcal{C})/D$$
equipped with its canonical right fibration $\mathrm{Cone}_{\mathcal{C}}(D)\to\mathcal{C}$. If $x\in\mathcal{C}$, the fiber is the ∞-groupoid denoted $\mathrm{Cone}_{\mathcal{C}}(x,D)$, whose objects are natural transformations $c_I(x)\to D$ in $\underline{\mathrm{Hom}}(I,\mathcal{C})$.
3. Given an an I-shaped diagram D in $\mathcal{C}$, a *limit* of D is a terminal object $(x,\lambda\colon c_I(x)\to D)$ of $\mathrm{Cone}_{\mathcal{C}}(D)$. We also use the notations
$$(x,\lambda) = \varprojlim D \qquad x = \varprojlim_{i\in I} D_i,$$
the right-hand one especially when λ is clear from context. Equivalently, $\varprojlim D$ is a choice of partial right adjoint to $c_I\colon \mathcal{C}\to\underline{\mathrm{Hom}}(I,\mathcal{C})$ at D.
4. Dually, we write $\mathrm{Cocone}_{\mathcal{C}}(D) = \mathcal{C}\times_{\underline{\mathrm{Hom}}(I,\mathcal{C})} D\backslash\underline{\mathrm{Hom}}(I,\mathcal{C})$ equipped with its canonical left fibration $\mathrm{Cocone}_{\mathcal{C}}(D)\to\mathcal{C}$, and the fiber over $x\in\mathcal{C}$ is denoted by $\mathrm{Cocone}(D,x)$. A *colimit of D* is an initial object of

$\mathrm{Cocone}_{\mathcal{C}}(D)$, of the form $(x, \lambda\colon D \to c_I(x))$, and we write

$$(x, \lambda) = \varinjlim D \qquad x = \varinjlim_{i \in I} D_i$$

as alternative notations. Equivalently $\varinjlim D$ is a choice of partial left adjoint to c_I at D. Note that

$$\begin{aligned}\mathrm{Cocone}_{\mathcal{C}}(D)^{\mathrm{op}} &= \mathcal{C}^{\mathrm{op}} \times_{\underline{\mathrm{Hom}}(I^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})} \underline{\mathrm{Hom}}(I^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}})/D^{\mathrm{op}} \\ &= \mathrm{Cone}_{\mathcal{C}^{\mathrm{op}}}(D^{\mathrm{op}}),\end{aligned}$$

so a colimit in $\mathcal{C}$ of D is the same thing as a limit in $\mathcal{C}^{\mathrm{op}}$ of D^{op}.

Here are the very immediate observations from the definition of limits (and colimits). For instance, we have a canonical equivalence

$$\mathrm{Cone}_{\mathcal{C}}(x, D) \xrightarrow{\sim} \mathcal{C}^I(c_I(x), D)$$

of ∞-categories for all x.

Remark 1.3.11 Let $\mathcal{C}$ be an ∞-category and D an I-shaped diagram in $\mathcal{C}$.

1. A morphism of cones $(c, \lambda) \to (c', \lambda')$ consists of a morphism $u\colon c \to c'$ in $\mathcal{C}$ and a 2-simplex of $\underline{\mathrm{Hom}}(I, \mathcal{C})$ recording commutativity of the triangle

$$\begin{array}{ccc} c_I(x) & & \\ {\scriptstyle u}\downarrow & \searrow^{\lambda} & \\ c_I(x') & \xrightarrow[\lambda']{} & D \end{array} \qquad\qquad \begin{array}{ccc} x & & \\ {\scriptstyle u}\downarrow & \searrow^{\lambda_i} & \\ x' & \xrightarrow[\lambda_i']{} & D_i \end{array}$$

   on the left-hand side above. In particular it consists of u with a map $\eta\colon \Delta^2 \times I \to \mathcal{C}$ such that η_i records commutativity of the right-hand triangles in $\mathcal{C}$ for each i. Since $\mathrm{Cone}(\mathcal{C}) \to \mathcal{C}$ is conservative, such a map is an equivalence of cones iff u is an equivalence in $\mathcal{C}$.

2. The space of limits of D is the full subcategory of $\mathrm{Cone}(D)$ spanned by terminal objects, which is either empty, or a contractible ∞-groupoid whenever D admits a limit in $\mathcal{C}$ (Cor. 1.1.30). In particular between any two limit cones there is an equivalence as described in (1). Conversely, if $(x, \lambda) = \varinjlim D$ and $u\colon x' \to x$ is an equivalence of $\mathcal{C}$, then we may extend it to an eqiuvalence of cones via $(x', c_I(x') \xrightarrow{\sim} c_I(x) \to D)$.

3. Suppose $\mathcal{C} = N(C)$ where C is a small ordinary category. Then the diagram D is induced from $\overline{D}\colon \tau(I) \to C$, so

$$\underline{\mathrm{Hom}}(I, \mathcal{C}) = N(\mathrm{Fun}(\tau(I), C)) \qquad \underline{\mathrm{Hom}}(I, \mathcal{C})/D = N(\mathrm{Fun}(\tau(I), C)/\overline{D}),$$

   and it follows that $\mathrm{Cone}_{\mathcal{C}}(D)$ is the nerve of the ordinary category of cones over $\overline{D}$. Therefore $\varprojlim_I D = \varprojlim_{\tau(I)} \overline{D}$, recovering the 1-category case.

4. If $\mathcal{C}$ is locally small and I is small, then we may use Lemma 1.3.4 to observe that $(x, \lambda) = \varprojlim D$ if and only if

$$\mathrm{Map}_{\mathcal{C}}(c, x) \to \mathrm{Map}_{\mathcal{C}^I}(c_I(c), c_I(x)) \to \mathrm{Map}_{\mathcal{C}^I}(c_I(c), D)$$

is invertible in $\mathcal{S}$ for all c. Alternatively we can write it without referring to the universe as saying that

$$\mathcal{C}(c, x) \xrightarrow{c_{I,*}} \mathcal{C}^I(c_I(c), c_I(x)) \xrightarrow{\lambda_*} \mathcal{C}^I(c_I(c), D)$$

is an isomorphism in $\mathcal{H}$ for all c. Then Prop. 1.3.6 says that c_I has a right adjoint iff all I-shaped diagrams in $\mathcal{C}$ have a limit, in which case there is an essentially unique choice of right adjoint which we call

$$\varprojlim_I \colon \underline{\mathrm{Hom}}(I, \mathcal{C}) \to \mathcal{C}$$

a limit functor. Moreover, the same result shows that we may prescribe for each diagram D the limit cone (x, λ), so that $\varprojlim_I(D) = x$, and the counit $c_I \varprojlim \to \mathrm{id}_{\mathcal{C}}$ records the limiting cone λ.

5. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and write $F_* D = FD$ for the composite I-shaped diagram in $\mathcal{D}$. Since F induces a compatible map on the legs of the defining pullback, we obtain an induced functor

$$F_* \colon \mathrm{Cone}_{\mathcal{C}}(D) \to \mathrm{Cone}_{\mathcal{D}}(F_* D)$$

on cone categories. For instance, by [HCHA, Cor. 2.3.29] this map is an equivalence of ∞-categories if $\mathcal{C} \to \mathcal{D}$ is an equivalence. We say that F *preserves the limit of* D if the functor above is final; if F preserves the limit of D of all I-shaped diagrams D, we say F preserves I-shaped limits. Note that strictly speaking we don't require $\mathcal{C}$ to actually have the limit; if it does, then this is simply asking that whenever (x, λ) is a limit cone, $(Fx, F_*\lambda) = \varprojlim F_* D$. It's equivalent to saying that the map above is a weak contravariant equivalence over $\mathcal{D}$, because $\mathrm{Cone}_{\mathcal{D}}(F_* D)$ is right fibrant, and if F is a right fibration then it's equivalent to saying that F_* above is an equivalence of ∞-categories.

Example 1.3.12 Take $I = \varnothing$, then D is the unique empty diagram and $\underline{\mathrm{Hom}}(I, \mathcal{C})/D = \underline{\mathrm{Hom}}(I, \mathcal{C}) = \varnothing$, so

$$\mathrm{Cone}_{\mathcal{C}}(\varnothing) = \mathrm{Cocone}_{\mathcal{C}}(\varnothing) = \mathcal{C}.$$

In particular a limit (resp. colimit) of $\varnothing$ is precisely a terminal (resp. initial) object of $\mathcal{C}$. If $F \colon \mathcal{C} \to \mathcal{D}$ is a functor, then we say F preserves terminal objects if it preserves limits of the empty diagram, or equivalently if F is final.

Remark 1.3.13 It is important to be careful with our notion of a functor preserving limits of a given diagram in the case that the limit doesn't exist, because it is not identical to the corresponding notion for a 1-category. For instance, a functor between nerves of ordinary categories preserving terminal objects means final, which is not equivalent to being final in the 1-categorical

sense, but rather it is stronger (and asserts weak contractibility of relative slices rather than mere connectedness).

Example 1.3.14 This is a more specific version of Remark 1.3.11(4) in the case that not all limits of a given shape exist. Namely, suppose $f\colon \mathcal{B} \to \underline{\mathrm{Hom}}(I, \mathcal{C})$ is a functor of ∞-categories such that the diagram $f(b)$ has a limit for all b. Then let us define

$$\mathcal{E} \subset (\mathcal{C} \times \mathcal{B}) \times_{\underline{\mathrm{Hom}}(\partial\Delta^1 \times I, \mathcal{C})} \underline{\mathrm{Hom}}(\Delta^1 \times I, \mathcal{C})$$

to be the full subcategory spanned by those pairs $(x, b, u\colon c_I(x) \to f(b))$ such that $(x, u) = \varprojlim f(b)$. Then Lemma 1.3.5 shows that the projection $\mathcal{E} \to \mathcal{B}$ is a trivial fibration. In particular, any choice of section $\mathcal{B} \to \mathcal{E}$ provides a functorial assignment of b to a limit cone $\varprojlim f(b)$, and we can further project to $\mathcal{C}$ to extract the actual limiting object. For instance if we let $\mathcal{B}$ be the full subcategory of $\underline{\mathrm{Hom}}(I, \mathcal{C})$ spanned by diagrams with a limit, then the aforementioned map is just a limit functor $\varprojlim\colon \mathcal{B} \to \mathcal{C}$.

Suppose $u\colon I' \to I$ is a morphism of simplicial sets. This induces a map $u^*\colon \mathrm{Cone}_{\mathcal{C}}(D) \to \mathrm{Cone}_{\mathcal{C}}(Du)$ on pullbacks over $\mathcal{C}$. Given a cone (x, λ), this gets sent to a restricted cone $(x, u^*\lambda)$. Note that we have a canonical cartesian square

$$\begin{array}{ccc} \mathrm{Cone}_{\mathcal{C}}(D) & \longrightarrow & \underline{\mathrm{Hom}}(I, \mathcal{C})/D \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Cone}_{\mathcal{C}}(u^*D) & \longrightarrow & \underline{\mathrm{Hom}}(I, \mathcal{C}) \underset{\underline{\mathrm{Hom}}(I', \mathcal{C})}{\times} \underline{\mathrm{Hom}}(I', \mathcal{C})/u^*D \end{array}$$

where the right-hand map is a right fibration (by [HCHA, Thm. 3.4.22]) since $\underline{\mathrm{Hom}}(I, \mathcal{C}) \to \underline{\mathrm{Hom}}(I', \mathcal{C})$ is a Joyal fibration. In particular u^* is a right fibration between right fibrant objects over $\mathcal{C}$.

Lemma 1.3.15 *Let $u\colon I' \to I$ be a cofinal map and D an I-shaped diagram of $\mathcal{C}$. Then the induced functor*

$$\mathrm{Cone}_{\mathcal{C}}(D) \to \mathrm{Cone}_{\mathcal{C}}(u^*D)$$

is a trivial fibration (over $\mathcal{C}$). In particular, a cone (x, λ) over D is a limit if and only if $(x, u^*\lambda)$ is a limit of Du.

Proof Since the map is a right fibration between right fibrant objects over $\mathcal{C}$, we may show the corresponding fiberwise statement that for all x, $\mathrm{Cone}_{\mathcal{C}}(x, D) \to \mathrm{Cone}_{\mathcal{C}}(x, Du)$ is an equivalence of ∞-groupoids. Consider the canonical equivalence

$$\mathrm{Cone}_{\mathcal{C}}(x, D) \xrightarrow{\sim} (x \backslash\backslash \underline{\mathrm{Hom}}(I, \mathcal{C}))_D \cong \underline{\mathrm{Hom}}(I, x \backslash\backslash \mathcal{C})_D = \underline{\mathrm{Hom}}_{\mathcal{C}}(I, x \backslash\backslash \mathcal{C}),$$

and likewise $\mathrm{Cone}_{\mathcal{C}}(x, Du) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{C}}(I', x\backslash\backslash\mathcal{C})$, which fit into:

$$\begin{array}{ccc} \mathrm{Cone}(x, D) & \xrightarrow{u^*} & \mathrm{Cone}(x, Du) \\ \simeq\downarrow & & \downarrow\simeq \\ \underline{\mathrm{Hom}}_{\mathcal{C}}(I, x\backslash\backslash\mathcal{C}) & \xrightarrow{u^*} & \underline{\mathrm{Hom}}_{\mathcal{C}}(I', x\backslash\backslash\mathcal{C}) \end{array}$$

Since $I' \to I$ is cofinal, it is a weak equivalence in $(\mathbf{sSet}/\mathcal{C})_{\mathrm{cov}}$ with structure maps the respective diagrams, so the lower horizontal map is an equivalence of ∞-groupoids by Lemma 1.1.3. □

As a special case, restricting a diagram along a weak categorical equivalence doesn't change a colimit/limit (Cor. 1.2.4). More generally, this mirrors the alternative characterization of cofinality/finality in ordinary category theory.

Example 1.3.16 Suppose that I has an initial object i_0, so that $\{i_0\} \to I$ is cofinal. Then for any I-shaped diagram D, $\mathrm{Cone}_{\mathcal{C}}(D) \to \mathrm{Cone}_{\mathcal{C}}(D_{i_0})$ is an equivalence of categories. By definition

$$\mathrm{Cone}_{\mathcal{C}}(D_{i_0}) = \mathcal{C}/D_{i_0},$$

so $\varprojlim D = D_{i_0}$. The limiting cone is given by any natural transformation $c_I(D_{i_0}) \to D$, i.e. which must pick out a initial map $D_{i_0} \to D_i$ for each i. For example, we see that if $\mathcal{C}$ has an initial object c_0, then $\varprojlim \mathrm{id}_{\mathcal{C}} = c_0$.

Example 1.3.17 Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories and D an I-shaped diagram in $\mathcal{C}$. If the limit of D exists, then F preserves the limit iff the limit of F_*D exists in $\mathcal{D}$, and the canonical map $F(\varprojlim D) \to \varprojlim F_*D$ is an equivalence in $\mathcal{D}$. *Proof* This is obvious once we interpret what it means. Namely, choose $(x, \lambda) = \varprojlim D$ and $(y, \eta) = \varprojlim F_*D$. Because $(Fx, F_*\lambda)$ is a cone, there is a contractible space of maps $(Fx, F_*\lambda) \to (y, \eta)$, which is what we mean by 'the' canonical map $Fx \to y$. The morphism of cones is an equivalence iff this map in $\mathcal{D}$ is an equivalence (Remark 1.3.11), so both conditions are equivalent to $F_*(x, \lambda)$ being terminal, and thus F_* being final.

Remark 1.3.18 As usual, there is a corresponding construction of the cone category using the fat slice. We define $\mathrm{Cone}^{\diamond}_{\mathcal{C}}(D)$ by

$$\mathrm{Cone}^{\diamond}_{\mathcal{C}}(D) \coloneqq \mathcal{C} \times_{\underline{\mathrm{Hom}}(I,\mathcal{C})} \underline{\mathrm{Hom}}(I, \mathcal{C})//D$$

so that the pullback map $\mathrm{Cone}^{\diamond}_{\mathcal{C}}(D) \to \mathcal{C}$ is again a right fibration. Since the map $\underline{\mathrm{Hom}}(I, \mathcal{C})/D \to \underline{\mathrm{Hom}}(I, \mathcal{C})//D$ is an equivalence of right fibrations over the functor category, it induces a canonical equivalence of right fibrations

$$\mathrm{Cone}_{\mathcal{C}}(D) \to \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D) \qquad (*)$$

over $\mathcal{C}$. Here is one reason this description is useful: We can paste the defining pullback to obtain:

$$\begin{array}{ccccc} \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D) & \longrightarrow & \underline{\mathrm{Hom}}(I,\mathcal{C})//D & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1 \times I, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathcal{C} & \longrightarrow & \underline{\mathrm{Hom}}(I,\mathcal{C}) & \xrightarrow{(1,\{D\})} & \underline{\mathrm{Hom}}(\partial\Delta^1 \times I, \mathcal{C}) \end{array}$$

It follows that a map $A \to \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D)$ is a map $A \diamond I \to \mathcal{C}$ restricting to D on I, or in other words a map $A \to \mathcal{C}_{//D}$. This provides a canonical identification $\mathcal{C}_{//D} \cong \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D)$, and hence we have a canonical zig-zag

$$\mathcal{C}_{/D} \xrightarrow{\sim} \mathcal{C}_{//D} \leftarrow \mathrm{Cone}_{\mathcal{C}}(D)$$

of equivalences, relating the cone category to the literal (join) slice. We know that the second map is a bijection on objects because $(*)$ is, and on the other hand an object of $\mathcal{C}_{/D}$ is a map $\Delta^0 * I \to \mathcal{C}$ under I, which is somewhat like a geometric cone from some object to the diagram D. Some final remarks:

1. This zig-zag is compatible with restriction along $u\colon I' \to I$, as is clear from the definitions. Namely, the induced map $\mathcal{C}_{/D} \to \mathcal{C}_{/u^*D}$ fits into a diagram of equivalences with $\mathrm{Cone}_{\mathcal{C}}(D) \to \mathrm{Cone}_{\mathcal{C}}(u^*D)$. In particular if u is cofinal, then $\mathcal{C}_{/D} \to \mathcal{C}_{/u^*D}$ is an equivalence of ∞-categories.
2. We also remark that this is the natural object appearing in Example 1.3.14: With respect to the trivial fibration $\mathcal{E} \to \mathcal{B}$ constructed in that example, the fiber $\mathcal{E}_b$ is precisely the full subcategory of $\mathrm{Cone}^{\diamond}_{\mathcal{C}}(f(b))$ consisting of terminal objects.

1.3.2 (Co)products and (co)cartesian squares

The next topic is some specific limits, namely products and pullbacks. Fix an ∞-category $\mathcal{C}$. Given a discrete family $\{x_i : i \in I\}$ of objects, we view this as a discrete diagram In $\mathcal{C}$. Since

$$\underline{\mathrm{Hom}}(\coprod_{i\in I} \Delta^0, \mathcal{C}) = \prod_{i\in I} \mathcal{C},$$

we conclude that

$$\mathrm{Cone}_{\mathcal{C}}(\{x_i : i \in I\}) = \mathcal{C} \times_{\mathcal{C}^I} \prod_i \mathcal{C}/x_i = \varprojlim_{i\in I} \left(\mathcal{C}/x_i \to \mathcal{C}\right).$$

In particular a cone over the diagram is an object c equipped with a collection of maps $f_i\colon c \to x_i$. A morphism of cones is a morphism $u\colon c \to c'$ equipped with commutative triangles

$$\begin{array}{ccc} c & \xrightarrow{u} & c' \\ & \searrow^{f_i} & \downarrow f_i' \\ & & x_i \end{array}$$

in $\mathcal{C}$ for all i. We can already see that because of the discreteness of the diagram, the condition is a lot like standard products, with the only new

addition is homotopy commutativity and homotopy uniqueness. The following is an instance of this:

Example 1.3.19 If $\mathcal{C}$ is an ∞-category, then $\mathcal{C} \to \mathrm{h}\mathcal{C}$ preserves products which exist in $\mathcal{C}$. We've seen that it preserves terminal objects which exist. To see this, let $x = \prod_i x_i$, meaning that we have maps $p_i \colon x \to x_i$ exhibiting $(x, \{p_i\})$ as the product of $\{x_i\}$. Then the (any choice of) induced map

$$\mathrm{Map}_{\mathcal{C}}(c, x) \to \mathrm{Map}_{\mathcal{C}^I}((c)_{i\in I}, (x_i)_{i\in I})$$

is an equivalence, so applying π_0 produces an isomorphism. The key point is that $\mathrm{h}(\mathcal{C}^I) = (\mathrm{h}\mathcal{C})^I$ because I is a discrete index set, and therefore the cone $(x, \{p_i\})$ viewed in $\mathrm{h}\mathcal{C}$ induces

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c, x) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{h}\mathcal{C}^I}(c, (x_i)_{i\in I}) = \prod_{i\in I} \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(c, x_i),$$

showing that that $x = \prod_i x_i$ in $\mathrm{h}\mathcal{C}$. There are two warnings:

1. The natural map $\mathcal{C} \to \mathrm{h}\mathcal{C}$ doesn't preserve products in our generalized sense, because it doesn't even preserve terminal objects in general, as $\mathcal{C} \to \mathrm{h}\mathcal{C}$ may not be final. For instance if $\mathcal{C}$ were an ∞-groupoid, then finality would imply that $\mathcal{C}$ is a 1-category (Remark 1.0.3).
2. By duality $\mathcal{C} \to \mathrm{h}\mathcal{C}$ also preserves coproducts which exist in $\mathcal{C}$, but it generally doesn't preserve (co)limits of non-discrete diagrams (unless $\mathcal{C}$ is a 1-category, in which case the map is an equivalence). Firstly, a cone over the diagram in the homotopy category may not admit a lift to an ∞-categorical cone, and even if it does, there may be two non-homotopic lifts. The failure of this property for pullbacks and pushouts is one of the things which will make stable ∞-categories much more powerful than triangulated categories.

Example 1.3.20 If $\mathcal{C}$ is an ∞-category and $e \in \mathcal{C}$ is a terminal object. For each object x, given any choice of map $t \colon x \to e$, the pair $x \xleftarrow{1_x} x \xrightarrow{t} e$ exhibits $x \simeq x \times e$. Indeed,

$$\mathrm{Cone}_{\mathcal{C}}(\{x, e\}) = \mathcal{C} \underset{\mathcal{C}\times\mathcal{C}}{\times} (\mathcal{C}/x \times \mathcal{C}/e) = \mathcal{C}/x \times_{\mathcal{C}} \mathcal{C}/e.$$

Since $\mathcal{C}/e \to \mathcal{C}$ is a trivial fibration, so too is $\mathrm{Cone}_{\mathcal{C}}(\{x, e\}) \to \mathcal{C}/x$, so any cone with the first component as 1_x will be terminal. Very slightly more generally, a cone $x \xleftarrow{u} x' \to e$ is limiting iff u is an equivalence, because an object (x', u) of $\mathcal{C}/x$ is terminal iff u is an equivalence.

Example 1.3.21 Let $x_0 \in \mathcal{C}$ be an object for which the product $x_0 \times c$ exists for all $c \in \mathcal{C}$. Apply Example 1.3.14 to the functor $\mathcal{C} \to \underline{\mathrm{Hom}}(\{0, 1\}, \mathcal{C})$ sending $c \mapsto (x_0, c)$, so we obtain a trivial fibration $\mathcal{E} \to \mathcal{C}$ where $\mathcal{E}$ is the space of triples (p, c, u) where u exhibits $p = x_0 \times c$. Choosing a section provides a functor assigning c to a product $x_0 \times c$ with its limit cone, and projecting to $\mathcal{C}$ we obtain a product functor $x_0 \times (-) \colon \mathcal{C} \to \mathcal{C}$.

For intsance, If $x_0 = e$ is a terminal object, then first choose a section of $\mathcal{C}//e \to \mathcal{C}$ to obtain a functorial map $t_\bullet \colon \mathcal{C} \to \mathrm{Arr}(\mathcal{C})$, then we obtain a functor

$$c \mapsto (c, c, (t_c, c))$$

where the third component means $(t_\bullet, \mathrm{id}_{\mathcal{C}}) \colon \mathcal{C} \to \underline{\mathrm{Hom}}(\Delta^1 \times \{0,1\}, \mathcal{C})$. By the previous example, the induced map factors through $\mathcal{C} \to \mathcal{E}$, and this gives a section of the trivial fibration $\mathcal{E} \to \mathcal{C}$. Since the space of sections is contractible, there is a natural equivalence between any two sections, and in particular $e \times (-) \simeq \mathrm{id}_{\mathcal{C}}$ for any choice of product functor $e \times (-)$.

In the case of products in a locally small ∞-category $\mathcal{C}$, we can re-express the universal property similarly to how, in ordinary categories, we may write $\mathrm{Hom}_C(a, b \times c) = \mathrm{Hom}_C(a,b) \times \mathrm{Hom}_C(a,c)$. The only difference is that we use $\mathcal{S}$ instead of sets.

Construction 1.3.22 Let I be a (discrete) **U**-small index set. We define the *product functors*

$$\prod_{i \in I} \mathcal{S}_\bullet \to \mathcal{S}_\bullet \qquad \prod_{i \in I} \mathcal{S} \to \mathcal{S}$$

as follows: They will be the two maps involved in classifying data for the left fibration $\prod_i \mathcal{S}_\bullet \to \prod_i \mathcal{S}$, so that we have a cartesian square on the left:

$$\begin{array}{ccc} \prod_{i\in I} \mathcal{S}_\bullet & \xrightarrow{\prod_I} & \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \prod_{i \in I} \mathcal{S} & \xrightarrow{\prod_I} & \mathcal{S} \end{array} \qquad \begin{array}{ccc} P & \longrightarrow & \prod_{i\in I} \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow \\ \Delta^n & \longrightarrow & \prod_{i \in I} \mathcal{S} \end{array}$$

But, we will give an explicit construction of these maps via Lemma 1.1.48, namely by specifying preferred **U**-small pullbacks in all diagrams of the form on the right above for all n-simplices in the product. In particular, the universal left fibration $\mathcal{S}_\bullet \to \mathcal{S}$ already provides pullbacks $P_i = \Delta^n \times_{f_i, \mathcal{S}} \mathcal{S}_\bullet$ where f_i is the ith factor. We form P as the pullback in

$$\begin{array}{ccccc} P & \longrightarrow & \prod_{i\in I} P_i & \longrightarrow & \prod_{i\in I} \mathcal{S}_\bullet \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^n & \longrightarrow & \prod_{i\in I} \Delta^n & \longrightarrow & \prod_{i\in I} \mathcal{S} \end{array}$$

uniquely so that $P \subset \prod_i P_i$ is a subobject of the product. Explicitly, it is the subobject on which all maps to Δ^n agree. For instance, if $\{X_i\}$ is a collection of **U**-small ∞-groupoids in $\mathcal{S}$, corresponding to the case that $n = 0$ above, so $P_i = X_i$, then $P = \prod_i X_i$. We also note that the property on simplices above generalizes to all maps $A \to \mathcal{S}$: if we have a family of maps $F_i \colon A \to \mathcal{S}$ classifying left fibrations $X_i \to A$, then pasting shows that

$$\prod_i (F_i) \coloneqq A \xrightarrow{(F_i)_{i\in I}} \prod_I \mathcal{S} \xrightarrow{\prod} \mathcal{S}$$

classifies the pullback projection $A \times_{A^I} \prod_i X_i \to A$. The projections $A \times_{A^I} \prod_i X_i \to X_i$ correspond (essentially uniquely) to natural transformations $\pi_i \colon \prod_i(F_i) \to F_i$.

Recall that there is a canonical choice of terminal object e_0 in $\mathcal{S}$, namely the one classifying Δ^0 with all of the chosen products $\Delta^n \times \Delta^0$ simply equal to Δ^n.

Proposition 1.3.23 *Let I be a small set, $\mathcal{C}$ a locally small ∞-category, and $p_i \colon x \to x_i$ an I-indexed family of maps in $\mathcal{C}$.*

1. *There is an essentially canonical natural transformation*

$$\Phi \colon \operatorname{Map}_{\mathcal{C}}(-, x) \to \prod_{i \in I} \big(\operatorname{Map}_{\mathcal{C}}(-, x_i) \big)$$

   in $\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$*, whose components* $\operatorname{Map}_{\mathcal{C}}(c, x) \to \prod_i \operatorname{Map}_{\mathcal{C}}(c, x_i)$ *are identified in $\mathcal{H}$ with the map induced by the maps* $\operatorname{Map}_{\mathcal{C}}(c, x) \to \operatorname{Map}_{\mathcal{C}}(c, x_i)$.

2. *The cone $(x, (p_i)_{i \in I})$ over the discrete diagram I is a limit cone if and only if the natural transformation of (1) is an equivalence.*

Proof We apply our description of $\operatorname{Cone}_{\mathcal{C}}(\{x_i\})$ above. Since $\mathcal{C}/x_i$ is homotopy classified $\operatorname{Map}_{\mathcal{C}}(-, x_i)$, it is clear that $\operatorname{Cone}(\{x_i\}) \to \mathcal{C}$ is homotopy classified by $\prod_i \operatorname{Map}_{\mathcal{C}}(-, x_i)$ as defined in Constr. 1.3.22. The cone projections $\operatorname{Cone}_{\mathcal{C}}(\{x_i\}) \to \mathcal{C}/x_i$ correspond to the product projections $\prod_i \operatorname{Map}_{\mathcal{C}}(-, x_i) \to \operatorname{Map}_{\mathcal{C}}(-, x_i)$ from the same construction.

Now by Lemma 1.1.3, the inclusion $\{1_x\} \hookrightarrow \mathcal{C}/x$ induces the trivial fibration

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}/x, \operatorname{Cone}_{\mathcal{C}}(\{x_i\})) \to \operatorname{Cone}_{\mathcal{C}}(\{x_i\})_x.$$

So if $\lambda \coloneqq (x, (p_i))$ is the given cone, this corresponds to an essentially unique map $\mathcal{C}/x \to \operatorname{Cone}(\{x_i\})$, and under the equivalence

$$\operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})\Big(\operatorname{Map}_{\mathcal{C}}(-, x), \prod_{i \in I} \operatorname{Map}_{\mathcal{C}}(-, x_i) \Big) \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}/x, \operatorname{Cone}_{\mathcal{C}}(\{x_i\}))$$

of Cor. 1.2.26, this corresponds to a canonical (up to homotopy) map Φ as in (1). The characterization of this map up to homotopy is by Lemma 1.A.7, because both the composition $\pi_i \circ \Phi$ as well as $p_{i,*} \colon \mathcal{X}(x) \to \mathcal{X}(x_i)$ send 1_x to p_i. For (2), recall by Cor. 1.A.8 that Φ is an equivalence iff the corresponding object under the Yoneda lemma is terminal. Under our homotopy identifications, this is precisely λ within $\operatorname{Cone}_{\mathcal{C}}(\{x_i\})$. □

Example 1.3.24 In the next section we will show that small (co)limits exist in $\mathcal{S}$. Therefore, suppose that objects x_i of $\mathcal{S}$ classify small ∞-groupoids X_i. Let x be the supposed categorical product, and apply the previous result; evaluating the natural equivalence Φ at a terminal object e, and using $\operatorname{Map}_{\mathcal{S}}(e, -) \simeq \mathrm{id}_{\mathcal{S}}$, we deduce that $x \simeq \prod_i x_i$, with the cone being supplied by $\pi_i \colon \prod_i x_i \to x_i$. In particular the product in $\mathcal{S}$ is the literal product $\prod_i X_i$ of ∞-groupoids.

Our next topic is pullbacks. Let $\mathcal{C}$ be an ∞-category. Recall that a *commutative square* in $\mathcal{C}$ is a functor $f\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$, which we usually represent as a diagram:

$$\begin{array}{ccc} f(0,0) & \longrightarrow & f(1,0) \\ \downarrow & & \downarrow \\ f(0,1) & \longrightarrow & f(1,1) \end{array}$$

Depending on context, we either view this as an object of $\mathcal{C}^{\square} \coloneqq \underline{\mathrm{Hom}}(\Delta^1 \times \Delta^1, \mathcal{C})$, or a morphism in $\mathrm{Arr}(\mathcal{C})$. If we use the description $\Delta^1 \times \Delta^1 = \Delta^2 \times_{\Delta^{\{0,2\}}} \Delta^2$, then we obtain the equivalent description

$$\mathcal{C}^{\square} \cong \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C}) \times_{\underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})} \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C}).$$

In other words a commutative square consists of four objects as the endpoints of a square, four morphisms between them, a diagonal map ($f_{00} \to f_{11}$), and 2-simplices witnessing that $f_{00} \to f_{10} \to f_{11}$ and $f_{00} \to f_{01} \to f_{11}$ compose to the diagonal. In particular it is clear that a commutative diagram induces a commutative diagram in $\mathrm{h}\mathcal{C}$.

There are two possible meanings of a diagram such as

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ \downarrow & & \downarrow & & \downarrow \\ x' & \longrightarrow & y' & \longrightarrow & z' \end{array}$$

in an ∞-category $\mathcal{C}$. First, we could mean two adjacent commutative squares which share the arrow $y \to y'$, i.e. a map $\Lambda^2_1 \times \Delta^1 \to \mathcal{C}$. This is also a pair of composable arrows in $\mathrm{Arr}(\mathcal{C})$. Alternatively, this could also mean the data of a map $\Delta^2 \times \Delta^1 \to \mathcal{C}$ such that e.g. $\Lambda^2_1 \times \Delta^1 \to \mathcal{C}$ is the shown restriction. We call such data a *commutative rectangle in* $\mathcal{C}$. Thus it comes with adjacent squares, but also a choice of composition for both rows, and commutavity data for the composite morphisms to form a commutative square. Since $\Lambda^2_1 \times \Delta^1 \to \Delta^2 \times \Delta^1$ is inner anodyne, the restriction map

$$\underline{\mathrm{Hom}}(\Delta^2 \times \Delta^1, \mathcal{C}) \to \underline{\mathrm{Hom}}(\Lambda^2_1 \times \Delta^1, \mathcal{C}) = \mathcal{C}^{\square} \times_{\mathrm{Arr}(\mathcal{C})} \mathcal{C}^{\square}$$

is a trivial fibration. Thus given adjacent commutative squares, there is a contractible ∞-groupoid of coherent composites to a rectangle.

Remark 1.3.25 In ordinary category theory, the data of a commutative square is the same as a cone over the lower-right cospan. This holds for ∞-categories but only up to equivalence. We use the $\mathrm{Cone}^{\diamond}_{\mathcal{C}}$ model. Given a cospan $a \to b \leftarrow c$ in $\mathcal{C}$, i.e. a diagram of shape Λ^2_2, a cone is a morphism from a constant diagram to the diagram in $\underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})$, or in other words adjacent commutative squares:

$$\begin{array}{ccccc} x & \xrightarrow{1_x} & x & \xleftarrow{1_x} & x \\ \downarrow & \searrow & \downarrow & \swarrow & \downarrow \\ a & \longrightarrow & b & \longleftarrow & c \end{array}$$

This is the data of a commutative square from x to the cospan precisely when the two triangles $x \to x \to b$ are both degenerate. Here's how to turn this into a map:

1. First, we have a map $\varphi\colon \Delta^1 \times (\Delta^1 \times \Delta^1) \to \Delta^1 \times \Delta^1$ defined by
$$\varphi\big(\epsilon, (i,j)\big) = \begin{cases} (0,0) & \epsilon = 0 \\ (i,j) & \epsilon = 1 \end{cases}$$
This is an order-preserving map: obviously its restrictions to $\epsilon = 0, 1$ are order preserving, and if $(0,(i,j)) \le (1,(i,j))$, then $(0,0) \le (i,j)$. This is also described as a homotopy from the inclusion of $(0,0)$ to the identity.

2. Write $\Lambda^2_2 = \Delta^1 \cup_{\{1\}} \Delta^1$, so that $\Delta^0 * \Lambda^2_2 = \Delta^2 \cup_{\Delta^{\{0,2\}}} \Delta^2 = \Delta^1 \times \Delta^1$. In particular we have the inclusion $\Lambda^2_2 \hookrightarrow \Delta^1 \times \Delta^1$, and we restrict along it to obtain $\varphi\colon \Delta^1 \times \Lambda^2_2 \to \Delta^1 \times \Delta^1$, this is, up to homotopy, the quotient map collapsing the spots where the 1_x's go in the diagram above. Indeed we have cocartesian squares
$$\begin{array}{ccccccc} 0 \times \Lambda^2_2 & \longrightarrow & (\Delta^0 \times \Lambda^2_2) \sqcup \Lambda^2_2 & = & \partial\Delta^1 \times \Lambda^2_2 & \longrightarrow & \Delta^1 \times \Lambda^2_2 \\ \downarrow & \ulcorner & \downarrow & & \downarrow & \ulcorner & \downarrow \\ \Delta^0 & \longrightarrow & \Delta^0 \sqcup \Lambda^2_2 & = & \Delta^0 \sqcup \Lambda^2_2 & \longrightarrow & \Delta^0 \diamond \Lambda^2_2 \end{array}$$
by definition. Under the canonical equivalence $\Delta^0 \diamond \Lambda^2_2 \xrightarrow{\sim} \Delta^0 \Lambda^2_2 = \Delta^1 \times \Delta^1$, the right-hand map goes precisely to φ, and the bottom map is the inclusion of $(0,0)$. In other words
$$\begin{array}{ccc} 0 \times \Lambda^2_2 & \hookrightarrow & \Delta^1 \times \Lambda^2_2 \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ \Delta^0 & \xrightarrow{(0,0)} & \Delta^1 \times \Delta^1 \end{array}$$
is homotopy cocartesian as the upper map is a monomorphism.

3. Thus for any ∞-category $\mathcal{C}$ we get a homotopy cartesian diagram which forms the lower square in the diagram:
$$\begin{array}{ccccc} \mathcal{C}^{\square} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})} D & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})//D & \longrightarrow & \{D\} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathcal{C}^{\square} & \xrightarrow{\varphi^*} & \underline{\mathrm{Hom}}(\Delta^1 \times \Lambda^2_2, \mathcal{C}) & \xrightarrow[\mathrm{ev}_1]{} & \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C}) \\ {\scriptstyle \mathrm{ev}_{0,0}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0} & & \\ \mathcal{C} & \xrightarrow{c_{\Lambda^2_2}} & \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C}) & & \end{array}$$
Now let $D = (a \to b \leftarrow c)$ be a cospan. Form the right-hand pullback above, and hence the top pullback. ev_1 is a Joyal fibration and so is $\mathrm{ev}_1\varphi^*$ because it is induced by the monomorphism $\Lambda^2_2 \hookrightarrow \Delta^1 \times \Delta^1$. Thus by the pasting lemma both upper squares are homotopy cartesian. Hence the

left-hand outer rectangle is homotopy cartesian. Since $\underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})//D \to \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})$ is a right fibration, the induced map

$$\mathcal{C}^{\square} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} \{D\} \to \mathcal{C} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} //D = \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D)$$

is an equivalence of ∞-categories. We write the LHS as $\mathcal{C}^{\square}_{a\to b\leftarrow c}$.

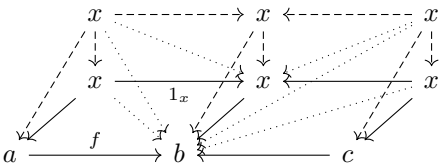
4. We may also fix the apex of the corresponding cone. Namely, since $\mathrm{ev}_{0,0}$ is also a Joyal fibration, so adding the pullback along $\{x\} \to \mathcal{C}$ and another application of the pasting lemma will show that

$$\mathcal{C}^{\square,x}_{a\to b\leftarrow c} \to \mathrm{Cone}^{\diamond}_{\mathcal{C}}(x, a \to b \leftarrow c)$$

is an equivalence of (∞-groupoids).

5. Also note that that both non-degenerate 2-simplices of $\Delta^1 \times \Delta^1$ are in the image of φ, so it is degreewise surjective. Thus the map φ^* in the diagram above is a monomorphism, and so too is its pullback over D. Thus the equivalence $\mathcal{C}^{\square}_{a\to b\leftarrow c} \to \mathrm{Cone}^{\diamond}_{\mathcal{C}}(a \to b \leftarrow c)$ composed with a projection is a monomorphism, so it is itself a trivial cofibration.

We note that this is not a fibration, so the canonical associated contractible space is the homotopy fiber as opposed to the strict fiber. However, if ever we are given a cone as adjacent commutative squares shown above, this comes with maps $x \to a, x \to b, x \to c$. Then We may always take a representative in the homotopy fiber with the same morphisms: We start with a bottom face diagram:



We want to find an extension so that the back face is constant at x, the standing triangles are all degenerate at the given maps $x \to a, b, c$, and both 2-simplices $x \to x \to b$ on the top face are degenerate at $x \to b$. Since we may glue along the middle 2-simplex, we can do each side individually, so focus on the left side.

Now we apply Example 1.A.13: this is exactly what we need because we have specified the bottom and back squares, assembling into $\Lambda^2_1 \times \Delta^1 \to \mathcal{C}$, and the left- and right-hand 2-simplices, defining the map on $\Delta^2 \times \partial\Delta^1$. Finally, $\Delta^{[00,01,21]}$ is exactly the 2-simplex $[x, x, b]$ lying on the top face, which we can set to be degenerate at $x \to b$. This is compatible with our choice of back- and right-face, so the result gives an extension to $\Delta^2 \times \Delta^1$ with the desired properties. Note that we have exhibited a morphism between the two cones in $\mathrm{Cone}(x, a \to b \leftarrow c)$; this is an ∞-groupoid, hence gives an equivalence in $\mathrm{Cone}(a \to b \leftarrow c)$, and thus in $\mathrm{Cone}^{\diamond}(a \to b \leftarrow c)$ as well.

Proposition 1.3.26 *Let $\mathcal{C}$ be an ∞-category and*

$$\begin{array}{ccc} x & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \longrightarrow & b \end{array}$$

be a commutative diagram in $\mathcal{C}$. The following conditions are equivalent:

1. *The square is a terminal object of $\mathcal{C}^{\square}_{a\to b\leftarrow c}$.*
2. *Its image in $\mathrm{Cone}^{\diamond}_{\mathcal{C}}(a \to b \leftarrow c)$ exhibits x as the limit of $a \to b \leftarrow c$.*
3. *The diagram $a \to b \leftarrow c$ in $\mathcal{C}$ has a limit $a \times_b c$, and the canonical map $x \to a \times_b c$ is an equivalence in $\mathcal{C}$.*

In this case we say that the square is cartesian.

Proof Clearly (1) = (2), because $\varphi^*\colon \mathcal{C}^{\square}_{a\to b\leftarrow c} \to \mathrm{Cone}^{\diamond}_{\mathcal{C}}(a \to b \leftarrow c)$ is an equivalence, hence preserves and reflects being terminal. 'The' canonical map of (3) is the one induced by taking the corresponding cone under φ^*, which then provides a morphism of cones $(x, \eta) \to (a \times_b c, \mu)$. We know that this being an equivalence is checked in $\mathcal{C}$ because $\mathrm{Cone}^{\diamond}_{\mathcal{C}}(a \to b \leftarrow c) \to \mathcal{C}$ is conservative, which finishes the proof. □

Example 1.3.27 Let $D \xrightarrow{\sim} D'$ be an equivalence of squares in $\mathcal{C}^{\square}$. Then one is cartesian iff the other is. To see this, let $r\colon \mathcal{C}^{\square} \to \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})$ be the canonical Joyal fibration, and extend the equivalence to a map $f\colon I_\sim \to \mathcal{C}^{\square}$. By composing with r, form the pullback $\mathcal{C}^{\square}_f := \mathcal{C}^{\square} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} I_\sim$. Then we have cartesian diagrams:

$$\begin{array}{ccccc} \mathcal{C}^{\square}_{r(D)} & \longrightarrow & \mathcal{C}^{\square}_f & \longleftarrow & \mathcal{C}^{\square}_{r(D')} \\ \downarrow & \lrcorner & \downarrow & \llcorner & \downarrow \\ 0 & \longrightarrow & I_\sim & \longleftarrow & 1 \end{array}$$

The two lower horizontal maps are equivalence, and the center map is a pullback of r, hence a Joyal fibration. Thus the upper horizontal maps are equivalences; by design, f induces a section $\widetilde{f}\colon I_\sim \to \mathcal{C}^{\square}_f$, which supplies an equivalence in $\mathcal{C}^{\square}_f$ between the image of D and the image of D' under the upper horizontal maps. Since they are equivalences they preserve and reflect terminal objects, proving the claim.

Remark 1.3.28 Thus there is a cartesian square including $a \to b \leftarrow c$ iff this cospan has limit in $\mathcal{C}$, and any such cartesian square canonically induces a limit cone. Our previous computation says the following thing: if $(a \times_b c, \mu) = \varprojlim(a \to b \leftarrow c)$, then μ in particular includes maps to a, b, and c, and we can arrange for a cartesian square

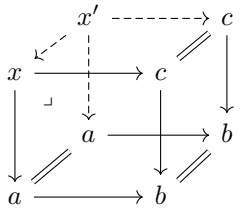
$$\begin{array}{ccc} a \times_b c & \xrightarrow{\mu_c} & c \\ {\scriptstyle \mu_a}\downarrow & \searrow^{\mu_b} & \downarrow \\ a & \longrightarrow & b \end{array}$$

with the three given maps whose image under φ^* is equivalent to $(a \times_b c, \mu)$. However we can not say anything about the two homotopies; for this reason it is usually more convenient to just present our pullbacks as cartesian squares, which include more refined (and less redundant) data than that of a limit cone.

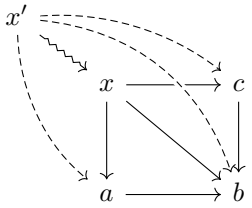
Example 1.3.29 Because of Prop. 1.3.26, if we have a cartesian square as shown on the left,

$$\begin{array}{ccc} x & \longrightarrow & c \\ \downarrow & \lrcorner & \downarrow \\ a & \longrightarrow & b \end{array} \qquad \begin{array}{ccc} x' & \longrightarrow & c \\ \downarrow & & \downarrow \\ a & \longrightarrow & b \end{array}$$

and a commutative diagram as on the right, then there is supposed to be an essentially unique from the right square to the left, fixing the cospan. In other words a diagram



of the form $\Delta^1 \times \Delta^1 \times \Delta^1 \to \mathcal{C}$. Here the bottom and right faces are degenerate at the corresponding maps, but a priori there's no reason that the embedded 2-simplices $x' \to c = c$, $x' \to b = b$, and $x' \to a = a$ have to be degenerate. For more refined data, we can view both squares as objects in $\mathcal{C}_{/D}$ where D is the diagram, in which case we know that x' is terminal. A morphism between them is a map $\Delta^1 * \Lambda^2_2 \to \mathcal{C}$ restricting to the two squares on the endpoints. Since these are both nerves of categories, the result is too; we can view this as $\Delta^0 * (\Delta^0 * \Lambda^2_2) = \Delta^0 * (\Delta^1 \times \Delta^1) \to \mathcal{C}$ (which is also a square in the coslice $x' \backslash \mathcal{C}$). This is also the same data as a pair of 3-simplices in $\mathcal{C}$ glued along a 2-simplex:



Here, the solid square is the cartesian one, which comes with the two 2-simplices in the middle. The dashed arrows are the other commutative diagram, which also come with two 2-simplices between those and the solid arrows $a \to b$ and $c \to b$. The morphism of squares consists of $x' \rightsquigarrow x$, equipped with three 2-simplices recording that $x' \rightsquigarrow x$ makes the maps in the commutative diagram factor through those of the cartesian square; and finally two 3-simplices recording an associativity constraints between the given data.

One useful observation is that we can still obtain strong functoriality statements if we force our cospan limit cones to come from cartesian squares. If we

let $\mathcal{C}^{\square}_{\mathrm{cart}} \subset \mathcal{C}^{\square}$ denote the full subcategory spanned by cartesian squares, then the following lemma reformulates Example 1.3.14.

Lemma 1.3.30 *Let $\mathcal{C}$ and $\mathcal{B}$ be ∞-categories and $f\colon \mathcal{B} \to \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})$ be a functor such that $f(b)$ has a pullback for each $b \in \mathcal{B}$. Then the projection*

$$\mathcal{C}^{\square}_{\mathrm{cart}} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} \mathcal{B} \to \mathcal{B}$$

is a trivial fibration.

Proof Consider the composite $\mathcal{C}^{\square}_{\mathrm{cart}} \to \mathcal{C}^{\square} \to \underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})$, which is a Joyal fibration because we know the latter one is and the former is an isofibration by Example 1.3.27. Thus the map in the lemma is a Joyal fibration. The homotopy cartesian square of Remark 1.3.25(3) shows that

$$\mathcal{C}^{\square} \to \mathcal{C} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C}),\mathrm{ev}_0} \underline{\mathrm{Hom}}(\Delta^1 \times \Lambda^2_2, \mathcal{C}) =: \mathcal{E}$$

is an equivalence of ∞-categories over ev_1 to $\underline{\mathrm{Hom}}(\Lambda^2_2, \mathcal{C})$. Since we have seen that both evaluation maps are Joyal fibrations, the induced map

$$\mathcal{C}^{\square} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} \mathcal{B} \to \mathcal{E} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} \mathcal{B}$$

is an equivalence (over $\mathcal{B}$). A pair (Q, b) in the left-hand side is a cartesian square (with lower-right cospan $f(b)$) if and only if the corresponding cone $\varphi^* Q$ over $f(b)$ is a limit cone, by Prop. 1.3.26. It follows that the equivalence restricts to an equivalence between the full subcategory of the LHS consisting of cartesians squares and of the RHS consisting of limit cones. In other words we have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\square}_{\mathrm{cart}} \times_{\underline{\mathrm{Hom}}(\Lambda^2_2,\mathcal{C})} \mathcal{B} & \xrightarrow{\simeq} & \mathcal{E}' \\ & \searrow \quad \swarrow & \\ & \mathcal{B} & \end{array}$$

where $\mathcal{E}'$ is the aforementioned full subcategory. Example 1.3.14 says that the right-hand map is a trivial fibration, so we conclude that the left-hand map is too by our observation at the start. □

Example 1.3.31 Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories, and $D\colon \Lambda^2_2 \to \mathcal{C}$ be a cospan. We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Cone}_{\mathcal{C}}(D) & \xrightarrow{\simeq} & \mathrm{Cone}_{\mathcal{D}}(D) & \xleftarrow{\simeq} & \mathcal{C}^{\square}_{D} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cone}_{\mathcal{D}}(F_* D) & \xrightarrow{\simeq} & \mathrm{Cone}^{\diamond}_{\mathcal{D}}(F_* D) & \xleftarrow{\simeq} & \mathcal{D}^{\square}_{F_* D} \end{array}$$

with horizontal equivalences. Since everything is a right fibrant object over $\mathcal{C}$, one vertical map is final iff all three are. I.e. preservation of pullbacks can be checked on (and applied to) cartesian squares. It is also clear that one map will preserve terminal objects that exist if all three do, so preservation of existing pullbacks is also checked on (and applied to) cartesian squares.

Example 1.3.32 Let $\mathcal{C}$ be an ∞-category with $b \in \mathcal{C}$. Let $u\colon a \to b$ and $v\colon c \to b$ be morphisms, and D the corresponding diagram $\Lambda^2_2 \to \mathcal{C}$. Then we have the canonical isomorphism $\mathcal{C}/D \cong (\mathcal{C}/d)/\{u, v\}$ because $(\Delta^0 \sqcup \Delta^0) * \Delta^0 = \Lambda^2_2$ Hence we obtain an equivalence of zig-zags:

$$\begin{array}{ccccc} (\mathcal{C}/b)/\{u,v\} & \xrightarrow{\simeq} & \mathrm{Cone}^{\diamond}_{\mathcal{C}/b}(\{u,v\}) & \xleftarrow{\simeq} & \mathrm{Cone}_{\mathcal{C}/b}(\{u,v\}) \\ \cong\downarrow & & & & \\ \mathcal{C}/D & \xrightarrow{\simeq} & \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D) & \xleftarrow{\simeq} & \mathrm{Cone}_{\mathcal{C}}(D) \end{array}$$

Let's unwind the definitions: An object of $\mathcal{C}/D$ is a map $\Delta^0 * \Lambda^2_2 = \Delta^1 \times \Delta^1 \to \mathcal{C}$, so by construction the map $\mathcal{C}/D \to \mathcal{C}//D = \mathrm{Cone}^{\diamond}_{\mathcal{C}}(D)$ is restriction along φ (sending a square to its cone). The vertical isomorphism views the two triangles in a diagram

$$\begin{array}{ccc} x & \longrightarrow & c \\ \downarrow & \searrow & \downarrow{\scriptstyle v} \\ a & \xrightarrow[u]{} & b \end{array}$$

as a pair of morphisms from $x \in \mathcal{C}/b$ (by the diagonal) to u and x to v. Then the first equivalence to $\mathrm{Cone}^{\diamond}_{\mathcal{C}/b}$ again views this as a pair of morphisms from $x \to b$ to u and to v. To summarize, a commutative square canonically induces a product cone in $\mathcal{C}/b$, and the square is cartesian iff the cone exhibits $x \to b$ as the product of u and v in $\mathcal{C}/b$. This is clearly reversible (at least on objects), i.e. commutative diagrams are in literal bijection with cones over $\{u, v\}$ in $\mathcal{C}/b$

Example 1.3.33 As a consequence of the previous example, we see that if $e \in \mathcal{C}$ is a terminal object, then we have a canonical equivalence $x \times_e y \xrightarrow{\sim} x \times y$. More precisely, choosing any map $x \to e$ and $y \to e$, the limit on one side exists iff the one on the other does, and the projections $x \times_e y \to x$ and $x \times_e y \to y$ induce an equivalence $x \times_e y \to x \times y$. This is because $\mathcal{C}/e \to \mathcal{C}$ is a trivial fibration. So, we may choose a cartesian square

$$\begin{array}{ccc} x \times_e y & \longrightarrow & y \\ \downarrow & \searrow & \downarrow \\ x & \longrightarrow & e \end{array}$$

This exhibits $x \times_e y$ as the product of x and y in $\mathcal{C}/e$, hence as the product of x and y in $\mathcal{C}$.

Example 1.3.34 Here is another consequence of Example 1.3.32. Given a commutative square

$$\begin{array}{ccc} x' & \longrightarrow & y' \\ \downarrow & & \downarrow{\scriptstyle \simeq} \\ x & \longrightarrow & y \end{array}$$

in $\mathcal{C}$ where $y' \to y$ is an equivalence, the square is cartesian iff $x' \to x$ is an equivalence. Indeed, it is cartesian iff it exhibits x' as the product of $x \to y$ and $y' \to y$ in $\mathcal{C}/y$, and the latter object is terminal. Thus by Example 1.3.20 this

happens iff the two simplex $[x', x, y]$ is an equivalence in $\mathcal{C}/y$, which happens iff $x' \to x$ is an equivalence because $\mathcal{C}/y \to \mathcal{C}$ is conservative.

Our final result will be the pasting lemma for ∞-categories. This will play an important role in the development of stable ∞-categories. We will prove it using a similar technique to the result above, namely by passing to the join-slice and using geometric properties there. Colloquially, the result says that $x' \times_{y'} (y' \times_{z'} z) \simeq x' \times_{z'} z$.

Proposition 1.3.35 *Let $\mathcal{C}$ be an ∞-category, and*

$$\begin{array}{ccccc} x & \longrightarrow & y & \longrightarrow & z \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ x' & \longrightarrow & y' & \longrightarrow & z' \end{array}$$

be a commutative rectangle $\Delta^2 \times \Delta^1 \to \mathcal{C}$ in which the right-hand square is cartesian. Then the left-hand square is cartesian if and only if the outer square is cartesian.

The result itself assumes a fixed rectangle restricting to the diagram shown. But even if we start with two adjacent diagrams, the condition of the outer square being cartesian is independent of such a choice: We can see it either because the choice is unique up to equivalence in $\underline{\mathrm{Hom}}(\Delta^2, \mathcal{C})$, and hence in $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})$, or alternatively the pasting lemma itself shows independence.

Proof (of Prop. 1.3.35) Let $D\colon \Delta^2 \times \Delta^1 \to \mathcal{C}$ be the rectangle in consideration. Let $K' \subset K \subset \Delta^2 \times \Delta^1$ be, respectively, the full subcategories spanned by objects except for $\{00, 10\}$ and for $\{00\}$. Lastly, let

$$R_1\colon \Lambda^2_2 \hookrightarrow K \qquad R_2, R_{12}\colon \Lambda^2_2 \hookrightarrow K'$$

be the left, right, and outer cospans, respectively. By Example 1.A.14, the maps R_1 and R_{12} are cofinal, so the maps

$$\mathcal{C}_{/D|_K} \to \mathcal{C}_{/D|_{R_1}} \qquad \mathcal{C}_{/D|_{K'}} \to \mathcal{C}_{/D|_{R_{12}}}$$

are equivalences by Remark 1.3.18. Note that $K = K' \cup \Delta^{\{1,2\}} \times \Delta^1$, and the intersection is precisely R_2. This gives the bicartesian square on the left,

$$\begin{array}{ccc} R_2 & \hookrightarrow & \{10\} * R_2 \\ \downarrow & \ulcorner & \downarrow \\ K' & \hookrightarrow & K \end{array} \qquad \begin{array}{ccc} \mathcal{C}_{/D|_K} & \longrightarrow & (\mathcal{C}_{/D|_{R_2}})_y \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C}_{/D|_{K'}} & \longrightarrow & \mathcal{C}_{/D|_{R_2}} \end{array}$$

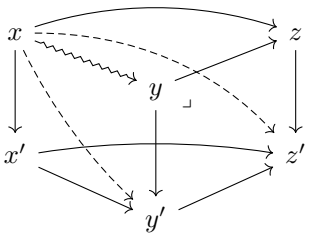
which translates to the one on the right by adjunction and associativity of join. Here, the object y of the slice refers to the object corresponding to the right-hand square in the rectangle, and since this square is cartesian the right-hand

vertical map is a trivial fibration. To summarize, all three maps in

$$\begin{array}{ccc} \mathcal{C}_{/D|_K} & \xrightarrow{\simeq} & \mathcal{C}_{/D|_{R_1}} \\ {\scriptstyle\simeq}\downarrow & & \\ \mathcal{C}_{/D|_{K'}} & \xrightarrow{\simeq} & \mathcal{C}_{/D|_{R_{12}}} \end{array}$$

are equivalences. Since $\{00\} * K = \Delta^2 \times \Delta^1$, the whole diagram D is an object of $\mathcal{C}_{/D|_K}$, and its two images in the diagram are the objects corresponding to the left-hand square and the outer square. In particular one is terminal iff the other is. □

Remark 1.3.36 In ordinary category theory, practical applications of the pasting lemma often consist a very slight variation on the statement above, which is that given a cartesian square and a factorization of the bottom arrow, we get a factorization of the whole square. The best possible translation of this to ∞-categories holds: Our setup is a diagram



in $\mathcal{C}$, where we are given the cartesian square on the right face, as well as the following data:

(i) The bottom face 2-simplex recording $x' \to y' \to z'$.

(ii) The back face commutative square with vertices $x, z, x'z'$.

(iii) (Optionally) The left-hand 2-simplex recording $x \to x' \to y'$.

The statement is that this data extends (essentially uniquely) to a rectangle $\Delta^2 \times \Delta^1 \to \mathcal{C}$. The pasting lemma then implies that if the back face square is cartesian, then we have the essentially canonical map $x \rightsquigarrow y$ making the left face cartesian. *Proof* If the left-hand 2-simplex of (*iii*) is not already chosen, then fill it. In the Δ^3 spanned by $[x, x', y', z']$, we have supplied the faces $[x, x', y']$, $[x', y', z']$, and $[x, x', z']$ (as part of the back face square). This corresponds to Λ^3_1, so we can fill to $[x, x', y', z']$. In particular this has a face $[x, y', z']$ that agrees with the given $[x, z, z']$ on the overlap $[x, z']$, which is the data of a commutative square with vertices x, z, y', z'. By the cartesian property and Example 1.3.29, we obtain a map $x \rightsquigarrow y$ along with (in particular) the following data:

1. A 2-simplex $[x, y, z']$, along which the following two 3-simplices are glued.
2. A 3-simplex $[x, y, y', z']$ such that $[x, y', z']$ agrees with the prior one.

3. A 3-simplex $[x, y, z, z']$ such that $[x, z, z']$ agrees with the prior one.

Thus under the identification $\Delta^2 \times \Delta^1 = \Delta^3 \cup_{\Delta^2} \Delta^3 \cup_{\Delta^2} \Delta^3$, the simplices

$$[x, x', y', z'] \qquad [x, y, y', z'] \qquad [x, y, z, z']$$

agree on intersections, hence glue to a unique commutative rectangle incorporating all of the given data.

1.3.3 More on adjunctions and (co)limits

In this subsection, we will prove significantly more powerful results than where we have stopped in the first section. We will give more precise criteria for adjunctions and (co)limits, and introduce techniques to generalize classical results of category theory, such as the (co)completeness of $\mathcal{S}$, that right adjoints preserve limits, and more. This is the most technical part of this section.

Appendix 1.A More on ∞-categories

1.A.1 Mapping spaces in the homotopy category

In this section there are some basic but annoying verifications we carry out, especially regarding mapping spaces. The premise is that there are a few different constructions for mapping spaces of an ∞-category, each with different functoriality properties. In all cases we have verified that taking π_0 recovers the right thing, which often suffices for the proofs. However there is usually something much stronger than that, which is that the constructions are well-defined on the equivalences classes in $\mathcal{H}$, the homotopy category of Kan complexes.

Throughout this section, we use the convention that if $\mathcal{C}$ is an ∞-category, then $\mathcal{C}(x, y)$ refers to both the space from earlier, and the corresponding object of the homotopy category $\mathcal{H}$, or $\mathcal{H}_{\mathrm{Joy}}$. But outside of this section - and because of the results proven here - we will be comfortable using $\mathrm{Map}_{\mathcal{C}}(x, y)$ as an object of the homotopy category. First we recall the relevant objects:

1. $\mathcal{C}(x, y) = \{(x, y)\} \times_{\underline{\mathrm{Hom}}(\partial\Delta^1, \mathcal{C})} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})$, and similarly we define
$$\mathcal{C}(x, y, z) = \{(x, y, z)\} \times_{\mathcal{C}\times\mathcal{C}\times\mathcal{C}} \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C}),$$
i.e. the space of all morphisms $x \to z$ equipped with homotopy coherent factorizations through y.

2. We have a canonical equivalence $\mathcal{C}/y \to \mathcal{C}//y$ of right fibrations over $\mathcal{C}$. Since the literal fiber of $\mathcal{C}/y$ over x is $\mathcal{C}(x, y)$, this induces a canonical equivalence $(\mathcal{C}/y)_x \to \mathcal{C}(x, y)$ for each x. In particular this provides a canonical isomorphism in $\mathcal{H}$. We remark the following from the definition: $\mathcal{C}/y \to \mathcal{C}//y$ is a bijection on objects, which in both categories are simply pairs (x, f) where $f\colon x \to y$ is a morphism in $\mathcal{C}$. A morphism in $\mathcal{C}/y$ between $(x, f) \to (x', f')$ is a 2-simplex σ with boundary
$$\begin{array}{ccc} x & \xrightarrow{u} & x' \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & y \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{u} & x' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ y & \xrightarrow{1_y} & y \end{array}$$
on the left above. A morphism in $\mathcal{C}//y$ between the same objects is a commutative diagram in $\mathcal{C}$ of the form to the right above. The natural functor $\mathcal{C}/y \to \mathcal{C}//y$ sends $u\colon (x, f) \to (x', f')$ to the corresponding commutative diagram where the diagonal $x \to y$ is f, one composite $1_y f \sim f$ is recorded by the degenerate 2-simplex $s_1(f)$. The other composite is recorded by the given simplex itself. Another way to phrase this is as precomposition along $\Delta^1 \times \Delta^1 \to \Delta^2$, induced by $(i, j) \mapsto i + j$.

3. Let $\mathbf{U}$ be a universe so that $\mathcal{C}$ is locally $\mathbf{U}$-small. Choose a mapping space $(\mathrm{Map}_{\mathcal{C}}, \sigma_{\mathcal{C}})$ for $\mathcal{C}$ valued in $\mathcal{S}$. This comes with the data of an equivalence $\mathrm{S}(\mathcal{C}) \to p^*_{\mathrm{univ}} \mathrm{Map}_{\mathcal{C}}$ of left fibrations over $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, and hence furthermore canonical equivalences
$$x\backslash\mathcal{C} \to {}_x\mathrm{S}(\mathcal{C}) \to p^*_{\mathrm{univ}} \mathrm{Map}_{\mathcal{C}}(x, -)$$

of left fibrations over $\mathcal{C}$. We usually write $\mathrm{Map}_{\mathcal{C}}(x,y)$ both for the object of $\mathcal{S}$, and for the left fibrant object (Kan complex) over Δ^0 classified by it, which is uniquely part of the data of this object. In particular this induces a canonical zig-zag

$$\mathcal{C}(x,y) \leftarrow (x\backslash\mathcal{C})_y \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}}(x,y)$$

of equivalences, and hence a canonical isomorphism $\mathcal{C}(x,y) \cong \mathrm{Map}_{\mathcal{C}}(x,y)$ in $\mathcal{H}$.

Next, we write down the various possible notions of composition morphisms. When there are two variances (e.g. $\mathcal{C}/x$ vs. $x\backslash\mathcal{C}$), we will just do one of the two.

1. *Simplicial composition law*: The inner anodyne extension $\Lambda^2_1 \hookrightarrow \Delta^2$ induces for each x,y,z a canonical trivial fibration $\mathcal{C}(x,y,z) \to \mathcal{C}(x,y) \times \mathcal{C}(y,z)$. In particular we may choose a section and compose:

$$\mathcal{C}(x,y) \times \mathcal{C}(y,z) \dashrightarrow \mathcal{C}(x,y,z) \to \mathcal{C}(x,z),$$

where the second map is induced by $\Delta^{\{0,2\}} \hookrightarrow \Delta^2$. Therefore this map is unique in $\mathcal{H}$. We will treat this as the default canonical composition map. If we apply π_0 to this map, then under the canonical identification $\pi_0\mathcal{C}(x,y) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y)$ it corresponds to the composition law

$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y) \times \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(y,z) \to \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,z)$$

in $\mathrm{h}\mathcal{C}$. Indeed, given $f\colon x \to y$ and $g\colon y \to z$, choose a composition $h \sim gf$ witnessed by $\sigma\colon \Delta^2 \to \mathcal{C}$. Thus σ maps to (f,g) under $\mathcal{C}(x,y,z) \to \mathcal{C}(x,y) \times \mathcal{C}(y,z)$, meaning that after applying π_0 the class of (f,g) maps to σ, and then to h.

2. *Fiber transport:* Next there is the special case of getting an induced map from a morphism of the base to a morphism on fibers of a left/right fibration. For this, we start with a morhpism $u\colon x \to x'$ in $\mathcal{C}$, and use the fact that we have maps

$$(\mathcal{C}/y)_x = \underline{\mathrm{Hom}}_{\mathcal{C}}(x, \mathcal{C}/y) \leftarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(u, \mathcal{C}/y) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(x', \mathcal{C}/y) = (\mathcal{C}/y)_{x'}$$

where e.g. $x \equiv (\Delta^0, x), u \equiv (\Delta^1, u)$ over $\mathcal{C}$. Since $x' \to u$ here means $\{1\} \to \Delta^1$, which is right anodyne, the right-hand map above is a trivial fibration by Lemma 1.1.3. Therefore we get an induced map

$$(\mathcal{C}/y)_{x'} \dashrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(u, \mathcal{C}/y) \to (\mathcal{C}/y)_x$$

of Kan complexes. This gives another notion of composition. We could also repeat the same thing but with $\mathcal{C}//y$ instead.

3. *Global transport 1:* Another construction is to get an actual morphism of right fibrations, and then take fibers. To do this, we consider a morphism $u\colon y \to y'$ in $\mathcal{C}$ and take the zig-zag

$$\mathcal{C}/y \leftarrow (\mathcal{C}/y')/(y,u) \to \mathcal{C}/y'.$$

The left-hand map is a trivial fibration by Lemma 1.1.22 and the right-hand map is a right fibration. Thus we get a composite $\mathcal{C}/y \dashrightarrow (\mathcal{C}/y')/(y,u) \to \mathcal{C}/y'$ of right fibrations over $\mathcal{C}$, inducing $(\mathcal{C}/y)_x \to (\mathcal{C}/y')_x$ for all x. We could do this for $\mathcal{C}//(-)$, but the middle term has to remain $(\mathcal{C}//y')/(y,u)$.

4. *Global transport 2:* The more intrinsic form of global transport is to slice over the morphism itself: Namely, given $u\colon y \to y'$, we have a zig-zag
$$\mathcal{C}/y \leftarrow \mathcal{C}/u \to \mathcal{C}/y',$$
and the first map is a trivial fibration by [HCHA, Lemma 3.4.21]. Thus we obtain a composite $\mathcal{C}/y \dashrightarrow \mathcal{C}/u \to \mathcal{C}/y'$. In this case the construction can fully be repeated using $\mathcal{C}//(-)$ instead.

5. *Mapping space functoriality:* With a fixed mapping space $\mathrm{Map}_{\mathcal{C}}\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$, we in particular have morphisms
$$\mathrm{Map}_{\mathcal{C}}(-,y) \to \mathrm{Map}_{\mathcal{C}}(-,y') \qquad \mathrm{Map}_{\mathcal{C}}(x',-) \to \mathrm{Map}_{\mathcal{C}}(x',-)$$
in $\mathrm{Fun}(\mathcal{C},\mathcal{S})$ and $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$ for $x \to x'$ and $y \to y'$. Upon evaluation at x or y we can bring them to $\mathcal{S}$. Moreover, under Cor. 1.2.26 we can view the pointwise evaluations as functors between Kan complexes, or maps of left/right fibrations over $\mathcal{C}$. We showed in Lemma 1.2.56 that the induced bijections
$$\mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,y) = \pi_0\mathcal{C}(x,y) \leftarrow \pi_0((\mathcal{C}/y)_x) \to \pi_0\,\mathrm{Map}_{\mathcal{C}}(x,y),$$
which agree with the ones using $x\backslash\mathcal{C}$, are natural in x and y.

Our first goal in this section is to ensure that, under the previously defined isomorphisms in $\mathcal{H}$, all of these composition constructions actually agree with each other in $\mathcal{H}$, or possibly in $\mathrm{LFib}(\mathcal{C})$ or $\mathrm{RFib}(\mathcal{C})$.

Remark 1.A.1 Here are some initial observations.

1. First we claim that (3) and (4) are *literally* the same, under canonical identification $(\mathcal{C}/y')/(y,u) \cong \mathcal{C}/u$. We simply check that the two maps coincide. By adjunction, the isomorphism is induced by the isomorphism of functors
$$(-*\Delta^0)*\Delta^0 \cong (-)*\Delta^1,$$
and the two point inclusions corresponding to either factor on the left, provide $(\mathcal{C}/y')/u \to \mathcal{C}/y$ or $(\mathcal{C}/y')/u \to \mathcal{C}/y'$.

2. The π_0-description of (1) above does not a priori characterize the composition maps in the homotopy category. However, applying $\pi_0\,\underline{\mathrm{Hom}}(K,-)$ for every simplicial set K does characterize it, since this equals $\mathrm{Hom}_{\mathcal{H}}(K,-)$

on Kan complexes. In particular, we obtain a zig-zag

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(K,\mathcal{C})(x,y,z) & \longrightarrow & \underline{\mathrm{Hom}}(K,\mathcal{C})(x,y)\times\underline{\mathrm{Hom}}(K,\mathcal{C})(y,z) \\ \downarrow & & \\ \underline{\mathrm{Hom}}(K,\mathcal{C})(x,z) & & \end{array}$$

after applying $\underline{\mathrm{Hom}}(K,-)$, and using the canonical identification on mapping spaces where the x,y,z are viewed as constant maps. These are the same maps we started with but for $\underline{\mathrm{Hom}}(K,\mathcal{C})$ instead of $\mathcal{C}$, so taking π_0 of this yields the composition law on $\mathrm{h}\,\underline{\mathrm{Hom}}(K,\mathcal{C})$.

We use these observations to prove the equivalences above. First we show the equivalence of fiber transport and mapping space functoriality, which is easy and true much more generally.

Lemma 1.A.2 *Let A be a simplicial set, $p\colon X\to A$ a left fibration, and $F\colon A\to\mathcal{S}$ a map with an equivalence of left fibrations $X\xrightarrow{\simeq} p^*_{\mathrm{univ}}F$ over A. If $u\colon a\to b$ is a morphism in A, the induced diagram*

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}_A((\Delta^1,u),X) & \longrightarrow & \underline{\mathrm{Hom}}_A(b,X) & \overset{\simeq}{\searrow} & \\ \downarrow & & & & \\ \underline{\mathrm{Hom}}_A(a,X) & \xrightarrow{\simeq} & F(a) & \xrightarrow{F(u)} & F(b) \end{array}$$

commutes in $\mathcal{H}$. In particular the fiber transport map $X_a\to X_b$ corresponds in $\mathcal{H}$ to $u_*\colon F(a)\to F(b)$.

Proof Because of the commutative diagram

$$\begin{array}{ccccc} \underline{\mathrm{Hom}}_A(a,X) & \longleftarrow & \underline{\mathrm{Hom}}_A((\Delta^1,u),X) & \longrightarrow & \underline{\mathrm{Hom}}_A(b,X) \\ \downarrow\simeq & & \downarrow\simeq & & \simeq\downarrow \\ \underline{\mathrm{Hom}}_A(a,p^*_{\mathrm{univ}}F) & \longleftarrow & \underline{\mathrm{Hom}}_A((\Delta^1,u),p^*_{\mathrm{univ}}F) & \longrightarrow & \underline{\mathrm{Hom}}_A(b,p^*_{\mathrm{univ}}F) \end{array}$$

whose vertical maps are equivalences of ∞-groupoids (by Cor. 1.1.7), we may assume that X is literally classified by F. Since the left-hand vertical map in the lemma is a trivial fibration, choose a section $s\colon X_a\to\underline{\mathrm{Hom}}_A(u,X)$. Consider the diagram:

$$\begin{array}{ccccccc} \{0\}\times X_a & \longrightarrow & W & \longrightarrow & X & \longrightarrow & \mathcal{S}_\bullet \\ \downarrow & \nearrow & \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \Delta^1\times X_a & \longrightarrow & \Delta^1 & \longrightarrow & A & \longrightarrow & \mathcal{S} \end{array}$$

The section corresponds by adjunction to a map $\Delta^1\times X_a\to X$ whose projection to Δ^1 over A corresponds to u. Thus it induces a unique map $\Delta^1\times X_a\to W$ over Δ^1, and the 1-fiber $X_a\to W_1=X_b$ is given by the composite

$$X_a\xrightarrow{s}\underline{\mathrm{Hom}}_A(u,X)\to X_b.$$

According to Constr. 1.2.13 this is exactly what we wanted to show. □

Corollary 1.A.3 *Let $\mathcal{C}$ be a locally small ∞-category, x an object, and $u\colon y \to y'$ a morphism in $\mathcal{C}$. The diagram*

$$\begin{array}{ccccc} \mathcal{C}(x,y) & \xleftarrow{\simeq} & (x\backslash\mathcal{C})_y & \xrightarrow{\simeq} & \mathrm{Map}_{\mathcal{C}}(x,y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}(x,y') & \xleftarrow{\simeq} & (x\backslash\mathcal{C})_{y'} & \xrightarrow{\simeq} & \mathrm{Map}_{\mathcal{C}}(x,y') \end{array}$$

commutes in $\mathcal{H}$, in which the three vertical maps are given by simplicial composition, fiber transport, and mapping space functoriality by u respectively. The corresponding results hold for $x\backslash\mathcal{C}$ replaced with $x\backslash\backslash\mathcal{C}$ or $\mathcal{C}/y$.

Proof The previous lemma shows that the right-hand square commutes. Lemma 1.2.56 shows that the outer diagram commutes if we apply π_0, so in particular the left-hand square does after π_0. The equivalences in the left-hand square are induced by $x\backslash\mathcal{C} \xrightarrow{\sim} x\backslash\backslash\mathcal{C}$, and it is clear from Cor. 1.1.7 that fiber transport for $x\backslash\mathcal{C}$ corresponds to fiber transport for $x\backslash\backslash\mathcal{C}$.

Therefore, it suffices to show that fiber transport $(x\backslash\backslash\mathcal{C})_y \to (x\backslash\backslash\mathcal{C})_{y'}$ equals simplicial composition in $\mathcal{H}$. Since we know this holds on π_0, it further suffices to see that for every simplicial set K, fiber transport for $x\backslash\backslash\mathcal{C}$ induces fiber transport for $\underline{\mathrm{Hom}}(K, x\backslash\backslash\mathcal{C}) = x\backslash\backslash\underline{\mathrm{Hom}}(K,\mathcal{C})$ over $\underline{\mathrm{Hom}}(K,\mathcal{C})$. The equality was shown in Prop. 1.1.18. This is purely formal, and follows from the canonical isomorphism

$$\underline{\mathrm{Hom}}(K, \underline{\mathrm{Hom}}_A(X,Y)) \cong \underline{\mathrm{Hom}}_{\underline{\mathrm{Hom}}(K,A)}\left(X, \underline{\mathrm{Hom}}(K,Y)\right)$$

applied to $A = \mathcal{C}$, $X = \Delta^0, \Delta^1$, and $Y = x\backslash\backslash\mathcal{C}$. □

From the original goal it remains to see that global transport agrees with composition, fiber transport, and mapping space functoriality. We will prove a slightly more general relative statement which is useful elsewhere.

Lemma 1.A.4 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between locally small ∞-categories and $u\colon Fx \to d$ be a morphism. The canonical diagram*

$$\begin{array}{ccccc} (\mathcal{C}/d)/(x,u) & \longrightarrow & \mathcal{C}/x & \longrightarrow & \mathcal{C}/Fx \xrightarrow{\simeq} p^*_{\mathrm{univ}}F^* \text{Ł}_{\mathcal{D}}(Fx) \\ \downarrow & & & & \downarrow{\scriptstyle u_*} \\ \mathcal{C}/d & & \xrightarrow{\qquad\simeq\qquad} & & p^*_{\mathrm{univ}}F^* \text{Ł}_{\mathcal{D}}(d) \end{array}$$

commutes in $\mathrm{RFib}(\mathcal{C})$. *As a consequence:*

1. *If $y \to y'$ is a morphism in $\mathcal{C}$, the global transport map $\mathcal{C}/y \to \mathcal{C}/y'$ is identified in* $\mathrm{RFib}(\mathcal{C})$ *with $p^*_{\mathrm{univ}} \text{Ł}_{\mathcal{C}}(y) \to p^*_{\mathrm{univ}} \text{Ł}_{\mathcal{C}}(y')$. In other words, the diagram*

$$\begin{array}{ccccc} \mathcal{C}(x,y) & \xleftarrow{\simeq} & (\mathcal{C}/y)_x & \xrightarrow{\simeq} & \mathrm{Map}_{\mathcal{C}}(x,y) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}(x,y') & \xleftarrow{\simeq} & (\mathcal{C}/y')_x & \xrightarrow{\simeq} & \mathrm{Map}_{\mathcal{C}}(x,y') \end{array}$$

commutes in $\mathcal{H}$, where the vertical maps are induced by simplicial composition, global transport, and mapping space functoriality respectively.

2. *If $u' \colon Fx' \to d$ is another morphism in $\mathcal{D}$, the mapping space of the relative slice $\mathrm{Map}_{\mathcal{C}/d}\big((x',u'),(x,u)\big)$ is equivalent to the homotopy fiber of the composite*

$$\mathrm{Map}_{\mathcal{C}}(x',x) \xrightarrow{F_*} \mathrm{Map}_{\mathcal{D}}(Fx',Fx) \xrightarrow{u_*} \mathrm{Map}_{\mathcal{D}}(Fx',d)$$

over the object corresponding to $u' \colon Fx' \to d$.

Proof We remark that both (1) and (2) are indeed immediate consequences: For (1), it suffices to show the right-hand square since the outer one already commutes by the previous result. This follows from taking $F = \mathrm{Id}_{\mathcal{C}}$ and the definition of global transport. For (2), we observe that $\mathcal{C}/x \to \mathcal{C}/Fx$ on the fiber over x' is $\mathcal{C}(x',x) \to \mathcal{D}(Fx',Fx)$, so the composite $\mathcal{C}/x \to p^*_{\mathrm{univ}}F^* \pitchfork_{\mathcal{D}}(d)$ indeed represents the displayed morphism on the fiber over x. Since $(\mathcal{C}/d)/(x,u) \to \mathcal{C}/x$ is a trivial fibration, this displayed map is isomorphic in $\mathcal{H}$ to the homotopy fiber of $\big((\mathcal{C}/d)/(x,u)\big)_{x'} \to (\mathcal{C}/d)_{x'}$, which is a right fibration. Then taking the fiber over u' computes the homotopy fiber, as well as the ∞-groupoid

$$\big((\mathcal{C}/d)/(x,u)\big)_{(x',u')} \simeq (\mathcal{C}/d)\big((x',u'),(x,u)\big),$$

proving the claim.

The object $1_{(x,u)}$ of $(\mathcal{C}/d)/(x,u)$ is terminal, so the corresponding right anodyne extension $\Delta^0 \to (\mathcal{C}/d)/(x,u)$ induces a trivial fibration

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_{\mathcal{C}}((\mathcal{C}/d)/(x,u), p^*_{\mathrm{univ}}F^* \pitchfork_{\mathcal{D}}(d)) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{C}}(\Delta^0, p^*_{\mathrm{univ}}F^* \pitchfork_{\mathcal{D}}(d)) \\ & & \| \\ & & \mathrm{Map}_{\mathcal{D}}(Fx,d) \end{array}$$

by Lemma 1.1.3. If we apply π_0 to both sides, the left produces $\mathrm{Hom}_{\mathrm{RFib}(\mathcal{C})}$, which means that to prove that the two directions of the rectangle coincide in $\mathrm{RFib}(\mathcal{C})$ we need only show that $1_{(x,u)}$ maps into the same component of $\mathrm{Map}_{\mathcal{D}}(Fx,d)$ over the x fiber. Taking the x fiber, then by Lemma 1.2.56, applying π_0 produces the diagram:

$$\begin{array}{ccccc} * & \xrightarrow{1_x} & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,x) & \to & \mathrm{Hom}_{\mathrm{h}\mathcal{D}}(Fx,Fx) \\ {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle u_*} \\ \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{D}}(Fx,d) & = & = & = & \underline{\mathrm{Hom}}_{\mathrm{h}\mathcal{D}}(Fx,d) \end{array}$$

This obviously commutes, which completes the proof. □

This finishes the proof that the five composition constructions agree in $\mathcal{H}$. There is one other useful compatibility construction, which will also be related to the map described in (2).

Example 1.A.5 Let us compute the mapping spaces of an arrow category $\mathrm{Arr}(\mathcal{C}) = \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})$. Let $u\colon a \to b$ and $u'\colon a' \to b'$ be morhpisms of $\mathcal{C}$. Since $\Delta^1 \times \Delta^1 = \Delta^2 \cup_{\Delta^{\{0,2\}}} \Delta^2$, we have a pullback diagram:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\Delta^1, \mathrm{Arr}(\mathcal{C})) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^1, \mathcal{C}) \end{array}$$

It follows that if we form the cartesian squares:

$$\begin{array}{ccc} S_u & \longrightarrow & \mathcal{C}(a,b,b') \\ \downarrow & \lrcorner & \downarrow \\ \{u\} \times \mathcal{C}(b,b') & \longrightarrow & \mathcal{C}(a,b) \times \mathcal{C}(b,b') \end{array} \qquad \begin{array}{ccc} T_{u'} & \longrightarrow & \mathcal{C}(a,a',b') \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C}(a,a') \times \{u'\} & \longrightarrow & \mathcal{C}(a,a') \times \mathcal{C}(a',b') \end{array}$$

Then by using $\mathcal{C}(a,b,b') \to \mathcal{C}(a,b') \leftarrow \mathcal{C}(a,a',b')$ we deduce $S_u \times_{\mathcal{C}(a,b')} T_{u'} = \mathrm{Arr}(\mathcal{C})(u,u')$. We observe that this is also a homotopy pullback: Indeed, e.g. $S_u = \{u\} \times_{\mathcal{C}(a,b)} \mathcal{C}(a,b,b')$ can also be written as the pullback of the cospan

$$\{u\} \times \mathcal{C}(a,b') \to \mathcal{C}(a,b) \times \mathcal{C}(a,b') \leftarrow \mathcal{C}(a,b,b'),$$

and this right-hand map is a Kan fibration because $\underline{\mathrm{Hom}}(\Lambda^2_0, \mathcal{C}) \to \underline{\mathrm{Hom}}(\Delta^2, \mathcal{C})$ is a Joyal fibration. Thus $S_u \to \mathcal{C}(a,b')$ is a Kan fibration, so $\mathrm{Arr}(\mathcal{C})(u,u') = S_u \times^{\mathbf{R}}_{\mathcal{C}(a,b')} T_{u'}$. Moreover, both right-hand maps in the squares above are trivial fibrations, so the left-hand ones are too. Choosing sections of the right-hand ones, commutativity in $\mathcal{H}$ of these diagrams shows that we may replace this with the homotopy pullback

$$\begin{aligned} \mathrm{Arr}(\mathcal{C})(u,u') = S_u \times^{\mathbf{R}}_{\mathcal{C}(a,b')} T_{u'} &\simeq \mathcal{C}(b,b') \times^{\mathbf{R}}_{\mathcal{C}(a,b')} \mathcal{C}(a,a') \\ &\simeq \mathrm{Map}_{\mathcal{C}}(b,b') \times^{\mathbf{R}}_{\mathrm{Map}_{\mathcal{C}}(a,b')} \mathrm{Map}_{\mathcal{C}}(a,a'), \end{aligned}$$

where the maps represent composition by u and u'. This makes sense: a morphism from u to u' can either be written as a commutative diagram between u and u', or a pair of maps whose compositions with u, u' coincide $a \to b'$.

Example 1.A.6 This is a relative version of the previous example. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories, and consider the induced evaluation-at-1 map

$$p\colon \mathcal{E} := \mathcal{C} \times_{\mathcal{D}, \mathrm{ev}_0} \underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \to \mathcal{D}.$$

Note that $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{D}) \to \mathcal{D} \times \mathcal{D}$ is a Joyal fibration by [HCHA, Cor. 3.6.4], therefore so is $\mathcal{E} \to \mathcal{D} \times \mathcal{C}$, and $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ is one too because $\mathcal{C}$ is an ∞-category, so p is a Joyal fibration. Suppose we're given objects $(c, u\colon Fc \to d)$ and $(c', u'\colon Fc' \to d')$ of $\mathcal{E}$. Borrowing the notation of the previous example, we have

$$\mathcal{E}(u,u') = S_u \underset{\mathcal{D}(Fc,d')}{\times} T_{u'} \underset{\mathcal{D}(Fc,Fc')}{\times} \mathcal{C}(c,c'),$$

and because $\mathcal{E}$ is defined using evaluation-at-0, we may group the latter term using:

$$\begin{array}{ccccc} T'_{u'} & \longrightarrow & T_{u'} & \longrightarrow & \mathcal{D}(Fc, Fc', d') \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathcal{C}(c,c') \times \{u'\} & \longrightarrow & \mathcal{D}(Fc, Fc') \times \{u'\} & \longrightarrow & \mathcal{D}(Fc, Fc') \times \mathcal{D}(Fc', d') \end{array}$$

Thus $\mathcal{E}(u, u') = S_u \times_{\mathcal{D}(Fc,d')} T'_{u'}$, and induced map on mapping spaces

$$\mathcal{E}(u, u') \to S_u \to \mathcal{D}(Fc, d, d') \to \mathcal{D}(d, d')$$

by p. As in Example 1.A.5, taking sections of the two trivial fibrations to the product spaces we obtain

$$\mathcal{E}(u, u') \simeq \mathcal{D}(d, d') \times^{\mathbf{R}}_{\mathcal{D}(Fc,d')} \mathcal{C}(c, c')$$

where $\mathcal{D}(d, d') \to \mathcal{D}(Fc, d')$ represents a composition by u map, and $\mathcal{C}(c, c') \xrightarrow{F_*} \mathcal{D}(Fc, Fc') \to \mathcal{D}(Fc, d')$ represents the other map. Thus $\mathcal{E}(u, u') \to \mathcal{D}(d, d')$ is given up to homotopy by the projection to that factor from the homotopy pullback.

Our last topic in this section is to complete some more checking work regarding the Yoneda lemma. Our main goals are to check that the constructions of Thm. 1.2.61 and Thm. 1.2.63 are actually the same (up to homotopy), and to give a canonical formula in $\mathcal{H}$ for computing the maps on ∞-groupoids. A nice feature will be that we don't need to re-examine the somewhat annoying and subtle constructions in that section.

Lemma 1.A.7 *Let $\mathcal{C}$ be a locally small ∞-category. For each object x of $\mathcal{C}$ and presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, the bijection*

$$\mathrm{Hom}_{\mathrm{hFun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{よ}_{\mathcal{C}}(x), F) = \pi_0 \mathrm{Map}_{\mathcal{C}}(\text{よ}_{\mathcal{C}}(x), F) \xrightarrow{\sim} \pi_0 F(x)$$

obtained by applying π_0 to the Yoneda isomorphism Thm. 1.2.63(2) sends a natural transformation $\alpha\colon \text{よ}_{\mathcal{C}}(x) \to F$ to $\alpha_x(1_x)$. In particular any two such transformations α, β are homotopic in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ iff $\alpha_x(1_x) = \beta_x(1_x)$ in $\pi_0 F(x)$.

Proof The second statement follows immediately from the first. By the naturality of Thm. 1.2.63 applied to the morphism α, we have a commutative diagram in $\mathrm{h}\mathcal{S} \cong \mathcal{H}$:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{よ}(x), \text{よ}(x)) & \xrightarrow{\simeq} & \mathrm{Map}_{\mathcal{C}}(x, x) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{よ}(x), F) & \xrightarrow{\simeq} & F(x) \end{array}$$

Applying π_0 and our previously established naturality properties provides another commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{hFun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɬ}(x), \text{ɬ}(x)) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(x,x) \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle (\alpha_x)_*} \\ \mathrm{Hom}_{\mathrm{hFun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɬ}(x), F) & \xrightarrow{\simeq} & \pi_0 F(x) \end{array}$$

of sets. In the top left, id maps to α, and in the other direction it maps to $\alpha_x(1_x)$, because Thm. 1.2.63 says that the upper map is inverse to the map induced by the Yoneda embedding. □

In light of this result, we can talk about 'the map $\text{ɬ}(x) \to F$' corresponding to an element of $F(x)$, and it gives a well-defined natural transformation up to homotopy. For instance:

Corollary 1.A.8 *Let $\mathcal{C}$ be a locally small ∞-category, and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ be a presheaf homotopy-classifying a right fibration $p\colon E \to \mathcal{C}$. An object $e \in E$ lying over $x \in \mathcal{C}$ is terminal in E if and only if the natural transformation*

$$\widetilde{e}\colon \text{ɬ}_{\mathcal{C}}(x) \to F \qquad \text{in } \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

corresponding to $e \in E_x \simeq F(x)$ under the Yoneda lemma is an equivalence.

Proof This statement makes sense because the being an equivalence is checked in $\mathrm{hFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$. Since $E \xrightarrow{\sim} p^*_{\mathrm{univ}} F$ preserves terminal objects, we may assume without loss of generality that E is literally classified by F. The induced map $E/e \to \mathcal{C}/x$ is a trivial fibration by Lemma 1.1.22 so we may choose a section $s\colon \mathcal{C}/x \to E/e$ sending $1_x \mapsto 1_e$, and then further compose to obtain $\mathcal{C}/x \to E$, sending $1_x \mapsto e$. By 2/3 this map is an equivalence of right fibrations iff $E/e \to E$ is, i.e. e is terminal. On the other hand it is also an equivalence iff the corresponding map $\widetilde{e}\colon \text{ɬ}(x) \to F$ of classifying maps is an equivalence. But there is a homotopy commutative diagram

$$\begin{array}{ccc} (\mathcal{C}/x)_x & \longrightarrow & E_x \\ \simeq\downarrow & & \downarrow\simeq \\ \text{ɬ}(x,x) & \xrightarrow{\widetilde{e}_x} & F(x) \end{array}$$

which shows that $\widetilde{e}_x(1_x)$ equals the image of e in $\pi_0 F(x)$. By Lemma 1.A.7 this means that $\widetilde{e}$ is the map corresponding to e under the Yoneda lemma. □

Lemma 1.A.9 *Let $\mathcal{C}$ be an essentially small ∞-category and let x be an object. The natural transformation*

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɬ}_{\mathcal{C}}(x), -) \to \mathrm{ev}_x$$

of functors $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ of Constr. 1.2.60 coincides with the Yoneda map at x of Thm. 1.2.63 in $\mathrm{hFun}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}), \mathcal{S})$.

Proof We are going to apply Lemma 1.A.7 to the locally small category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})^{\mathrm{op}}$. We can do this beacuse by definition

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɬ}_{\mathcal{C}}(x), -) = \text{ɬ}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})^{\mathrm{op}}}\big(\text{ɬ}_{\mathcal{C}}(x)\big),$$

so it suffices to check that both constructions in the statement, when we evaluate at $\text{ɬ}_{\mathcal{C}}(x)$ as in

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɬ}_{\mathcal{C}}(x), \text{ɬ}_{\mathcal{C}}(x)) \to \mathrm{Map}_{\mathcal{C}}(x, x),$$

and take the image of $1_{\text{ɬ}_{\mathcal{C}}}$, that both constructions coincide in $\pi_0 \mathrm{Map}_{\mathcal{C}}(x, x)$. For the Yoneda map of Thm. 1.2.63, this is clear because it is required to be a homotopy inverse of the induced map

$$\text{ɬ}_{\mathcal{C}} \colon \mathrm{Map}_{\mathcal{C}}(x, x) \to \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})}(\text{ɬ}_{\mathcal{C}}(x), \text{ɬ}_{\mathcal{C}}(x))$$

which sends $[1_x] \mapsto [1_{\text{ɬ}(x)}]$ on homotopy classes. For Constr. 1.2.60, we showed precisely this statement in Thm. 1.2.61. □

In the following construction, we recall that the equivalence $\mathcal{S}(X, Y) \simeq \underline{\mathrm{Hom}}(X, Y)$ for small ∞-groupoids X and Y, technically depends on a universal choice as a map of Kan complexes, but is canonical in $\mathcal{H}$.

Construction 1.A.10 Let $\mathcal{C}$ be a locally small ∞-category. The Yoneda emebdding $\text{ɬ}_{\mathcal{C}} \colon \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ induces functor $\text{ɬ} \colon \mathrm{h}\mathcal{C} \to \mathrm{hFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ which is unique up to natural isomorphism. Thus we obtain a canonical bifunctor

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})(\text{ɬ}_{\mathcal{C}}(-), -) \colon \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{hFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{H}.$$

On the other hand, $\mathrm{ev} \colon \mathcal{C}^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{S}$ induces a bifunctor

$$\mathrm{ev} \colon \mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{hFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{h}\mathcal{S} \hookrightarrow \mathcal{H}.$$

There is a canonical morphism in $\mathcal{H}$

$$\begin{aligned} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})(\text{ɬ}_{\mathcal{C}}(x), F) &\xrightarrow{\mathrm{ev}_{x,*}} \mathcal{S}\big(\mathrm{Map}_{\mathcal{C}}(x, x), F(x)\big) \\ &\xrightarrow{\sim} \underline{\mathrm{Hom}}\big(\mathcal{C}(x, x), F(x)\big) \\ &\xrightarrow{1_x^*} F(x) \end{aligned}$$

for each x in $\mathcal{C}$ and $F \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$. On objects, a natural transformation $\alpha \colon \text{ɬ}_{\mathcal{C}}(x) \to F$ is sent to $\alpha_x(1_x) \in F(x)$.

Corollary 1.A.11 *Let $\mathcal{C}$ be a locally small ∞-category. The family of morphisms in $\mathcal{H}$ of Constr. 1.A.10 form a natural isomorphism*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})\big(\text{ɬ}_{\mathcal{C}}(-), -\big) \xrightarrow{\sim} \mathrm{ev}$$

of bifunctors $\mathrm{h}\mathcal{C}^{\mathrm{op}} \times \mathrm{hFun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathcal{H}$. Moreover, componentwise this isomorphism is canonically identified with the images of the equivalences of Thm. 1.2.61 and Thm. 1.2.63 in $\mathcal{H}$.

Proof Under the identifications in $\mathcal{H}$ of functorial mapping spaces $\mathrm{Map}_{\mathcal{D}}$ with simplicial mapping spaces $\mathcal{D}(-,-)$, it is clear that on components the map in question coincides with that of Constr. 1.2.60. By Lemma 1.A.9 it also coincides with the fully functorial Yoneda map of Thm. 1.2.63. This implies both that it is a natural transformation of bifunctors, and that it is an isomorphism, completing the proof. □

1.A.2 Lifting properties and problems

Lemma 1.A.12 *Let $\mathcal{D}$ be an ∞-category without any non-identity equivalences (e.g. the nerve of a poset).*

1. *Any Joyal trivial cofibration $i\colon A \to \mathcal{D}$ is an inner anodyne extension.*
2. *Any inner fibration $p\colon \mathcal{C} \to \mathcal{D}$ is a Joyal fibration.*

Proof (2) is obvious by the assumption, because all identities lift so p is an isofibration. For (1), factor i as $A \to \mathcal{C} \to \mathcal{D}$ into an inner anodyne extension followed by an inner fibration. It suffices to show that i is a retract of $A \to \mathcal{C}$, whence that $\mathcal{C} \to \mathcal{D}$ is a trivial fibration by the retract lemma. But it is a Joyal fibration by (2) and a weak categorical equivalence by 2-of-3. □

Example 1.A.13 Label the vertices of $\Delta^2 \times \Delta^1$ by $(i,j) \equiv ij$. The inclusion

$$K \coloneqq \Lambda^2_1 \times \Delta^1 \cup \Delta^2 \times \partial\Delta^1 \cup \Delta^{\{00,01,21\}} \hookrightarrow \Delta^2 \times \Delta^1$$

is an inner anodyne extension. Indeed, since $\Lambda^2_1 \hookrightarrow \Delta^2$ is inner anodyne, so too is

$$K_0 \coloneqq \Lambda^2_1 \times \Delta^1 \cup \Delta^2 \times \partial\Delta^1 \hookrightarrow \Delta^2 \times \Delta^1$$

by [HCHA, Cor. 3.2.4]. Now $K_0 \cap \Delta^{\{00,01,21\}} = \Delta^{\{00,01\}} \cup \Delta^{\{01,21\}}$, so we have a bicartesian square:

$$\begin{array}{ccc} \Delta^{\{00,01\}} \cup \Delta^{\{01,21\}} & \longrightarrow & K_0 \\ \downarrow & & \downarrow \\ \Delta^{\{00,01,21\}} & \longrightarrow & K \end{array}$$

It follows from 2-of-3 that $K \hookrightarrow \Delta^2 \times \Delta^1$ is a weak categorical equivalence, and hence a trivial cofibration. Lemma 1.A.12 then shows that it is in fact inner anodyne, because $\Delta^2 \times \Delta^1$ is a poset.

The following two cases are used in the proof of the pasting lemma. They concern the same simplicial set $\Delta^2 \times \Delta^1$ as in the previous example.

Example 1.A.14 Let $K' \subset K \subset \Delta^2 \times \Delta^1$ be, respectively, the full subcategories spanned by objects except for $\{00, 10\}$ and for $\{00\}$. Then:

1. The inclusion $(01 \to 11 \leftarrow 10) \hookrightarrow K$ is left anodyne.
2. The inclusion $(01 \to 21 \leftarrow 20) \hookrightarrow K'$ is left anodyne.

Proof Since both K and K' exclude 00, they don't have any non-degenerate 3-simplices. It is thus clear from looking at maximal chains that

$$K' = (00 \to 21 \leftarrow 20) \cup \Delta^2 \times 1 \qquad K = \Delta^1 \times 1 \cup \Delta^{\{1,2\}} \times \Delta^1$$

(2) Follows immediately, because the intersection of these two pieces is $\Delta^{\{0,2\}} \times 1$, and $\Delta^{\{0,2\}} \hookrightarrow \Delta^2$ is left anodyne because it preserves the initial object. For (1), we split the inclusion up into the two pieces

$$(01 \to 11 \leftarrow 10) \hookrightarrow \Delta^{\{0,1\}} \times 1 \cup \Delta^{\{1,2\}} \times 1 \hookrightarrow K.$$

The second map is a pushout of $\Lambda^2_1 \times 1 \to \Delta^2 \times 1$, hence is inner anodyne. The first map is a pushout of $1 \times \Delta^1 \to \Delta^{\{1,2\}} \times \Delta^1$, which is left anodyne because $\{0\} \to \Delta^1$ is.

We next prove some simple simplicial lemmas whose proof *is* in Cisinski, but we write them down here for fun and to preserve an extra variant. Recall that, if $S \subset [n]$, we define generalized horns as

$$\Lambda^n_S \coloneqq \bigcup_{i \notin S} \Delta^{[n]\setminus i} = \bigcup_{\substack{S \subset E \subset [n] \\ |E| = n-1}} \Delta^E \subset \partial\Delta^n,$$

so that $\Lambda^n_{\{k\}} = \Lambda^n_k$, and $\Lambda^n_\varnothing = \partial\Delta^n$. This allows us to select any of the faces of the boundary.

Lemma 1.A.15 *Let $\varnothing \subsetneq S \subsetneq [n]$ and consider $j \colon \Lambda^n_S \hookrightarrow \Delta^n$.*

1. *j is an anodyne extension, and if $0 \in S$ (resp. $n \in S$) then j is a left (resp. right) anodyne extension.*
2. *If $[n] \setminus S$ is not an interval, then j is inner anodyne.*

Proof For any $t \in [n]$, the inclusion of $\Lambda^n_S \cap \Delta^{[n]\setminus t} \hookrightarrow \Delta^{[n]\setminus t}$ is isomorphic to $\Lambda^{n-1}_{S'} \hookrightarrow \Delta^{n-1}$, where S' is the preimage of S under $d^i \colon [n-1] \hookrightarrow [n]$. Thus we have a bicartesian square:

$$\begin{array}{ccc} \Lambda^{n-1}_{S'} & \longrightarrow & \Delta^{n-1} \\ \downarrow & \ulcorner & \downarrow \\ \Lambda^n_S & \longrightarrow & \Lambda^n_{S\setminus t} \end{array}$$

We induct on n, ignoring the claim about right anodyne extensions because it is obtained by duality. For $n = 1$, (1) is clear (and (2) is vacuous): the only generalized horn inclusions are horn inclusions. For $n = 2$, (2) is clear because the only valid S is $\{1\}$.

In the general case, we induct on the size of S as well, as both (1) and (2) are known in the case of horn inclusions. If $|S| > 1$, then the pushout above shows that j is anodyne, because $\Lambda^n_{S\setminus t} \hookrightarrow \Delta^n$ by choosing any $t \in S$. For the left anodyne case, just choose $t \neq 0$. For the inner anodyne case, choose $a < b < c$

where $a, c \notin S$ and $b \in S$. Then choose $t \in S \setminus \{b\}$, so then $[n] \setminus (S \setminus t)$ is not an interval, and neither is S'. □

Remark 1.A.16 We have the following converse statement to inner anodyne statement above: If an inclusion $\Lambda^n_S \subset \Delta^n$ is inner anodyne, then $[n] \setminus S$ is not an interval.

Lemma 1.A.17 *Let $n \geq 2$. Then the inclusions*

$$\Delta^{n-1} \cup \Delta^{\{n-1,n\}} \hookrightarrow \Delta^n \qquad I^n \hookrightarrow \Delta^n$$

are inner anodyne extensions.

Proof Begin with the first one, which we show by induction on n, where $n = 2$ is just $\Lambda^2_1 \hookrightarrow \Delta^2$. Assuming the n case shows that $\Delta^{[n]\setminus 0} \cup \Delta^{\{n,n+1\}} \hookrightarrow \Delta^{[n+1]-0}$ is inner anodyne. The bicartesian square

$$\begin{array}{ccc} \Delta^{[n]\setminus 0} \cup \Delta^{\{n,n+1\}} & \hookrightarrow & \Delta^{[n+1]\setminus 0} \\ \downarrow & & \downarrow \\ \Delta^n \cup \Delta^{\{n,n+1\}} & \hookrightarrow & \Delta^{[n+1]\setminus 0} \cup \Delta^{[n+1]\setminus(n+1)} \end{array}$$

then implies that the lower map is inner anodyne, and the pushout is precisely Δ^{n+1}_S where $S = [n+1] \setminus \{0, n+1\}$. Since $\{0, n+1\}$ is not an interval, we conclude by Lemma 1.A.15.

For the second one, we also induct on n which has the same base case $n = 2$. Assuming the $n-1$ case, we consider the two inclusions

$$I^n \subset \Delta^{n-1} \cup \Delta^{\{n-1,n\}} \subset \Delta^n,$$

so we need only prove that the first is inner anodyne. But since $I^n = I^{n-1} \cup \Delta^{\{n-1,n\}}$, this is a pushout of $I^{n-1} \hookrightarrow \Delta^{n-1}$. □

We remark that even the (less frequently stated) first part of this result also has some independent interest. It means that in an ∞-category, if we're given a homotopy-coherent composition of $n-1$ composable morphisms (i.e. an $(n-1)$-simplex), and another single composable map, we can extend this data to a a homotopy-coherent composition of all n morphisms, and such an extension is unique up to contractible ∞-groupoid.

Example 1.A.18 We can use this to give a specialized criterion that is sometimes useful when arguing about ∞-categories. Let M be a saturated class of monomorphisms of simplicial sets satisfying the following properties:

1. M contains the monomorphisms

$$\varnothing \to \Delta^0 \qquad \varnothing \to \Delta^1 \qquad \Lambda^n_k \hookrightarrow \Delta^n$$

for $0 < k < n$.

2. If $i\colon A \hookrightarrow B$ is a monomorphism and $\varnothing \to A$ and $\varnothing \to B$ are in M, then i is in M.

Then M contains $\varnothing \to A$ for all simplicial sets A. *Proof* First of all, it follows directly from the definition of saturated and condition (2) that the class $\{A : (\varnothing \to A) \in M\}$ of simplicial sets is saturated by monomorphisms. Thus it suffices to show that M contains $\varnothing \to \Delta^n$ for all $n \geq 0$. By (1) and Lemma 1.A.17, it suffices to show that $\varnothing \to I^n$ is in M for $n \geq 0$. This is by induction on $n \geq 2$, since $0, 1$ are assumed. But $I^{n-1} \to I^n$ is a pushout of $\Delta^0 \hookrightarrow \Delta^1$, and condition (2) guarantees that the latter is in M. Thus $\varnothing \to I^n$.

We next prove a lemma which generalizes Lemma 1.2.44 and will be used in the next section (TODO: remove the original lemma). Suppose $p\colon X \to A$ is a right fibration and we have a lifting problem of the form:

$$\begin{array}{ccc} K & \xrightarrow{a} & X \\ \downarrow & & \downarrow p \\ L & \xrightarrow{\sigma} & A \end{array}$$

We define the filler space $\mathrm{Fill}_p(\sigma, a)$ via the cartesian diagram:

$$\begin{array}{ccccc} \mathrm{Fill}_p(\sigma,a) & \longrightarrow & \underline{\mathrm{Hom}}_A(L,X) & \longrightarrow & \underline{\mathrm{Hom}}(L,X) \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \{a\} & \longrightarrow & \underline{\mathrm{Hom}}_A(K,X) & \longrightarrow & \underline{\mathrm{Hom}}(K,X) \underset{\underline{\mathrm{Hom}}(K,A)}{\times} \underline{\mathrm{Hom}}(L,A) \end{array}$$

Thus by Lemma 1.1.3 $\mathrm{Fill}_q(\sigma, a)$ is a Kan complex whose objects are the solutions of the lifting problem defined above. The point is to relate the homotopy theory of this space to that of the fibers of p.

Lemma 1.A.19 *Let $p\colon X \to A$ be a right fibration and*

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{a} & X \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\sigma} & A \end{array}$$

be a lifting problem.

1. *If $\pi_{n-1}(X_{\sigma(0)}, a(0)) = 0$ for $n \geq 1$, then the square has a lift.*
2. *If $f\colon \Delta^n \to X$ is a lift for any $n \geq 0$, there is an equivalence*

$$\big(\mathrm{Fill}_p(\sigma, a), f\big) \simeq \Omega^n\big(X_{\sigma(0)}, a(0)\big)$$

of pointed Kan complexes.

Proof We prove this by induction on $n \geq 0$. In the base case $\mathrm{Fill}_p(\sigma, a) = X_{\sigma(0)}$ so the claim is clear. For $n \geq 1$, we use $\partial\Delta^n = \Lambda^n_n \cup_{\partial\Delta^{n-1}} \Delta^{n-1}$ to obtain the

following cartesian diagram, where $\tau \coloneqq a|_{\Delta^{n-1}}$:

$$\begin{array}{ccccc}
\mathrm{Fill}_p(\sigma,a) & \hookrightarrow & \mathrm{Fill}_p(\sigma,a|_{\Lambda^n_n}) & \to & \underline{\mathrm{Hom}}_A(\Delta^n,X) \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\Delta^0 & \xrightarrow{\tau} & \mathrm{Fill}_p(d_n\sigma,\partial\tau) & \to & \underline{\mathrm{Hom}}_A(\partial\Delta^n,X) \to \underline{\mathrm{Hom}}_A(\Delta^{n-1},X) \\
 & & \downarrow & \lrcorner & \downarrow \quad\lrcorner \quad\downarrow \\
 & & \{a|_{\Lambda^n_n}\} & \longrightarrow & \underline{\mathrm{Hom}}_A(\Lambda^n_n,X) \to \underline{\mathrm{Hom}}_A(\partial\Delta^{n-1},X)
\end{array}$$

Since $\Lambda^n_n \hookrightarrow \Delta^n$ is a right anodyne extension, Lemma 1.1.3(2) shows that the third column composite is a trivial fibration, hence $\mathrm{Fill}_p(\sigma,a|_{\Lambda^n_n})$ is contractible. Part (1) of the same lemma shows that the first map in the second column is a Kan fibration, so the left-most square is homotopy cartesian. By the induction hypothesis and the fact that τ is a lift of $\partial\tau$, we have

$$\big(\mathrm{Fill}_p(d_n\sigma,\partial\tau),\tau\big) \simeq \Omega^{n-1}\big(X_{\sigma(0)},a(0)\big). \tag{$*$}$$

Now we prove the two claims: (1) Taking π_0 of the above equation shows that the LHS is connected, so the long exact sequence associated to the Kan fibration above contains an exact sequence of pointed sets:

$$\begin{array}{ccccc}
\pi_0\mathrm{Fill}_p(\sigma,a) & \longrightarrow & \pi_0\mathrm{Fill}_p(\sigma,a|_{\Lambda^n_n}) & \longrightarrow & \pi_0\mathrm{Fill}_p(d_n\sigma,\partial\tau) \\
 & & \| & & \| \\
 & & * & & *
\end{array}$$

In particular $\mathrm{Fill}_p(\sigma,a) \neq \varnothing$ as desired. (2) The image of f gives a point in $\mathrm{Fill}_p(\sigma,a|_{\Lambda^n_n})$ lying over τ; because $\mathrm{Fill}_p(\sigma,a)$ is a homotopy fiber, we have

$$\begin{aligned}
\big(\mathrm{Fill}_p(\sigma,a),f\big) &\simeq \{\tau\} \times^{\mathbf{R}}_{\mathrm{Fill}_p(d_n\sigma,\partial\tau)} \mathrm{Fill}_p(\sigma,a|_{\Lambda^n_n}) \\
&\simeq \{\tau\} \times^{\mathbf{R}}_{\mathrm{Fill}_p(d_n\sigma,\partial\tau)} \{\tau\} \\
&\simeq \Omega\big(\mathrm{Fill}_p(d_n\sigma,\partial\tau),\tau\big)
\end{aligned}$$

by Example 1.1.25. Combined with $(*)$ we conclude. □

Remark 1.A.20 Suppose in the lemma above that we take $A = \Delta^0$, so that X is a Kan complex. Furthermore take $a = x_0$ to be the constant map at a point x_0. There is a canonical filler $\Delta^n \to X$ which is the constant map x_0. Then this filler space, denoted $\mathrm{Fill}^n_X(x_0)$, satisfies $\big(\mathrm{Fill}^n_X, 1_{x_0}\big) \simeq \Omega^n(X,x_0)$, and therefore

$$\pi_n(X,x_0) \cong \pi_0\mathrm{Fill}^n_X(x_0).$$

In other words we obtain an identification with relative homotopy classes of maps those maps $\Delta^n \to X$ sending $\partial\Delta^n \mapsto x_0$, or also pointed classes $\Delta^n/\partial\Delta^n \to (X,x_0)$. Thus this coincides with $\pi_0\,\underline{\mathrm{Hom}}_*(\Delta^n/\partial\Delta^n,X)$ where X is a pointed space. This gives a similar description as in topology. This is distinct from a different criterion we use frequently, which is that $\pi_k(X) = 0$ (at all basepoints) iff every map $\partial\Delta^{k+1} \to X$ extends to Δ^{k+1}. One direction follows immediately from Lemma 1.A.19(1), and the other follows from [HCHA,

Prop. 3.8.10] and the fact that a pointed map $\partial\Delta^{n+1} \to \Delta^n \to X$ has trivial homotopy class, because Δ^n is contractible.

Lemma 1.A.21 *Let A be a simplicial set. The following conditions are equivalent:*

(*i*) *$\tau(A)$ is a groupoid.*

(*ii*) *For each ∞-category $\mathcal{C}$, the inclusion $\underline{\mathrm{Hom}}(A, \mathcal{C}^{\simeq}) \subset \underline{\mathrm{Hom}}(A, \mathcal{C})^{\simeq}$ is an equality.*

Proof The inclusion $\underline{\mathrm{Hom}}(A, \mathcal{C}^{\simeq}) \subset \underline{\mathrm{Hom}}(A, \mathcal{C})$ factors through the core because $\underline{\mathrm{Hom}}(A, \mathcal{C}^{\simeq})$ is an ∞-groupoid. (*ii*) $\Rightarrow$ (*i*): Taking objects and $\mathcal{C} = N(C)$ for all small categories C, we deduce that $\mathrm{Hom}(A, N(C^{\simeq})) = \mathrm{Hom}(A, N(C))$, hence $\mathrm{Fun}(\tau A, C^{\simeq}) = \mathrm{Fun}(\tau A, C)$. In particular the identity map $\tau A \to \tau A$ factors through $\tau A^{\simeq}$, so $\tau A = \tau A^{\simeq}$.

(*i*) $\Rightarrow$ (*ii*) The inclusion of simplicial sets $\underline{\mathrm{Hom}}(A, \mathcal{C})^{\simeq} \subset \underline{\mathrm{Hom}}(A, \mathcal{C})$ corresponds to $\underline{\mathrm{Hom}}(A, \mathcal{C})^{\simeq} \times A \to \mathcal{C}$, and it suffices to show that it factors through $\mathcal{C}^{\simeq}$. By definition of the core, this happens iff the factorization occurs after applying τ; we conclude because τ preserves products, so τ of the source is a groupoid. □

1.A.3 Homotopy n-categories of an ∞-category

In this section, we will discuss skeleta/coskeleta, give an explicit model for the homotopy bicategory of an ∞-category (in the classical sense), and then construct some homotopy truncations of ∞-categories and ∞-groupoids. Most of this section consists of some elementary simplicial constructions and verifications.

We freely use the notion of the n-skeleton $\mathrm{Sk}_n(X)$ of a simplicial set, which is the largest subcomplex containing all non-degenerate simplices of dimension $\leq n$. In other words, a m-simplex in $\mathrm{Sk}_n(X)$ is of the form $f^*\sigma$ where $\sigma \in X_k$ for $k \leq n$, and $f\colon [m] \to [k]$. A simplicial set X is called *r-skeletal* if the inclusion $\mathrm{Sk}_r X \to X$ is an equality. For instance, we recall that

$$\mathrm{Sk}_r \Delta^n = \begin{cases} \partial\Delta^n & r = n-1 \\ \Delta^n & r \geq n \end{cases}$$

so Δ^n is r-skeletal for $r \geq n$. It is clear from the definition that $\mathrm{Sk}_n \circ \mathrm{Sk}_n = \mathrm{Sk}_n$, so $\mathrm{Sk}_r X$ is always r-skeletal.

Example 1.A.22 Let $r \geq 0$ and $f\colon X \to Y$ be a morphism of simplicial sets. Then the following are equivalent:

(*i*) $f_k\colon X_k \to Y_k$ is a bijection for each $k \leq r$.

(*ii*) The induced map $\mathrm{Sk}_r X \to \mathrm{Sk}_r Y$ is an isomorphism.

(*iii*) For each r-skeletal simplicial set K, the induced map

$$\mathrm{Hom}_{\mathbf{sSet}}(K, X) \to \mathrm{Hom}_{\mathbf{sSet}}(K, Y)$$

is a bijection.

Proof (*i*) $\Rightarrow$ (*ii*) We may show that if $X \to Y$ is a map of r-skeletal simplicial sets for which f_k is a bijection when $k \leq r$, then $X \to Y$ is an isomorphism. It suffices by the Eilenberg-Zilberg lemma to prove that it induces a bijection between non-degenerate simplices. Clearly f preserves degnerate simplices, hence reflects non-degnerate ones. If σ is a non-degnerate simplex (hence in degree $k \leq r$), then $f(\sigma)$ must be non-degenerate, since f_i is an isomorphism for $i \leq k$. (*ii*) $\Rightarrow$ (*iii*) By definition we may write K as $\mathrm{Sk}_r K$; then every map $\mathrm{Sk}_r K \to X$ factors (uniquely) through $\mathrm{Sk}_r X \subset X$ since the image of a cell of dimension $> r$ is by definition degenerate. But then $\mathrm{Hom}_{\mathbf{sSet}}(K, \mathrm{Sk}_r X) \to \mathrm{Hom}_{\mathbf{sSet}}(K, \mathrm{Sk}_r Y)$ is a bijection by assumption. (*iii*) $\Rightarrow$ (*i*) Take $K = \Delta^k$ for $k \leq r$.

A map f satisfying the equivalent conditions of the example above is called a *(skeletal) r-isomorphism*. It is easy to see that the class of r-isomorphisms is both a saturated and cosaturated class of morphisms by (*i*). We implicitly saw in the proof that the construction $X \mapsto \mathrm{Sk}_r X$ is a right adjoint to the inclusion $\mathbf{sSet}_{\mathrm{Sk}_r} \hookrightarrow \mathbf{sSet}$ of the full subcategory of r-skeletal simplicial sets. When its target is viewed as $\mathbf{sSet}$, it has a right adjoint as well.

Construction 1.A.23 Let X be a simplicial set. We define a simplicial set $\mathrm{cosk}_r(X)$ as given by the formula

$$[n] \mapsto \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{Sk}_r\, \Delta^n, X),$$

so that $\mathrm{Sk}_r\, \Delta^n \to \Delta^n$ induces a canonical map $X \to \mathrm{cosk}_r(X)$. Since $\mathrm{Sk}_r\, \Delta^n = \Delta^n$ for $n \leq r$, it follows that $X \to \mathrm{cosk}_r(X)$ is an r-isomorphism, i.e. we have $\mathrm{Sk}_r\, X \xrightarrow{\sim} \mathrm{Sk}_r\, \mathrm{cosk}_r(X)$. Moreover, since $\mathrm{Sk}_r\, \Delta^n$ is r-skeletal, we have

$$(\mathrm{cosk}_r\, X)_n = \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{Sk}_r\, \Delta^n, \mathrm{Sk}_r\, X) = (\mathrm{cosk}_r\, \mathrm{Sk}_r\, X)_n,$$

i.e. $\mathrm{cosk}_r\, \mathrm{Sk}_r\, X \xrightarrow{\sim} \mathrm{cosk}_r\, X$. Under these two identifications, it is clear that

$$\mathrm{Sk}_r\, X \to X \qquad X \to \mathrm{cosk}_r\, X$$

form the counit and unit of an adjunction $\mathrm{Sk}_r \dashv \mathrm{cosk}_r$, since they tautologically satisfy the triangle identities. As a consequence, because Sk_r is idempotent, $\mathrm{cosk}_r\, X \to \mathrm{cosk}_r\, \mathrm{cosk}_r\, X$ is also an isomorphism. In general, we say that X is *r-coskeletal* if $X \to \mathrm{cosk}_r\, X$ is an isomorphism.

We have the following simple characterization of a simplicial set being r-coskeletal, which explains what it is supposed to represent.

Proposition 1.A.24 *Let X be a simplicial set. The following are equivalent:*

(*i*) *X is r-coskeletal.*

(*ii*) *For $n > r$, each map $\partial\Delta^n \to X$ extends uniquely to an n-simplex of X.*

(*iii*) *For any r-isomorphism $A \to B$ of simplicial sets, the induced map*

$$\underline{\mathrm{Hom}}(B, X) \to \underline{\mathrm{Hom}}(A, X)$$

is an isomorphism of simplicial sets.

Proof Clearly (*iii*) $\Rightarrow$ (*ii*) by applying it to $\partial\Delta^n \hookrightarrow \Delta^n$ and taking 0-simplices. (*i*) $\Rightarrow$ (*iii*) Then $\Delta^n \times A \to \Delta^n \times B$ is also an r-isomorphism. This means that

$$\begin{aligned}\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times B, X) &= \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{Sk}_r(\Delta^n \times B), X) \\ &\cong \mathrm{Hom}_{\mathbf{sSet}}(\mathrm{Sk}_r(\Delta^n \times A), X) \\ &= \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times A, X)\end{aligned}$$

where we have used the adjunction property twice.

(*ii*) $\Rightarrow$ (*i*) Consider the commutative diagram:

$$\begin{array}{ccccc}\mathrm{Hom}(\Delta^n, X) & \longrightarrow & \mathrm{Hom}(\Delta^n, \mathrm{cosk}_r\, X) & = & \mathrm{Hom}(\mathrm{Sk}_r\, \Delta^n, X) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(\partial\Delta^n, X) & \longrightarrow & \mathrm{Hom}(\partial\Delta^n, \mathrm{cosk}_r\, X) & = & \mathrm{Hom}(\mathrm{Sk}_r\, \partial\Delta^n, X)\end{array}$$

Suppose the left-hand map is a bijection for all $n > r$. We prove by induction that the top row is a bijection for all $n \geq 0$, which will prove that $X \to \mathrm{cosk}_r\, X$

is an isomorphism. We already know it is an r-isomorphism, so assume now that $n > r$ and that $\mathrm{Hom}(\Delta^k, X) \to \mathrm{Hom}(\Delta^k, \mathrm{cosk}_r X)$ is a bijection for $k \leq n-1$. Thus $X \to \mathrm{cosk}_r X$ is an $(n-1)$-isomorphism, so by Example 1.A.22 we conclude that the lower row is a bijection, because $\partial\Delta^n$ is $(n-1)$-skeletal. Since the left-hand map is a bijection, so too is the top one. □

What the result says is that r-coskeletal is an 'orthogonal' condition to r-isomorphism; more interestingly, there are two different senses in which we might consider simplices in a simplicial set X to 'depend' on lower simplices. One is in the sense of degenerate simplices, which is skeletality, but another is in the categorical sense of composites determining higher composites, which is coskeletality.

Example 1.A.25 Akin to the case of the category of r-skeletal objects, we can also form the inclusion $\mathbf{sSet}_{\mathrm{cosk}_r} \hookrightarrow \mathbf{sSet}$ of the full subcategory of r-coskeletal objects. Then cosk_r forms a left adjoint: indeed, we apply (iii) of the previous result to the r-isomorphism $X \to \mathrm{cosk}_r X$ and take 0-cells.

Example 1.A.26 Suppose X is 0-coskeletal. Since $\mathrm{Sk}_0 \Delta^n = \{0, \ldots, n\}$, we see immediately that $X_n = (X_0)^n$ for all n, which is the nerve of the indiscrete groupoid on the set of objects X_0: between any two objects there is a unique morphism. We remark however that if X is r-coskeletal for $r \geq 1$, it need not be a nerve or even an ∞-category. Nonetheless we will give some interesting results below to see when it is. In general, we see certainly that if X is a Kan complex and r-coskeletal, then it is $(r-1)$-truncated, meaning that $\pi_i(X, x_0) = 0$ for all x_0 and $i > r-1$. This follows immediately from (ii) in the previous result.

Proposition 1.A.27 *Let X be a simplicial set.*

1. *For each $r \geq 0$ and $y \in X$, the canonical diagrams*

$$\begin{array}{ccc} \mathrm{Sk}_{r-1} X & \longrightarrow & (\mathrm{Sk}_{r-1} X) * \Delta^0 \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Sk}_r X & \longrightarrow & \mathrm{Sk}_r(X * \Delta^0) \end{array} \qquad \begin{array}{ccc} (\mathrm{cosk}_r X)/y & \longrightarrow & \mathrm{cosk}_{r-1}(X/y) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{cosk}_r X & \longrightarrow & \mathrm{cosk}_{r-1} X \end{array}$$

   are cocartesian and cartesian respectively.

2. *If X is an ∞-category (resp. ∞-groupoid), then $\mathrm{cosk}_r X$ is an ∞-category (resp. ∞-groupoid) as well.*

Proof (1) Consider applying $\mathrm{Hom}(K, -)$ to the right-hand diagram: by the (co)skeletal and join-slice adjunctions, we obtain $\mathrm{Hom}(-, X)$ applied to the the left-hand square (with K instead of X). In particular the right-hand diagram for all X implies the left-hand one for all X, and the left hand one for $X = \Delta^n$ implies the right-hand one for all X.

Thus consider $\mathrm{Sk}_r(\Delta^{n+1})$; a face of Δ^{n+1} of dimension at most r either contains the final vertex $\{n+1\}$ or not. If it doesn't, then it belongs to $\mathrm{Sk}_r \Delta^n$, and if it does then it is of the form $\Delta^E *$ for E a dimension $\leq r-1$ face, i.e. it belongs to $(\mathrm{Sk}_{r-1} \Delta^n) * \Delta^0$. The face can be in both sets iff it in fact has dimension $\leq r-1$.

(2) We have seen that $\mathrm{Sk}_0 X$ is always the nerve of a groupoid, so assume $r \geq 1$. The horn inclusions $\Lambda^n_k \to \Delta^n$ are r-isomorphisms for $r \leq n-2$, so $\mathrm{cosk}_r X$ has unique horn-lifting for $n \geq r+2$ by Prop. 1.A.24. By adjunction, it remains to consider surjectivity of

$$\mathrm{Hom}(\mathrm{Sk}_r \Delta^n, X) \to \mathrm{Hom}(\mathrm{Sk}_r \Lambda^n_k, X)$$

for $n \leq r+1$. But then Λ^n_k is $(n-1)$- (hence r-) skeletal, so the RHS above is just $\mathrm{Hom}(\Lambda^n_k, X)$. In particular if a map lifts to Δ^n, we may restrict it back to $\mathrm{Sk}_r \Delta^n$. □

Example 1.A.28 Let $\mathcal{C}$ be an ∞-category. If we take fibers over x in part (1) of the previous result, we obtain

$$((\mathrm{cosk}_r \mathcal{C})/y)_x \cong (\mathrm{cosk}_{r-1}(\mathcal{C}/y))_x \cong \mathrm{cosk}_{r-1}(\mathcal{C}/y)_x$$

(where the last step is because cosk_{r-1} preserves pullbacks), and these isomorphisms commute with the structure map from $(\mathcal{C}/y)_x$. Therefore the map $(\mathcal{C}/y)_x \to ((\mathrm{cosk}_r \mathcal{C})/y)_x$ is an isomorphism is an isomorphism iff $(\mathcal{C}/y)_x$ is $(r-1)$-coskeletal. In particular the latter condition is true if $\mathcal{C}$ itself is r-coskeletal. Therefore:

1. If $\mathcal{C}$ is r-coskeletal, then $\mathcal{C}(x,y)$ is equivalent to an $(r-1)$-cokseletal ∞-groupoid for each x and y, and hence is $(r-2)$-truncated.
2. In particular, if $\mathcal{C}$ is 2-coskeletal, this means that its mapping spaces are 0-truncated, hence $\mathcal{C}$ is a 1-category by Remark 1.0.3.

This suggests that there should be a good theory of ∞-categories/∞-groupoids equivalent to coskeletal ones, which should represent some kind of truncation. This theory should also generalize our observations Remark 1.0.3. We will develop this theory momentarily, but first we pause to give a quick construction of the left adjoint to the nerve functor on an arbitrary simplicial set, whose general description is not included in detail in Cisinski. Nonetheless, this is all review. We write the adjunction as

$$\tau \colon \mathbf{sSet}_{\mathrm{Joy}} \rightleftarrows (\mathbf{Cat})_{\mathrm{can}} : N.$$

The basic construction is as follows.

Construction 1.A.29 Let X be a simplicial set. Let $\widetilde{X}_{01}$ be the free category on the directed graph with vertices X_0 and edges X_1, so that $\mathrm{Ob}(\widetilde{X}_{01}) = X_0$ and morphisms are $x_0 \to x_r$ are finite composable sequences

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} x_r$$

built from the morphisms in X, where the empty string $\{x_0\}$ is the identity. We define a categorical congruence on $\widetilde{X}_{01}$ which is specified on $\mathrm{Hom}_{\widetilde{X}_{01}}(x, z)$ as follows:

1. If $z = x$, then $(x) \sim 1_x$ where $1_x \colon x \to x$ is the degenerate 1-simplex.
2. For each 2-simplex σ with $\sigma|_{\Lambda^2_1} = x \xrightarrow{f} y \xrightarrow{g} z$ and $\sigma|_{\Delta^{\{0,2\}}} = x \xrightarrow{h} z$, we define $(x \to y \to z) \sim x \xrightarrow{h} z$ in $\mathrm{Hom}_{\widetilde{X}_{01}}(x, z)$.

Let $\sim$ be the congruence relation on $\widetilde{X}_{01}$ specified by the above relations. We define $\tau X \coloneqq \widetilde{X}_{01}/\!\sim$. This comes with a specified map $X \to N(\tau X)$ defined on n-simplices as

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \to \mathrm{Hom}_{\mathbf{sSet}}(I^n, X) \to \mathrm{Fun}([n], \tau X)$$

where the first map is induced by $I^n \to \Delta^n$, and the second sends a string of n composable maps to the corresponding string in τX.

Proposition 1.A.30 *Using the notions of Constr. 1.A.29, the maps $X \to N(\tau X)$ for each X are well-defined, and they exhibit uniquely $X \mapsto \tau X$ as a left adjoint to $N \colon$ **Cat** $\to$ **sSet** with counit $X \to N(\tau X)$.*

Proof To show the map is a morphism of simplicial sets, we have to use the relations. Let σ be an n-simplex with spine

$$x_0 \xrightarrow{f_1} \cdots x_{i-1} \xrightarrow{f_i} x_i \xrightarrow{f_{i+1}} x_{i+1} \to \cdots \xrightarrow{f_n} x_n,$$

which maps to this same string in τX. The face operator d_i on $N(\tau X)$ replaces the middle term with $(x_{i-1} \to x_i \to x_{i+1})$ i.e. the composite in τX, to obtain a string of length $n-1$. On the other hand $d_i\sigma$ replaces this with the morphism $x_{i-1} \to x_{i+1}$ given by the third edge in $\sigma|_{\Delta^{\{i-1,i,i+1\}}}$. But then by the relations on τX this 2-simplex shows that the two d_i's agree.

Likewise, a similar argument works for the degeneracy operators s_i: on the nerve side, it adds $x_i \to x_i \to x_{i+1}$ where the first map is the identity in τX, and in X the new face $s_i(\sigma)|_{\Delta^{\{i,i+1\}}}$ is precisely the degenerate 1-simplex at x_i. By the other relation in τX, this shows that the two s_i's agree, completing the proof that $X \to N(\tau X)$ is a simplicial map.

Now the universal property: Let C be a small category and $f \colon X \to N(C)$ a simplicial map. By definition of the free category, this gives a functor $\widetilde{X}_{01} \to C$ sending $x \mapsto f(x)$ and $u \mapsto f(u)$. The degenerate simplex 1_x maps to $f(x) \xrightarrow{1_x} f(x)$ (because f is a simplicial morphism), as does the empty string (x). Moreover if σ is a 2-simplex with inner horn $x \to y \to z$, then $f(\sigma)$ is the composable string $f(x) \to f(y) \to f(z)$; its remaining face is the composite $f(x) \to f(z)$ in C, so the second relation is also satisfied. Thus the functor factors through $f^\flat \colon \tau X \to C$, and it is clear that if $X \to N(\tau X) \to C$ is f then $f^\flat$ is certainly unique with this property because as a functor it is determined by the 0 and 1-simplices.

It remains to show that $X \to N(\tau X) \xrightarrow{N(f^\flat)} C$ is indeed f. As explained we already know they are equal on 0 and 1-simplices. But for any map $g\colon X \to N(C)$ and σ an n-simplex for $n \geq 2$, we have

$$g(\sigma) = g(\sigma(0)) \xrightarrow{g(\sigma|_{\Delta\{0,1\}})} g(\sigma(1)) \to \cdots \to g(\sigma(n)).$$

Thus by definition of the nerve, agreement on 0 and 1-simplices suffices. □

There are some obvious corollaries from the construction itself. For instance, $\tau(X)^{\mathrm{op}} = \tau(X^{\mathrm{op}})$, and the next example. We note also that in τX, we have

$$x_0 \to \cdots \to x \xrightarrow{1_x} x \to \cdots \to x_r = x_0 \to \cdots x \to \cdots \to x_r,$$

because we can write this as a composition of three morphsims, with the middle one $1_x\colon x \to x$ which we have declared to be the identity. In particular, all morphisms can be written as a string of non-identity morphisms.

Example 1.A.31 If X is a simplicial set, then by construction τX depends only on X_0, X_1, X_2, so in particular $\tau(\mathrm{Sk}_2 X) \xrightarrow{\sim} \tau X$. In particular,

$$\mathrm{Hom}(X, N(C)) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Sk}_2 X, N(C))$$

by adjunction for all small categories C, so $N(C) \xrightarrow{\sim} \mathrm{cosk}_2\, N(C)$. I.e. the nerve of a category is 2-coskeletal. Combined with Example 1.A.28, we deduce yet another characterization of 1-categories, which is being equivalent to a 2-coskeletal ∞-category.

Remark 1.A.32 Here are some basic computations with computing τ of non-∞-categories.

1. Since $\partial\Delta^n \to \Delta^n$ is a 2-isomorphism for all $n \geq 3$, we have $\tau(\partial\Delta^n) \xrightarrow{\sim} [n]$ for $n \geq 3$. Since $\Lambda^n_k \to \Delta^n$ is a 2-isomorphism for $n \geq 4$, $\tau\Lambda^n_k \to [n]$ for $n \geq 4$. We know that Λ^2_0 and Λ^2_2 are already nerves of categories.
2. Now let's focus on inner horns. $\mathrm{Hom}(\Delta^2, N(C)) \to \mathrm{Hom}(\Lambda^2_1, N(C))$ is a bijection for any C, because in a category any two morphisms have a unique composite. So $\tau(\Lambda^2_1) \xrightarrow{\sim} [2]$ as well. The remaining inner horns are Λ^3_1 and Λ^3_2, and we claim that their τ is also $[3]$. One is obtained from the other by taking opposites so let's do the former, i.e.

$$\Lambda^3_1 = \Delta^{\{012\}} \cup \Delta^{\{013\}} \cup \Delta^{\{123\}}.$$

It has 4 vertices, and there is only 1 map $0 \to 1, 1 \to 2$ and $2 \to 3$. By construction of τ, the three maps which are just composites of the three 2-simplices making up Λ^3_1 form only one map. The only remaining possible maps in $\tau\Lambda^3_1$ are from $0 \to 3$ and they are

$$0 \to 3 \qquad 0 \to 2 \to 3 \qquad 0 \to 1 \to 2 \to 3.$$

Now $0 \to 3 = 0 \to (1 \to 3) = 0 \to (1 \to 2 \to 3)$. Likewise $0 \to 2 = 0 \to 1 \to 2$, so the middle arrow is also equal to $0 \to 1 \to 2 \to 3$. We have shown that for $i \leq j$, there is a unique map $i \to j$ in $\tau\Lambda^3_1$, thus it is [3].

3. There is the following consequence of (2): we have shown that τ applied to all inner horn inclusions is an isomorphism; since τ preserves colimits, the collection of such maps is saturated, hence $\tau A \to \tau B$ is an isomorphism for any inner anodyne extension $A \to B$. For instance, this means that if $A \hookrightarrow \Delta^n$ is inner anodyne, then $I^n \subset A$: indeed, $\tau A = [n]$, so we must have $A_0 = (\Delta^n)_0$, and since there is a map from $i \to i+1$ we must have $\Delta^{\{i,i+1\}} \subset A$. This shows that I^n is the unique smallest inner anodyne subcomplex of Δ^n.

4. Since τ preserves colimits, $\tau(\varnothing) = \varnothing$, $\tau(\partial\Delta^1) = [0] \sqcup [0]$. The universal property says that a map $\tau(\partial\Delta^2) \to C$ is a string $x \to y \to z$ of morphisms and a string $x \to z$. In other words, $\partial\Delta^2$ is the free category on the directed graph $0 \to 1 \to 2$ and $0 \to 2$.

Example 1.A.33 We can use what we have said to give a simple proof of the following fact. The following are equivalent for a simplicial set X:

(*i*) For each $n \geq 2$, $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$ is bijective for $0 < k < n$.

(*ii*) For each $n \geq 2$, $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(I^n, X)$ is bijective.

(*iii*) The canonical map $X \to N(\tau X)$ is an isomorphism.

(*iv*) X is isomorphic to the nerve of a small category.

Proof Since $\mathrm{Hom}_{\mathbf{sSet}}(-, X) \colon \mathbf{sSet} \to \mathbf{Set}^{\mathrm{op}}$ preserves colimits, the collection of morphisms $A \to B$ for which $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is an isomorphism is saturated. Thus if (*i*) holds, it includes all inner anodyne maps, so (*ii*) holds by Lemma 1.A.17. If (*ii*) holds, then consider the map of (*iii*) on simplices,

$$\mathrm{Hom}(\Delta^n, X) \xrightarrow{\sim} \mathrm{Hom}(I^n, X) \to \mathrm{Fun}([n], \tau X),$$

so we just have to show that the latter map is bijective. In particular, we may show that the set of morphisms $x \to y$ in τX coincides with the set of morphisms $x \to y$ in X. $n = 2$ implies that any composable maps have a unique composite, so every map in τX can be represented by an element of X_1, and we have to check that these are distinct. But $n = 3$ implies that composition is associative, so we may form the category C with objects X_0 and morphisms X_1 using the composition law. The defining property gives a map $\tau X \to C$, proving uniqueness of representatives. To conclude, (*iii*) $\Rightarrow$ (*iv*) is obvious, and (*iv*) $\Rightarrow$ (*i*) holds because we showed $\tau(\Lambda^n_k) \xrightarrow{\sim} \tau(\Delta^n)$ for all $0 < k < n$. □

Now we return to generalities about coskeleta. We separately treat the case of Kan complexes and ∞-categories, as the former is a bit simpler.

Proposition 1.A.34 *Let X be a Kan complex and $r \geq 0$. The following conditions are equivalent:*

(*i*) *For each* $x_0 \in X$, $\pi_k(X, x_0) = 0$ *for all* $k \geq r$, *or* X *is empty.*

(*ii*) X *is equivalent to an* r*-coskeletal Kan complex.*

(*iii*) *The canonical map* $X \to \operatorname{cosk}_r X$ *is a homotopy equivalence.*

(*iv*) *The canonical map* $X \to \operatorname{cosk}_r X$ *is a trivial fibration.*

In this case we say that X *is* $(r-1)$*-truncated.*

Under the convention that $\operatorname{Sk}_{-1} Y$ should be $\varnothing$, we see that $\operatorname{cosk}_{-1} X = \Delta^0$ for all X. Thus one usually extends the above definition to say that X is (-2)-truncated if $X \to \Delta^0$ is an equivalence, i.e. if X is contractible (and thus nonempty). In contrast, we see that X being (-1)-truncated means that it is contractible or empty.

Proof (of Prop. 1.A.34) We remark that $\operatorname{cosk}_r(\varnothing) = \varnothing$ from the definition, which is why (*i*) includes the possibility of X being empty. The implications (*iv*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*i*) are all clear (the final one is Example 1.A.26).

(*i*) $\Rightarrow$ (*iv*) Assume X is nonempty, and consider a lifting problem:

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & X \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & \operatorname{cosk}_r X \end{array}$$

Since $X \to \operatorname{cosk}_r X$ is an r-isomorphism, we may assume that $n > r$. The existence of a solution is equivalent to surjectivity of the map

$$\begin{aligned} \operatorname{Hom}(\Delta^n, X) &\to \operatorname{Hom}(\partial\Delta^n, X) \underset{\operatorname{Hom}(\partial\Delta^n, \operatorname{cosk}_r X)}{\times} \operatorname{Hom}(\Delta^n, \operatorname{cosk}_r X) \\ &= \operatorname{Hom}(\partial\Delta^n, X) \underset{\operatorname{Hom}(\operatorname{Sk}_r \partial\Delta^n, X)}{\times} \operatorname{Hom}(\operatorname{Sk}_r \Delta^n, X) \\ &= \operatorname{Hom}(\partial\Delta^n, X), \end{aligned}$$

where the final step is because $n > r$ gives $\operatorname{Sk}_r \partial\Delta^n = \operatorname{Sk}_r \Delta^n$. This map is surjective because $\pi_{n-1}(X) = 0$ as $n - 1 \geq r$. □

The generalization of this result to ∞-categories is similar, but needs an extra lemma to give us the upgrade to being a trivial fibration.

Lemma 1.A.35 *Let* $p\colon X \to A$ *be a right fibration with* k*-truncated fibers for any* $k \geq -2$. *Then* p *has the RLP against* $\partial\Delta^n \to \Delta^n$ *for* $n > k+1$.

Proof If $k = -2$, so that p has contractible fibers, then we already know p is a trivial fibration by Example 1.1.8; so assume $k \geq -1$. Suppose we have a lifting problem

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{a} & X \\ \downarrow & & \downarrow p \\ \Delta^n & \xrightarrow{\sigma} & A \end{array}$$

where $n > k+1$. Then $n \geq 1$ and $n-1 > k$, so $\pi_{n-1}(X_{\sigma(0)}, a(0)) = 0$, and then Lemma 1.A.19 provides a lift. □

Theorem 1.A.36 *Let $\mathcal{C}$ be an ∞-category and $r \geq 0$. The following are equivalent:*

(*i*) *For all objects x, y of $\mathcal{C}$, the ∞-groupoid $\mathcal{C}(x,y)$ is $(r-2)$-truncated.*

(*ii*) *$\mathcal{C}$ is equivalent to an r-coskeletal ∞-category.*

(*iii*) *The canonical map $\mathcal{C} \to \mathrm{cosk}_r \mathcal{C}$ is an equivalence of ∞-categories.*

(*iv*) *The canonical map $\mathcal{C} \to \mathrm{cosk}_r \mathcal{C}$ is a trivial fibration.*

(*v*) *$\mathcal{C}$ has the RLP against $\partial\Delta^n \hookrightarrow \Delta^n$ for all $n > r$.*

If any of these equivalent conditions hold then $\mathcal{C}$ is called an $(r-1)$-category

Proof (*v*) $\Rightarrow$ (*iv*) is the same as the proof of (*i*) $\Rightarrow$ (*iv*) in Prop. 1.A.34. (*iv*) $\Rightarrow$ (*iii*) $\Rightarrow$ (*ii*) are clear. For $r \geq 1$, (*ii*) $\Rightarrow$ (*i*) follows from Example 1.A.28. For $r = 0$, Example 1.A.26 showed that 0-coskeletal implies contractible ∞-groupoid, hence (*i*) follows.

(*i*) $\Rightarrow$ (*v*) For each object y of $\mathcal{C}$, the assumption means that the right fibration $\mathcal{C}/y \to \mathcal{C}$ has $(r-2)$-truncated fibers. Thus it has the RLP against $\partial\Delta^n \to \Delta^n$ for all $n > r-1$ by the previous lemma. We have a lifting correspondence

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \mathcal{C}/y \\ \downarrow & \nearrow & \downarrow p \\ \Delta^n & \longrightarrow & \mathcal{C} \end{array} \qquad \begin{array}{ccc} \partial\Delta^n * \Delta^0 \cup \Delta^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow & \downarrow \\ \Delta^n * \Delta^0 & \longrightarrow & \Delta^0 \end{array}$$

(where the top map on the right side should map $\varnothing * \Delta^0$ to y), by [HCHA, Lemma 3.4.20]. The inclusion on the left of the right-hand square is precisely $\partial\Delta^{n+1} \hookrightarrow \Delta^{n+1}$ ([HCHA, Prop. 3.4.17]). Thus $\mathcal{C}$ has the RLP against $\partial\Delta^{n+1} \to \Delta^{n+1}$ for $n > r-1$. □

Some immediate consequences of the definition: An ∞-groupoid $\mathcal{D}$ is an n-category iff it is n-truncated. It also has the pleasant consequence that, for ∞-categories, changing the RLP to the unique RLP precisely corresponds to changing $\mathcal{C} \to \mathrm{cosk}_r \mathcal{C}$ from an equivalence to an isomorphism. Moreover:

1. By Remark 1.0.3 it follows that an ∞-category $\mathcal{C}$ is a n-category for $n = 1$ iff it is a 1-category in the sense defined prior.
2. An ∞-category $\mathcal{C}$ is a 0-category iff its mapping spaces are either contractible are either empty or contractible. In particular it is equivalent to $N(C)$ where for any objects a, b, $\mathrm{Hom}_C(a,b)$ is either empty or the one-point set. Thus a 0-category is something equivalent to the nerve of a preorder, or likewise it is equivalent to the nerve of a poset. This means that a Kan complex is 0-truncated iff it is equivalent to a discrete groupoid, i.e. a 'set'.

3. A nonempty ∞-category $\mathcal{C}$ is a -1-category iff its mapping spaces are all contractible ∞-groupoids, i.e. it is equivalent to the nerve of a contractible groupoid, or equivalently to Δ^0. This has the same caveat as before: the empty ∞-category is also a (-1)-category, so we define a (-2)-category to be equivalent to Δ^0.

Example 1.A.37 Let $\mathcal{C}$ be an ∞-category and $\mathcal{B} \subset \mathcal{C}$ an ∞-subcategory, as defined in Remark 1.0.8. If $\mathcal{C}$ is an n-category (for $n \geq -1$), then $\mathcal{B}$ is also one. Indeed, by Remark 1.0.8, for all $x, y \in \mathcal{B}$ the induced map

$$\mathcal{B}(x, y) \to \mathcal{C}(x, y)$$

is both a monomorphism and a Kan fibration. All nonempty fibers are Δ^0, hence $\pi_k\big(\mathcal{B}(x, y), f\big) \xrightarrow{\sim} \pi_k\big(\mathcal{C}(x, y), f\big)$ for all $f\colon x \to y$ in $\mathcal{B}$ by the long exact sequence. In particular, the core $\mathcal{C}^{\simeq}$ of an n-category is n-truncated.

Our next topic is truncations and truncated objects. We again begin with the case of Kan complexes. The key point is the following lemma:

Lemma 1.A.38 *Let $p\colon X \to Y$ be a map of Kan complexes with and $n \geq 0$ and Y n-truncated. The following conditions are equivalent:*

(*i*) *The induced map $p_*\colon \pi_k(X, x) \to \pi_k\big(Y, p(x)\big)$ is an isomorphism for all x and for each $k \leq n$.*

(*ii*) *For each n-truncated Kan complex Z, the induced map*

$$\underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z)$$

is an equivalence of ∞-groupoids.

(*iii*) *There is commutative diagram in $\mathcal{H}$ of the form*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{cosk}_{n+1} X \\ & \searrow^{p} & \big\downarrow \cong \\ & & Y \end{array}$$

where the right-hand map is an isomorphism.

(*iv*) *The induced map $\mathrm{cosk}_{n+1} X \to \mathrm{cosk}_{n+1} Y$ is an equivalence.*

Proof Consider the commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathrm{cosk}_{n+1} X & \longrightarrow & \mathrm{cosk}_{n+1} Y \end{array}$$

Since each map to the $(n+1)$-cokseleta is an $(n+1)$-isomorphism, it induces an isomorphism on π_k for $k \leq n$. Both coskeleta are n-truncated. Thus the lower map is an equivalence iff it induces isomorphisms on π_k for $k \leq n$, which

happens iff the upper map does. This shows $(i) = (iv)$, and moreover since obviously $(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ we deduce (iii) is equivalent too.

Now let Z be an n-truncated Kan complex; by naturality and the fact that Z is equivalent to an $(n+1)$-coskeletal Kan complex, it suffices to prove the equivalence when Z is n-coskeletal. Consider the commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(\mathrm{cosk}_{n+1} Y, Z) & \longrightarrow & \underline{\mathrm{Hom}}(Y, Z) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(\mathrm{cosk}_{n+1} X, Z) & \longrightarrow & \underline{\mathrm{Hom}}(X, Z) \end{array}$$

Because $X \to \mathrm{cosk}_{n+1} X$ and $Y \to \mathrm{cosk}_{n+1} Y$ are $(n+1)$-isomorphisms, Prop. 1.A.24 implies that the horizontal maps are isomorphisms. Thus the left-hand map is an equivalence iff the right-hand one is. In particular $(iii) \Rightarrow (ii)$. For the other direction, apply π_0 and consider the full subcategory of $\mathcal{H}$ spanned by n-truncated Kan complexes. The Yoneda lemma shows that $\mathrm{cosk}_{n+1} X \to \mathrm{cosk}_{n+1} Y$ is an isomorphism in this subcategory. □

Definition 1.A.39 Let X be a Kan complex. An *n-truncation* of X is an n-truncated Kan complex $\tau_{\leq n} X$ equipped with a map $p\colon X \to \tau_{\leq n} X$ satisfying the equivalent conditions of Lemma 1.A.38.

Note that of course this exists, because $X \to \mathrm{cosk}_{n+1} X$ is always a (functorial) n-truncation. The lemma shows that it is unique up to equivalence, and comes with a universal property for maps to an n-truncated Kan complex. We also see that the n-truncation is well-defined directly on the level of the homotopy category. Now let us repeat this story with ∞-categories, which will have a slightly different universal property.

Proposition 1.A.40 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of ∞-categories where $\mathcal{D}$ is an n-category for $n \geq 0$. The following are equivalent:*

(*i*) *F is essentially surjective, and for all objects x, y of $\mathcal{C}$, the induced map*

$$F_*\colon \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$$

is an $(n-1)$-truncation of Kan complexes.

(*ii*) *For each n-category $\mathcal{E}$, the induced map*

$$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

is an equivalence of ∞-categories.

(*iii*) *For each n-category $\mathcal{E}$, the induced map*

$$F^*\colon \mathrm{Fun}(\mathcal{D}, \mathcal{E})^{\simeq} \to \mathrm{Fun}(\mathcal{C}, \mathcal{E})^{\simeq}$$

is an equivalence of ∞-groupoids.

(*iv*) *There is a commutative diagram in* $\mathcal{H}_{\mathrm{Joy}}$ *of the form*

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \operatorname{cosk}_{n+1}\mathcal{C} \\ & \searrow^{F} & \big\vert\cong \\ & & \mathcal{D} \end{array}$$

where the right-hand map is an isomorphism.

(*v*) *The map* $\operatorname{cosk}_{n+1}\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{D}$ *is an equivalence of* ∞*-categories.*

Proof In Example 1.A.28, we showed that $\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{C}$ induces on slice mapping spaces the map

$$(\mathcal{C}/y)_x \to \operatorname{cosk}_n\big((\mathcal{C}/y)_x\big)$$

up to isomorphism. By Lemma 1.A.38, this map is an $(n-1)$-truncation; it is also isomorphic in the $\mathcal{H}$ to the induced maps $\mathcal{C}(x,y) \to \big(\operatorname{cosk}_{n+1}\mathcal{C}\big)(x,y)$, so this is also an $(n-1)$-truncation.

(*i*) = (*iv*) Since $\mathcal{D}$ is an n-category, $\mathcal{D} \to \operatorname{cosk}_{n+1}\mathcal{D}$ is an equivalence. Thus essential surjectivity of $\mathcal{C} \to \mathcal{D}$ is equivalent to essential surjectivity of $\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{D}$, which is equivalent to essential surjectivity of $\operatorname{cosk}_{n+1}\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{D}$ (because $\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{C}$ is bijective on objects). The diagram

$$\begin{array}{ccc} \mathcal{C}(x,y) & \longrightarrow & \mathcal{D}(Fx,Fy) \\ \downarrow & & \downarrow \\ \big(\operatorname{cosk}_{n+1}\mathcal{C}\big)(x,y) & \longrightarrow & \big(\operatorname{cosk}_{n+1}\mathcal{D}\big)(Fx,Fy) \end{array}$$

is commutative. If (*iv*) holds, then by Lemma 1.A.38(iii) the upper map is an $(n-1)$-truncation, because the left-hand map is, hence (*i*). If (*i*) holds, then all maps except for the bottom one are $(n-1)$-truncations, so the bottom is an isomorphism on homotopy groups below $(n-1)$; since domain and codomain are $(n-1)$-truncated, it is an equivalence. Thus $\operatorname{cosk}_{n+1}\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{D}$ is essentially surjective and fully faithful.

The implication (*v*) $\Rightarrow$ (*ii*) is identical to (*iv*) $\Rightarrow$ (*ii*) in Lemma 1.A.38 because an n-category is equivalent to an $(n+1)$-coskeletal ∞-category. (*ii*) $\Rightarrow$ (*iii*) is obvious, and then (*iii*) $\Rightarrow$ (*v*) is identical to (*ii*) $\Rightarrow$ (*iv*) of Lemma 1.A.38 because $\pi_0\mathrm{Fun}(-,-)^{\simeq}$ corresponds to Hom's in $\mathcal{H}_{\mathrm{Joy}}$.

Finally, clearly (*v*) $\Rightarrow$ (*iv*), and we will show that (*iv*) $\Rightarrow$ (*ii*). We already know that $\mathcal{C} \to \operatorname{cosk}_{n+1}\mathcal{C}$ satisfy the other four conditions, and it is clear that if we postcompose with an equivalence of ∞-categories $\operatorname{cosk}_{n+1}\mathcal{C} \xrightarrow{\sim} \mathcal{D}'$ then the other four continue to be satisfied (e.g. by (*ii*)). Thus it suffices to show that if $F \simeq F'$ are naturally equivalent and F satisfies (*ii*) then F' satisfies (*ii*).

Indeed, if $\alpha\colon F \to F'$ is an equivalence, viewed as a map $J \to \mathrm{Fun}(\mathcal{C},\mathcal{D})$ i.e. $J \times \mathcal{C} \to \mathcal{D}$, then precomposition gives a map

$$\mathrm{Fun}(\mathcal{D},\mathcal{E}) \to \mathrm{Fun}(J, \mathrm{Fun}(\mathcal{C},\mathcal{E})).$$

By adjunction we obtain $J \to \mathrm{Fun}\big(\mathrm{Fun}(\mathcal{D}, \mathcal{E}), \mathrm{Fun}(\mathcal{C}, \mathcal{E})\big)$, which represents an equivalence from F^* to $(F')^*$. □

Definition 1.A.41 Let $\mathcal{C}$ be an ∞-category. A *homotopy n-category of* $\mathcal{C}$ is an n-category $\mathrm{h}_n\mathcal{C}$ equipped with a map $\mathcal{C} \to \mathrm{h}_n\mathcal{C}$ satisfying the equivalent conditions of Prop. 1.A.40.

So, we see that it always exists and can be constructed as $\mathcal{C} \to \mathrm{cosk}_{n+1}\mathcal{C}$. We also see that if $\mathcal{D}$ is an ∞-groupoid, then its n-truncation coincides with its homotopy n-category. We saw previously then that $\mathrm{h}_1\mathcal{C} \simeq \mathrm{h}\mathcal{C}$. Moreover, part (iv) says that it is well-defined up to isomorphism (of arrows) in $\mathcal{H}_{\mathrm{Joy}}$.

We remark that the construction h_n is homotopy functorial independent of the chosen model. More precisely, choose homotopy n-categories for $\mathcal{C}$ and $\mathcal{D}$. If $F\colon \mathcal{C} \to \mathcal{D}$ is any functor, then the composite $\mathcal{C} \to \mathrm{h}_n\mathcal{D}$ is a map to an n-category, hence factors through $\mathrm{h}_n\mathcal{C}$ up to homotopy. This is equivalent in $\mathcal{H}_{\mathrm{Joy}}$ to the map induced by cosk_{n+1}.

Remark 1.A.42 Sometimes a property of ∞-categories can be checked in its in homotopy n-categories for all n. For instance, it is easy to see that a functor $F\colon \mathcal{C} \to \mathcal{D}$ is fully faithful iff $\mathrm{h}_n\mathcal{C} \to \mathrm{h}_n\mathcal{D}$ is fully faithful for all n. This is because it induces $\tau_{\leq n-1}\mathcal{C}(x, y) \to \tau_{\leq n-1}\mathcal{D}(Fx, Fy)$ on mapping spaces, so in particular

$$\pi_{n-1}\mathcal{C}(x, y) \to \pi_{n-1}\mathcal{D}(Fx, Fy)$$

is an isomorphism for all n.

Here is an interesting observation:

Lemma 1.A.43 *Let $n \geq -2$.*

1. *If X is an n-truncated Kan complex, then $\underline{\mathrm{Hom}}(A, X)$ is an n-truncated complex for all simplicial sets A.*
2. *If $\mathcal{C}$ is an n-category, then $\underline{\mathrm{Hom}}(A, \mathcal{C})$ is an n-category for all simplicial sets A.*

Proof (1) is obviously a special case of (2). By Thm. 1.A.36, we may replace $\mathcal{C}$ with $\mathrm{cosk}_{n+1}\mathcal{C}$ and assume that $\mathcal{C}$ is $(n+1)$-coskeletal. Since $\partial\Delta^m \to \Delta^m$ is an $(n+1)$-isomorphism for $m > n+1$, so too is $\partial\Delta^m \times A \to \Delta^m \times A$. By Prop. 1.A.24 we deduce that

$$\mathrm{Hom}(\Delta^m, \underline{\mathrm{Hom}}(A, \mathcal{C})) \xrightarrow{\sim} \mathrm{Hom}(\partial\Delta^m, \underline{\mathrm{Hom}}(A, \mathcal{C}))$$

by adjunction for $m > n+1$. The same result then shows that $\underline{\mathrm{Hom}}(A, \mathcal{C})$ is $(n+1)$-coskeletal, as desired. □

Based on this, we make an internal definition for what truncated should mean.

Definition 1.A.44 Let $\mathcal{C}$ be an ∞-category and $r \geq -2$

1. An object y_0 of $\mathcal{C}$ is *r-truncated* if for each $x \in \mathcal{C}$, $\mathcal{C}(x, y_0)$ is an r-truncated Kan complex.
2. We define $\mathcal{C}_{\leq r} \subset \mathcal{C}$ to be the full subcategory of $\mathcal{C}$ spanned by the r-truncated objects.

Here are the immediate remarks from the definition:

1. $\mathcal{C}$ is an $(r+1)$-category iff $\mathcal{C}_{\leq r} = \mathcal{C}$. $(\mathcal{C}_{\leq r})_{\leq s} = \mathcal{C}_{\leq \min\{r,s\}}$. Also, since the condition of being r-truncated is clearly invariant under equivalence, $\mathcal{C}_{\leq r} \to \mathcal{C}$ is a Joyal fibration.
2. An object $e \in \mathcal{C}$ is terminal iff it is (-2)-truncated. Being (-1)-truncated can behave a bit more weirdly, e.g. in the category of (**U**-small) sets, $\varnothing$ is a (-1)-truncated object.
3. An object y_0 of $\mathcal{C}$ is r-truncated iff $\mathcal{C}/y_0 \to \mathcal{C}$ is a right fibration with r-truncated fibers, which we have seen is equivalent to this map having the RLP against $\partial\Delta^n \to \Delta^n$ for $n > r+1$, or to the property that for each $\mathcal{C}$ has the RLP against $\partial\Delta^n \to \Delta^n$ for $n > r$, where the map $\partial\Delta^n \to \mathcal{C}$ sends $n \mapsto y_0$. Thus this is a simple generalization of Thm. 1.1.27.
4. Let y_0 be an object of $\mathcal{S}$ classifying a small ∞-groupoid Y. Then for each x classifying X, we know that $\mathrm{Map}_{\mathcal{S}}(x, y_0) \simeq \underline{\mathrm{Hom}}(X, Y)$. Thus y_0 is r-truncated iff Y is r-truncated, by the previous lemma. In particular if $\mathcal{C}$ is locally small, an object y_0 is truncated iff its Yoneda functor $\text{ɣ}_{\mathcal{C}}(y_0)\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ factors through $\mathcal{S}_{\leq r}$.

Here is a natural definition to make based on this. Let $i\colon \mathcal{C}_{\leq r} \hookrightarrow \mathcal{C}$ be the inclusion; a partial right adjoint at some object x is thus an object $\tau_{\leq r}x$ equipped with a map $x \to \tau_{\leq r}x$ so that

$$\mathrm{Map}_{\mathcal{C}}(\tau_{\leq r}x, y) \to \mathrm{Map}_{\mathcal{C}}(x, y)$$

is an equivalence for each r-truncated object y. We call this an r-truncation of the object x. Thus all objects of $\mathcal{C}$ admit an r-truncation iff $\mathcal{C}_{\leq r} \hookrightarrow \mathcal{C}$ has a left adjoint, whose unit map specifies the truncation.

Example 1.A.45 The protoypical instance of this is $\mathcal{S}$. Based on Lemma 1.A.38, we have a left adjoint $\tau_{\leq r}\colon \mathcal{S} \to \mathcal{S}_{\leq r}$ as long as we check that each **U**-small Kan complex X has an r-truncation which is also **U**-small. In fact $\mathrm{cosk}_{r+1} X$ is already **U**-small, because Δ^n and hence $\mathrm{Sk}_{r+1}\Delta^n$ are **U**-small.

We want to address the same question for presheaves, and ask when a presheaf is truncated as an object of its functor category. It's useful to come up with some more general theory first. For $r \geq -2$, let us define an r-truncated morphism of simplicial sets to be one with the right lifting property against $\{\partial\Delta^n \hookrightarrow \Delta^n : n \geq r+2\}$. So, as expected, a (-2)-truncated morphism is a trivial fibration.

Lemma 1.A.46 *Let $r \geq -2$. The following classes of monomorphisms of presheaves are equal:*

(*a*) *The class of monomorphisms which are $(r+1)$-isomorphisms.*

(*b*) *The smallest saturated class containing $\{\partial\Delta^n \hookrightarrow \Delta^n : n \geq r+2\}$.*

(*c*) *The class of morphisms with the LLP against r-truncated morphisms.*

Proof $(b) = (c)$ by the small object argument. (b) is contained in (a) because all such boundary inclusions are $(r+1)$-isomorphisms. Finally let $A \hookrightarrow B$ be an injective $(r+1)$-isomorphism; it will suffice to show that $A \cup \mathrm{Sk}_{k-1} B \to A \cup \mathrm{Sk}_k B$ is contained in (b) for all k, by closure under transfinite compositions. Whenever $k \leq r+1$, $\mathrm{Sk}_{k-1} B$ and $\mathrm{Sk}_k B$ are already in A, so this map is just 1_A, and we may assume $k \geq r+2$. The skeletal filtration gives $A \cup \mathrm{Sk}_{k-1} B \to A \cup \mathrm{Sk}_k B$ as a pushout of

$$\coprod_\alpha \partial\Delta^k_\alpha \to \coprod_\alpha \Delta^k_\alpha,$$

which is in (b) because $k \geq r+2$. □

Because of this, we have our usual favorite closure properties which apply to r-truncated morphisms. These in turn apply to r-truncated Kan complexes and r-categories, because they are the case of the map $X \to \Delta^0$.

Example 1.A.47 For instance, again let $r \geq -2$ and suppose that $A \to B$ is an injective $(r+1)$-isomorphism and $U \hookrightarrow V$ is any monomorphism. Then the pushout product

$$A \times V \cup B \times U \hookrightarrow B \times V$$

is again an injective $(r+1)$-isomorphism. Indeed, in degrees $k \leq r+1$, we have

$$(A \times V \cup B \times U)_k = B_k \times V_k \cup B_k \times U_k = B_k \times V_k$$

since $A_k = B_k$. Therefore by the usual lifting correspondence:

1. If $U \to V$ is a monomorphism and $X \to Y$ is an r-truncated morphism, then the induced map

$$\underline{\mathrm{Hom}}(V, X) \to \underline{\mathrm{Hom}}(U, X) \times_{\underline{\mathrm{Hom}}(U,Y)} \underline{\mathrm{Hom}}(V, Y)$$

is an r-truncated morphism.

2. If $A \hookrightarrow B$ is an $(r+1)$-isomorphism and $X \to Y$ is an r-truncated morphism, then the induced map

$$\underline{\mathrm{Hom}}(B, X) \to \underline{\mathrm{Hom}}(A, X) \times_{\underline{\mathrm{Hom}}(A,Y)} \underline{\mathrm{Hom}}(B, Y)$$

is a trivial fibration.

Example 1.A.48 A right (or left) fibration $p\colon X \to Y$ is r-truncated if and only if it has r-truncated fibers. Indeed one direction is clear by pullback stability,

and the other direction is precisely Lemma 1.A.35. We call such a map an r-truncated right fibration. We can reformulate the previous example as follows: Suppose $Y \to A$ is an r-truncated right fibration, $X' \hookrightarrow X$ is a monomorphism over A, and consider the induced map

$$\underline{\mathrm{Hom}}_A(X, Y) \to \underline{\mathrm{Hom}}_A(X', Y)$$

of Kan complexes. By Lemma 1.1.3 this is a Kan fibration. Since it is pullback of the restriction maps in Example 1.A.47, it is also r-truncated. If $X' \hookrightarrow X$ is furthermore an $(r+1)$-isomorphism, then it is a trivial fibration.

Example 1.A.49 Let Y be an r-truncated Kan complex. Then a Kan fibration $p\colon X \to Y$ is r-truncated iff X is r-truncated. Indeed, one direction is clear because the class of r-truncated Kan fibrations is closed under composition. For the other direction, note that for each $y \in Y$, the fiber X_y is r-truncated by the long exact sequence. Thus Example 1.A.48 implies that $X \to Y$ is r-truncated.

Example 1.A.50 Let A be a small simplicial set. A presheaf $F\colon A^{\mathrm{op}} \to \mathcal{S}$ is r-truncated as an object of $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})$ iff for each $a \in A$, $F(a)$ is r-truncated in $\mathcal{S}$ (or equivalently classifies an r-truncated ∞-groupoid). Indeed, we translate this problem under Cor. 1.2.26 to showing that $\underline{\mathrm{Hom}}_A(X, Y)$ is r-truncated for all right fibrations $X \to A$ with small fibers iff $Y \to A$ is an r-truncated right fibration with small fibers. One direction follows from the previous example (applied to $\varnothing \hookrightarrow X$). For the other direction, choose a right anodyne extension $(\Delta^0, a) \to X$ giving a fibrant replacement for $a \in A$ in $(\mathbf{sSet}/A)_{\mathrm{con}}$, with X having small fibers via Remark 1.1.55. Then $Y_a \simeq \underline{\mathrm{Hom}}_A(a, Y) \simeq \underline{\mathrm{Hom}}_A(X, Y)$ is r-truncated (Lemma 1.1.3), so $Y \to A$ is an r-truncated right fibration. In other words $\underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S})_{\leq r} = \underline{\mathrm{Hom}}(A^{\mathrm{op}}, \mathcal{S}_{\leq r})$. Moreover:

1. It is clear that if $A = \mathcal{C}$ is a locally small ∞-category, then we can dispense with the smallnes above: a has a fibrant replacement by $\mathcal{C}/a$ which is right fibrant with essentially small fibers. (Equivalently, the Yoneda lemma.) Moreover, what we have said means that $x \in \mathcal{C}$ is r-truncated iff $\text{ɬ}_{\mathcal{C}}(x) \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is an r-truncated presheaf.

2. It also follows from the above that $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ has truncation functors, namely $\tau_{\leq r}\colon \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}_{\leq r})$ simply given by postcomposition with $\tau_{\leq r}$. If we consider the truncation $\text{ɬ}(x) \to \tau_{\leq r}\text{ɬ}(x)$ (which also corresponds to $1_x \in \tau_{\leq r}\,\mathrm{Map}_{\mathcal{C}}(x, x)$), it induces

$$\mathrm{Map}(\tau_{\leq r}\text{ɬ}(x), \text{ɬ}(y)) \xrightarrow{\sim} \mathrm{Map}(\text{ɬ}(x), \text{ɬ}(y)) \simeq \mathrm{Map}_{\mathcal{C}}(x, y)$$

for all r-truncated objects y of $\mathcal{C}$. If $\tau_{\leq r}\text{ɬ}(x)$ is representable by some t (equipped with an object in $\tau_{\leq r}\,\mathrm{Map}_{\mathcal{C}}(t, x)$), then t is r-truncated, and the case above of $y = t$ corresponding to the equivalence $\tau_{\leq r}\text{ɬ}(x) \simeq \text{ɬ}(t)$ provides a map $u\colon x \to t$. This satisfies the property that the induced map $\mathrm{Map}_{\mathcal{C}}(\text{ɬ}(t), \text{ɬ}(y)) \to \mathrm{Map}_{\mathcal{C}}(\text{ɬ}(x), \text{ɬ}(y))$ is an equivalence for all $y \in \mathcal{C}_{\leq r}$,

so by the Yoneda lemma $x \to t$ is a truncation. We remark however that the converse doesn't always hold (e.g. it doesn't hold for $\mathcal{S}$).

Our last result of this section will discuss the r-truncated objects of $\mathbf{Cat}_\infty$, which turn out not to be simply n-categories or $(n+1)$-categories, but rather something in between.

Proposition 1.A.51 *Let $\mathcal{C}$ be an ∞-category and $r \geq -2$. The following conditions are equivalent:*

(*i*) $\underline{\mathrm{Hom}}(A, \mathcal{C})^{\simeq}$ *is an r-truncated ∞-groupoid for all simplicial sets A.*

(*ii*) $\underline{\mathrm{Hom}}(\Delta^n, \mathcal{C})^{\simeq}$ *is an r-truncated ∞-groupoid for $n = 0, 1$.*

(*iii*) $\mathcal{C}$ *is an $(r+1)$-category and $\mathcal{C}^{\simeq}$ is r-truncated.*

Proof Clearly, (*i*) $\Rightarrow$ (*ii*); we show (*ii*) $\Leftrightarrow$ (*iii*). We have to show that if $\mathcal{C}^{\simeq}$ is r-truncated, then $\mathcal{C}$ is an $(r+1)$-category iff $\underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})^{\simeq}$ is r-truncated. Indeed, the Kan fibration

$$\underline{\mathrm{Hom}}(\Delta^1, \mathcal{C})^{\simeq} \to \mathcal{C}^{\simeq} \times \mathcal{C}^{\simeq}$$

is r-truncated iff its fibers $\mathcal{C}(x, y)$ are r-truncated, iff the domain is r-truncated by Example 1.A.49.

(*ii*) $\Rightarrow$ (*i*) Let M be the class of monomorphisms $A \hookrightarrow B$ for which the induced Kan fibration

$$\underline{\mathrm{Hom}}(B, \mathcal{C})^{\simeq} \to \underline{\mathrm{Hom}}(A, \mathcal{C})^{\simeq}$$

is r-truncated. Then M satisfies the following properties:

1. M is saturated.
2. M contains $\varnothing \to \Delta^0, \varnothing \to \Delta^1$ and inner horn inclusions.
3. If $A \hookrightarrow B$ and $\varnothing \to A$ and $\varnothing \to B$ are in M, then $A \hookrightarrow B$ is in M.

Indeed, (2) is by the assumption and the fact that the induced map $\underline{\mathrm{Hom}}(\Delta^n, \mathcal{C})^{\simeq} \to \underline{\mathrm{Hom}}(\Lambda^n_k, \mathcal{C})^{\simeq}$ is a trivial fibration for $0 < k < n$. (3) is a consequence of Example 1.A.49. (1) is because, by the standard lifting correspondence, this relevant map having the RLP against $\partial\Delta^m \to \Delta^m$ is equivalent to $A \hookrightarrow B$ having the LLP against $h(\Delta^m, \mathcal{C}) \to h(\partial\Delta^m, \mathcal{C})$. Thus by Example 1.A.18 we conclude that M contains $\varnothing \to A$ for all A. □

In particular, for the ∞-category $\mathbf{Cat}_\infty$ of small ∞-categories, it follows that $\mathcal{C}$ is r-truncated iff the conditions of the result above hold, namely that $\mathcal{C}$ is an $(r+1)$-category with r-truncated core.

We would like to have an intrinsic theory of r-truncated *maps* in an ∞-category as well. To do this we need some more homotopy-invariant statements on the level of Kan complexes.

Lemma 1.A.52 *Let $f\colon X \to Y$ be a map of Kan complexes. The following conditions are equivalent for $r \geq -2$:*

(*i*) *f has r-truncated homotopy fibers.*

(*ii*) *For each $x_0 \in X$, the induced map $f_*\colon \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is injective for $k = r+1$, and an isomorphism for $k > r+1$.*

(*iii*) *Given any Kan fibration $f'\colon X' \to Y'$ between Kan complexes for which there is commutative diagram in $\mathcal{H}$*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

with vertical isomorphisms, the map f' is r-truncated.

In this case, we say that f is homotopy r-truncated.

Proof The long exact sequence of homotopy groups associated to a homotopy fiber of f immediately shows that (*i*) and (*ii*) are equivalent. Assuming (*iii*), the condition of (*ii*) clearly passes to f', so f' has r-truncated fibers, and hence is r-truncated by Example 1.A.48. The converse is clear because we can take a fibrant replacement of f to compute its homotopy fibers. □

Remark 1.A.53 Here are some basic properties of this definition:

1. A homotopy pullback of a homotopy r-truncated map is homotopy r-truncated. This is clear by criterion (*i*) above.
2. For $r \geq -1$, a map $f\colon X \to Y$ of Kan complexes is homotopy r-truncated iff the homotopy diagonal map $X \to X \times^h_Y X$ is homotopy $(r-1)$-truncated. We may check this after replacing f with a Kan fibration:
   (*a*) Then note that f is r-truncated iff either projection $X \times_Y X \to X$ is r-truncated. One direction is by pullback stability, and the other is because the fibers of the projection are $X_{f(x)}$ by the pasting lemma, hence the fibers of f are either empty (i.e. (-1)-truncated, so r-truncated) or r-truncated.
   (*b*) The diagonal $X \to X \times_Y X$ is a section of either projection, hence induces split monomorphisms on all homotopy groups. Hence it is homotopy $(r-1)$-truncated iff its retract $\pi_k(X \times_Y X) \to \pi_k(X)$ is a monomorphism for $k \geq r+1$, in which case both maps are isomorphisms.
3. We note that being homotopy (-2)-truncated is equivalent to all homotopy groups inducing isomorphisms, i.e. to $X \to Y$ being an equivalence. Thus along wiht (2) this gives a complete inductive characterization.

We note that because of Lemma 1.A.52, being homotopy r-truncated is a condition which depends only on the corresponding class in $\mathcal{H}$. In particular it makes sense to speak about a morphism in h$\mathcal{S}$ being homotopy r-truncated as well.

Corollary 1.A.54 *Let $\mathcal{C}$ be an ∞-category with a map $u\colon y \to y'$, and $r \geq -2$. The following conditions are equivalent:*

(*i*) *For each $x \in \mathcal{C}$, the map $\mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x,y')$ is homotopy r-truncated in $\mathcal{S}$.*

(*ii*) *(y,u) is an r-truncated object of $\mathcal{C}/y'$.*

In this case, we say that u is an r-truncated morphism of $\mathcal{C}$. Moreover:

1. *If e is a terminal object of $\mathcal{C}$, then an object y of $\mathcal{C}$ is r-truncated iff any terminal map $y \to e$ is r-truncated.*
2. *A morphism $u\colon y \to y'$ is (-2)-truncated iff it is an equivalence.*
3. *Let $u\colon y \to y'$ be a morphism for which the pullback $y \times_{y'} y$ exists. For $r \geq -1$, u is r-truncated iff the diagonal $y \to y \times_{y'} y$ is $(r-1)$-truncated.*

Proof It follows from Lemma 1.A.4 that

$$\mathrm{Map}_{\mathcal{C}/y'}\big((x,v),(y,u)\big) \simeq \mathrm{hofib}_v\left(\mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{u_*} \mathrm{Map}_{\mathcal{C}}(x,y')\right).$$

Therefore the equivalence $(i) = (ii)$ is immediate from the definitions.

For the consequences: (1) This is clear by (ii) under $\mathcal{C}/e \simeq \mathcal{C}$. (2) This holds iff (y,u) is terminal in $\mathcal{C}/y'$, i.e. u is an equivalence. (3) Applying $\mathrm{Map}_{\mathcal{C}}(x,-)$ to the diagonal gives the homotopy diagonal

$$\mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x,y) \times^h_{\mathrm{Map}_{\mathcal{C}}(x,y')} \mathrm{Map}_{\mathcal{C}}(x,y).$$

The diagonal of u is $(r-1)$-truncated iff the above map is $(r-1)$-truncated for all x, iff $\mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x,y')$ is r-truncated for all x by Remark 1.A.53, iff $y \to y'$ is r-truncated. □

Example 1.A.55 Suppose $\mathcal{C}$ has finite limits and $F\colon \mathcal{C} \to \mathcal{D}$ preserves them. Then whenever $u\colon y \to y'$ is an r-truncated morphism, $F(u)$ is an r-truncated morphism in $\mathcal{D}$. Indeed this follows by induction, where the base case is that F preserves equivalences. If F preserves $(r-1)$-truncated morphisms, then since it preserves diagonal morphisms, it also preserves r-truncated morphisms. A basic corollary is the following: if $\mathcal{D}$ is any ∞-category with finite limits, then a morphism of presheaves $F \to G$ in $\underline{\mathrm{Hom}}(A^{\mathrm{op}},\mathcal{D})$ is r-truncated iff $F(a) \to G(a)$ is r-truncated in $\mathcal{D}$ for all a. In particular $\underline{\mathrm{Hom}}(A^{\mathrm{op}},\mathcal{D})_{\leq r} = \underline{\mathrm{Hom}}(A^{\mathrm{op}},\mathcal{D}_{\leq r})$. This generalizes (and removes the size assumption in) Example 1.A.50.

2 Stable ∞-categories

2.1 Theory of stable ∞-categories

2.1.1 Stability, (co)fibers, and path/loop functors

We begin with the notion of zero object in an ∞-category. This behaves exactly like the corresponding case of a zero object in an ordinary category.

Definition 2.1.1 Let $\mathcal{C}$ be an ∞-category. A *zero object* of $\mathcal{C}$ is an object $0 \in \mathcal{C}$ which is both terminal and initial. We denote by $Z_{\mathcal{C}} \subset \mathcal{C}$ the full subcategory of $\mathcal{C}$ spanned by the zero objects. $\mathcal{C}$ is *pointed* if it contains a zero object.

Our convention is that pointedness is an existence statement as opposed to the structure of a specific zero object. Note that the condition of an object being zero is one up to equivalence, because both initiality and terminality are. In particular in a pointed ∞-category, $Z_{\mathcal{C}}$ coincides with the full subcategory of initial objects and of terminal objects, and is in particular a contractible ∞-groupoid by Cor. 1.1.30. From the definition (and Cor. 1.1.29), it follows that a zero object represents a zero object of $h\mathcal{C}$.

Example 2.1.2 The remark immediately above implies that between any two objects $x, y \in \mathcal{C}$ there's a unique homotopy class in $\pi_0\mathcal{C}(x, y)$ corresponding to a zero morphism. Here is a much stronger statement; assume $\mathcal{C}$ has a zero object 0, and consider the following pullback:

$$\begin{array}{ccccc} \mathcal{C}(-,0,-) & \longrightarrow & \mathcal{C}(-,Z_{\mathcal{C}},-) & \longrightarrow & \underline{\mathrm{Hom}}(\Delta^2,\mathcal{C}) \\ \downarrow & & \downarrow & \lrcorner & \downarrow \\ \mathcal{C}\times\mathcal{C} & \xrightarrow{(\mathrm{id},0,\mathrm{id})} & \mathcal{C}\times Z_{\mathcal{C}}\times\mathcal{C} & \longrightarrow & \mathcal{C}\times\mathcal{C}\times\mathcal{C} \end{array}$$

Thus the vertical maps are all Joyal fibrations, and the lower left horizontal map is a trivial cofibration, since it's a section of a pullback of $Z_{\mathcal{C}} \to \Delta^0$. Since all objects are fibrant, the upper left horizontal map is an equivalence. The left-most map factors as in the diagram:

$$\begin{array}{ccccc} \mathcal{C}(-,0,-) & \longrightarrow & \mathcal{C}(-,0)\times\mathcal{C}(0,-) & \longrightarrow & \mathcal{C}\times\mathcal{C} \\ \downarrow & \lrcorner & \downarrow & & \\ \underline{\mathrm{Hom}}(\Delta^2,\mathcal{C}) & \longrightarrow & \underline{\mathrm{Hom}}(\Lambda^2_1,\mathcal{C}) & & \end{array}$$

The lower map is a trivial fibration since $\mathcal{C}$ is an ∞-category, and $\mathcal{C}(-,0) = \mathcal{C}//0$ and $\mathcal{C}(0,-) = 0\backslash\backslash\mathcal{C}$. Since 0 is initial and terminal, the upper right map is the product of trivial fibrations, so $\mathcal{C}(-,0,-) \to \mathcal{C}\times\mathcal{C}$ is a trivial fibration. Combining these observations shows that the projection

$$\mathcal{C}(-,Z_{\mathcal{C}},-) \to \mathcal{C}\times Z_{\mathcal{C}}\times\mathcal{C} \to \mathcal{C}\times\mathcal{C}$$

on the left is a trivial fibration, as is the one on the right since $Z_{\mathcal{C}}$ is a contractible ∞-groupoid. The takeaways are:

1. If we fix $x, y \in \mathcal{C}$, then $\mathcal{C}(x, Z_{\mathcal{C}}, y) \simeq Z_{\mathcal{C}}$ is contractible, so there is an essentially unique zero morphism (after projecting to $\mathcal{C}(x, y)$) from x to y. By also fixing $0 \in Z_{\mathcal{C}}$ we can make this factor through any 0 object.
2. Choosing a section and projecting gives

$$\mathcal{C} \times \mathcal{C} \dashrightarrow \mathcal{C}(-, Z_{\mathcal{C}}, -) \to \mathcal{C}(-, -) = \mathrm{Arr}(\mathcal{C}),$$

so in other words we may choose a functorial arrow $(x, y) \mapsto (x \to 0 \to y)$ which is essentially unique. We can also do this after fixing $0 \in Z_{\mathcal{C}}$, which gives us a functor assigning factorizations through the same zero object for all x and y.

The latter property of (2) is very convenient: it means that we can prescribe a zero object 0 and and arrange for the rectangles

$$\begin{array}{ccccc} x & \longrightarrow & 0 & \longrightarrow & y \\ \downarrow & & \| & & \downarrow \\ x' & \longrightarrow & 0 & \longrightarrow & y' \end{array}$$

to commute (as $\Delta^1 \times \Delta^2 \to \mathcal{C}$) functorially in $x \to x'$ and $y \to y'$.

Definition 2.1.3 Let $\mathcal{C}$ be a pointed ∞-category. A *triangle* in $\mathcal{C}$ is a commutative square $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & z \end{array}$$

where 0 is a zero object of $\mathcal{C}$. We often write this as $x \to y \to z$, with the data of the maps $x \to 0 \to y$, $x \to z$, and the homotopies recording both composites suppressed. Moreover:

1. We say the triangle is a *fiber sequence* if the square above is cartesian, and a *cofiber sequence* if the square is cocartesian.
2. If it is a fiber sequence, we call $(x, x \to y)$ the *fiber* of $y \to z$. If it is a cofiber sequence, we call $(z, y \to z)$ the cofiber of $x \to y$.

Appendix 2.A More on stable ∞-categories

2.A.1 Recollections on Dold-Kan

Here we recall some material regarding the Dold-Kan correspondence, and verify its interactions with the Kan-Quillen model structure on simplicial sets. Initial references are in the author's expository paper *Extending the Dold-Kan Correspondence*. An alternative reference is [HA]. For the following constructions, we work mostly over $\mathbf{Ab} = \mathrm{Mod}(\mathbb{Z})$. Recall that a *simplicial abelian group* is a functor $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{Ab}$, and the corresponding functor category is written $\mathbf{sAb}$.

Remark 2.A.1 Here we recall the three major constructions of chain complexes associated to a simplicial abelian group A.

1. *Moore complex*: We denote this simply by $A_* \in \mathrm{Ch}_{\geq 0}(\mathbb{Z})$, as its nth term is simply A_n, with differential $\partial_n \colon A_n \to A_{n-1}$ given by the formula $\partial_n = \sum_{i=0}^{n} (-1)^i d_i$. This gives a well-defined chain complex and defines a functor $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$.

2. *Reduced complex*: Within A_*, there is a subcomplex $(DA)_* \subset A_*$ where $DA_n \subset A_n$ is the abelian subgroup generated by the degenerate simplices Of A_n. The *reduced complex* $(A/DA)_* = A_*/DA_*$ is the quotient complex by such degenerate simplices, and is likewise functorial in A with quotient map a natural transformation.

3. *Normalized complex*: For each $n \geq 0$, we define a subcomplex $N_*A \subset A_*$ by $N_0A = A_0$, and
$$N_nA \coloneqq \bigcap_{i=1}^{n} \ker d_i \subset A_n,$$
so that the restricted boundary map $\partial_n \colon N_nA \to N_{n-1}A$ coincides with d_0. This is again functorial in A with inclusion a natural transformation.

The initial relationship between these three constructions is that the map $N_*A \oplus DA_* \to A_*$ induced by the inclusions is an isomorphism, so that in particular we have a canonical isomorphism $N_*A \cong A_*/DA_*$.

Example 2.A.2 One uses the following elementary argument to show that the inclusion $N_*A \to A_*$ is in fact a homotopy equivalence. Indeed we already have the chain retract $p \colon A_* \to A_*/DA_* \xrightarrow{\sim} NA_*$, so we just need to check that $ip \simeq \mathrm{Id}_A$. We define a decreasing sequence of subcomplexes $\{F_*^r\}$ where
$$F_n^r = \bigcap_{i=\max\{1,n-r+1\}} \ker d_i,$$
which are indeed subcomplexes by the simplicial identities. Then $F_*^0 = A_*$, and for $r \geq n$ we have $F_n^r = N_nA$. One shows that each inclusion $F^r \hookrightarrow F^{r-1}$ is a chain homotopy equivalence by explicitly defining the inverse using face and degneracy maps. The degreewise stabilization shows that this sequence provides the map p above and completes the proof. We usually won't use this result or A_*, since it is usually too big of an object to work with directly.

Example 2.A.3 Since the free abelian group functor $\mathbb{Z}[-]$ is left adjoint to the forgetful functor $\mathbf{Ab} \to \mathbf{Set}$, the same degreewise functor $\mathbb{Z}[-] \colon \mathbf{sSet} \to \mathbf{sAb}$ is left adjoint to the forgetful functor. If X is a simplicial set, we define $C_*(X)$ or $C_*(X; \mathbb{Z})$ to be $N_*\mathbb{Z}[X]$, and $H_n(X; \mathbb{Z}) \coloneqq H_n(C_*(X))$ to be the homology of X. Here are some preliminary remarks:

1. We usually like to think of this in the model A/DA instead; this is because the subcomplex $D\mathbb{Z}[X]$ in degree n is precisely the free abelian group on the degenerate n-simplices, and $(\mathbb{Z}[X])_*$ is the free abelian group

on all n-simplices. Thus $(\mathbb{Z}[X]/D\mathbb{Z}[X])_n$ is the free abelian group on non-degenerate simplices. For small simplicial sets this is reasonable to compute, as shown below.

2. If $W \subset X$ is a simplicial subset, then the induced inclusion $C_*(W) \subset C_*(X)$ corresponds to (on the basis) identifying the non-degenerate n-simplices of W as those in X. We define $C_*(X, W) \coloneqq C_*(X)/C_*(W)$ as the quotient, and $H_k(X, W) \coloneqq H_k(C_*(X, W))$ to be the *relative homology* of (X, W). In particular $C_*(X, W)$ may be thought of as spanned by non-degenerate simplices of X not contained in W. Moreover we have a long exact sequence

$$\cdots \to H_k(W) \to H_k(X) \to H_k(X, W) \to H_{k-1}(W) \to \cdots$$

of the corresponding homology groups.

3. For instance, $C_n(\Delta^n) = \mathbb{Z}$ spanned by the single non-degenerate n-simplex and we see that $C_*(\Delta^n, \partial\Delta^n) = \mathbb{Z}[-n]$ (here we are using homological shifts). Similarly, $C_*(\Delta^n/\partial\Delta^n) = \mathbb{Z}[-n-1]$, so in particular $H_k(\Delta^n/\partial\Delta^n) = 0$ for all $k \neq n+1$, and $\mathbb{Z}$ for $k = n+1$.

Before even using the Dold-Kan correspondence, we can say interesting things about this definition of homology.

Proposition 2.A.4 *Let $f\colon X \to Y$ be a weak homotopy equivalence of simplicial sets. Then $C_*(X) \to C_*(Y)$ is a quasi-isomorphism.*

Proof The class of quasi-isomorphisms in $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ contain the isomorphisms and satisfy two-out-of-three. By Ken Brown's lemma ([HCHA, Prop. 2.2.7]), it suffices to prove that C_* sends anodyne extensions to quasi-isomorphisms. We first claim that the class of monomorphisms whose image under C_* is a quasi-isomorphism is saturated.

A retract of a quasi-isomorphism is a quasi-isomorphism. If $\{A_\alpha\}$ is an increasing sequence of simplicial sets, it induces an increasing sequence of inclusions on n-simplices, and thus it is clear that $\varinjlim_\alpha C_*(A_\alpha) \cong C_*(\bigcup_\alpha A_\alpha)$. If all of the transition maps induce quasi-isomorphisms, it follows that the transfinite composition is a quasi-isomorphism because homology commutes with filtered colimits. Third, suppose $A \hookrightarrow B$ induces a quasi-isomorphism and we have a pushout diagram:

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & B' \end{array}$$

Such a pushout gives an identification $B'_n = (B_n \setminus A_n) \cup A'_n$ which preserves non-degenerate simplices. Hence the non-degenerate simplices of B' not in A' are canonically identified with those of B not in A. This provides an isomorphism $C_*(B', A') \cong C_*(B, A)$, which is a chain map by commutativity of the diagram above. The assumption is equivalent to $C_*(B, A)$ being acyclic, so $C_*(B', A')$ is also acyclic, i.e. $A' \to B'$ induces a quasi-isomorphism.

We have now reduced to showing that $\Lambda^n_k \hookrightarrow \Delta^n$ induces a quasi-isomorphism for all $n \geq 1$ and $0 \leq k \leq n$. The simplices in Δ^n which aren't in Λ^n_k are precisely the unique non-denderate n-simplex, as well as the face corresponding to $d_k \colon \Delta^{n-1} \to \Delta^n$. Thus $C_*(\Delta^n, \Lambda^n_k)$ is precisely the complex

$$0 \to \mathbb{Z} \xrightarrow{(-1)^k} \mathbb{Z} \to 0$$

with the non-trivial terms in degrees n and $n-1$ from left to right. This complex is contractible, completing the proof. □

Example 2.A.5 This shows that the induced functor $\mathbf{sSet}_{\mathrm{KQ}} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to D^-(\mathbb{Z})$ factors through $\mathcal{H} \to D^-(\mathbb{Z})$. In particular we have well-defined homology functors on the homotopy category, so e.g. homotopic maps between Kan complexes induce the same map on homology. Some new applications:

1. It is clear from the definition that $C_*(\Delta^0) = \mathbb{Z}[0]$, so it only has 0th non-trivial homology. In particular, the same is true for Δ^n, and any weakly contractible simplicial set (such as ∞-category with an initial or terminal object).
2. Our computation in Example 2.A.3(3) implies, based on (1), that the map $\mathbb{Z} = H_n(\Delta^n, \partial\Delta^n) \to H_{n-1}(\partial\Delta^n)$ is an isomorphism, and that $H_k(\partial\Delta^n) = 0$ for $k \neq n-1$ for $n \geq 2$. Unwinding the definitions, a generator is $\sum_i (-1)^i [\Delta^{[n]\setminus i}]$. For $n = 1$, the map is $\mathbb{Z} \to H_0(\partial\Delta^1) = \mathbb{Z}^2$ sending $1 \mapsto [0] - [1]$.

We remark that this is our first real proof that $\partial\Delta^n$ is not weakly equivalent to Δ^n, so that $\partial\Delta^n \hookrightarrow \Delta^n$ isn't anodyne which was not obvious previously.

The *Dold-Kan functor* or *Eilenberg-MacLane* functor is a functor

$$\mathrm{DK} \colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{sAb}$$

assigning a chain complex to a simplicial abelian group. For $n \geq 0$, we define

$$\mathrm{DK}_n(C_*) \coloneqq \bigoplus_{[n] \twoheadrightarrow [k]} A_k$$

where the sum is taken over all surjections in $\mathbf{\Delta}$ out of $[n]$. The corresponding morphism is given by factorization of simplicial operators, but we will not write this down explicitly here because we don't need it. The classical Dold-Kan correspondence says that the pair

$$N_* \colon \mathbf{sAb} \rightleftarrows \mathrm{Ch}_{\geq 0}(\mathbb{Z}) : \mathrm{DK}$$

forms an equivalence of categories. Here is our first simple consequence:

Lemma 2.A.6 *A map $f \colon A \to B$ of simplicial abelian groups is a Kan fibration if and only if $N_k f \colon N_k A \to N_k B$ is surjective for $k > 0$. In particular, any simplicial abelian group is a Kan complex.*

Proof Denote by $U\colon \mathbf{sAb} \to \mathbf{sSet}$ the forgetful functor. Then $U(0) = \Delta^0$, and $U(A) \to \Delta^0$ is $U(A \to 0)$, which assuming the result is always a Kan fibration because $N_*A \to N_*0 = 0$ is degree-wise surjective.

For the main result, we have a lifting correspondence

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & U(A) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & U(B) \end{array} \qquad \begin{array}{ccc} \mathbb{Z}[\Lambda^n_k] & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow \\ \mathbb{Z}[\Delta^n] & \longrightarrow & B \end{array} \qquad \begin{array}{ccc} C_*(\Lambda^n_k) & \longrightarrow & N_*A \\ \downarrow & \nearrow & \downarrow \\ C_*(\Delta^n) & \longrightarrow & N_*B \end{array}$$

where the first is by the adjunction $\mathbb{Z}[-] \dashv U$, and the second is by the Dold-Kan correspondence. We have already seen that $C_n(\Delta^n, \Lambda^n_k)$ is the contractible complex $\mathbb{Z} \xrightarrow{(-1)^k} \mathbb{Z}$. It follows that the above lifting problems are also in correspondence with:

$$\begin{array}{ccc} 0 & \longrightarrow & N_*A \\ \downarrow & \nearrow & \downarrow \\ (\mathbb{Z} \xrightarrow{(-1)^k} \mathbb{Z})[-n] & \longrightarrow & N_*B \end{array}$$

The contractible complex is in degrees $[n-1, n]$, and a map from it to a chain complex is the same data as a choice of degree n element (equipped with $(-1)^k$ times its boundary). Thus this lifting problem expresses surjectivity in degree n, regardless of the chosen horn (provided that $n \geq 1$). □

Our next goal is to study the homotopy theory of a simplicial abelian group A, and show that it coincides with the homology theory of the corresponding chain complex $N_*A \cong (A/DA)_*$. Let us make a few preliminary observations:

1. Every simplicial abelian group A comes with a distinguished point $0 \in A_0$, which we view as a canonical basepoint. In particular we write $\pi_n(A) \coloneqq \pi_n(A, 0)$. Since all maps are required to preserve 0, this is functorial on simplicial abelian groups.

2. Note that $\pi_0(A)$, is now automatically an abelian group. Indeed, since A is a Kan complex, points a_0, a_1 have the same class iff $d_1(\sigma) = a_0$ and $d_0(\sigma) = a_1$ for some $\sigma \in A_1$. In other word, this happens iff $a_0 - a_1 \in \operatorname{im}(d_1 - d_0)$, so $\pi_0(A) = A_0/\operatorname{im}(d_1 - d_0)$. On the other hand, note that $N_0A = A_0$, and $N_1A \cong A_1/(\operatorname{im} s_0 + s_1)$, and the differential $N_1A \to N_0A$ is precisely $d_0 - d_1$, so canonically $\pi_0(A) = H_0(N_*A)$ as quotients of A_0.

3. If B is a simplicial abelian group and X is a simplicial set, then $\underline{\mathrm{Hom}}(X, B)$ is a simplicial abelian group; this is clear by its n-simplices $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n \times X, B)$ being abelian groups by pointwise addition, and simplicial operators acting through the first component by composition

4. For the same reason, most of our natural constructions behave well with respect to the group structure. For example, if A is a simplicial abelian

group and $Y \to A$ is a map of simplicial sets, then A/Y and $A//Y$ are both again simplicial abelian groups, because the assignments

$$\mathrm{Hom}_{Y\backslash\mathbf{sSet}}(\Delta^n * Y, A) \qquad \mathrm{Hom}_{Y\backslash\mathbf{sSet}}(\Delta^n \diamond Y, A)$$

make n-simplices abelian groups and simplicial operators act through the first component by composition.

Remark 2.A.7 Let A be a simplicial abelian group, then by (3) above we see that $\Omega(A) \coloneqq A(0,0)$ is canonically a simplicial abelian group, thus so too is $\Omega^n(A)$ for all n. By (2) above, $\pi_0\Omega^n(A)$ is a group in two ways, one from the simplicial abelian group structure and another from the structure of $\pi_n(A) = \mathrm{Hom}_{\mathrm{h}\Omega^{n-1}(A)}(0,0)$. In fact these two structures are the same: first by replacing A with $\Omega^{n-1}(A)$ we may assume $n = 1$. Since A is an abelian group object in **sSet** and $\tau\colon$ **sSet** $\to$ **Cat** preserves products, it follows that $\mathrm{h}A$ is an abelian group object in **Cat**, i.e. the group operation $\mathrm{h}A \times \mathrm{h}A \to \mathrm{h}A$ is a functor. The $0 \in \pi_0\Omega(A)$ comes from the zero of $\Omega(A)_0$ which is $s_0(0) = 1_0$, i.e. the zero in A_1. Functoriality above implies that the composition group structure on $\mathrm{Hom}_{\mathrm{h}A}(0,0)$ satisfies the interchange law with respect to the group strutcure on $\pi_0(\Omega A)$, so Eckman-Hilton shows that the structures coincide.

Remark 2.A.8 Let Y be a simplicial set, A be a simplicial abelian group, and $Y \to A$ a map of simplicial sets. Then canonical equivalence $A/Y \xrightarrow{\sim} A//Y$ is a morphism of simplicial abelian groups. Indeed, this is obvious, because in degree n this map is

$$\mathrm{Hom}_{Y\backslash\mathbf{sSet}}(\Delta^n * Y, A) \to \mathrm{Hom}_{Y\backslash\mathbf{sSet}}(\Delta^n \diamond Y, A)$$

which is precomposition by $\Delta^n \diamond Y \to \Delta^n * Y$, hence a group homomorphism. As a particular case of this, we will temporarily use the following notation: let $M^0(A) = A$ and inductively $M^n(A) = (M^{n-1}(A)/0)_0$. Then this says that the equivalence $M^n(A) \xrightarrow{\sim} \Omega(M^{n-1}(A))$ is one of simplicial abelian groups, and hence inductively we have a pointed equivalence $M^n(A) \xrightarrow{\sim} \Omega^n(A)$. The previous remark shows that the induced map $\pi_0 M^n(A) \cong \pi_n(A)$ is an isomorphism of abelian groups, not just sets.

We will use the two preceding remarks to show that under the Dold-Kan correspondence, homotopy groups are mapped to homology groups. In the next lemma we will use the smart truncation $\tau_{\geq 0}$ defined in [DcAG].

Lemma 2.A.9 *Let A be a simplicial abelian group. There is a canonical isomorphism of chain complexes*

$$N_*\big(M^n(A)\big) \cong \tau_{\geq 0}\big(N_*(A)[-n]\big)$$

natural in A. In other words, for any $C_* \in \mathrm{Ch}_{\geq 0}(\mathbb{Z})$, there is a canonical isomorphism $M_n(\mathrm{DK}(C_*)) \cong \mathrm{DK}\left(\tau_{\geq 0}(C_*[-n])\right)$.

Proof The second statement indeed follows immediately from the Dold-Kan correspondence. For the first statement, note that

$$\tau_{\geq 0}\big((\tau_{\geq 0}N_*(A)[-1])[-n+1]\big) = \tau_{\geq 0}(N_*(A)[-n]),$$

so inductively it suffices to prove the case that $n = 1$. It follows from the definition that

$$M^1(A)_n = \{\sigma \in A_{n+1} : d_{n+1}(\sigma) = 0, \sigma(n+1) = 0\},$$

and that the group structure is inherited as well from A_{n+1}. Hence $N_*M^1(A)$ adds the relations $d_1(\sigma) = \cdots = d_n(\sigma) = 0$. Note that for $n \geq 1$, the vertex $n+1$ is contained in $d_1\sigma$, so the condition is vacuous and we conclude that $N_n(M^1A) = N_{n+1}(A)$ for $n \geq 1$. The differential remains d_0 in all degrees ≥ 1.

For $n = 0$ however, we have $N_0(M^1A) = \{\sigma \in A_1 : d_1(\sigma) = d_0(\sigma) = 0\}$, which in other words is $\ker(N_1A \xrightarrow{d_0} N_0A)$. Thus $N_*(M^1A)$ is precisely the complex

$$\cdots \to N_3A \xrightarrow{d_0} N_2A \xrightarrow{d_0} \ker(N_1A \to N_0A) \to 0$$

in degrees $2, 1, 0, -1$. We conclude the lemma by adding a sign of $(-1)^n$ to the nth differential to match the standard convention of shifts. □

Corollary 2.A.10

1. *The functors* $(-)_*, N_* \colon \mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$ *send weak homotopy equivalences to quasi-isomorphisms, and* $\mathrm{DK} \colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{sAb}$ *sends quasi-isomorphisms to weak homotopy equivalences. The functor* $\mathbb{Z}[-] \colon \mathbf{sSet} \to \mathbf{sAb}$ *preserves weak homotopy equivalences.*

2. *For all* $n \geq 0$*, there are canonical quasi-isomorphisms*

$$\tau_{\geq 0}(N_*(A)[-n]) \xrightarrow{\sim} N_*(\Omega^n(A))$$

natural in $A \in \mathbf{sAb}$*, and weak homotopy equivalences*

$$\mathrm{DK}\,\tau_{\geq 0}(C_*[-n]) \xrightarrow{\sim} \Omega^n(\mathrm{DK}(C_*))$$

natural in $C_* \in \mathrm{Ch}_{\geq 0}(\mathbb{Z})$*.*

3. *Applying* H_0 *and* π_0 *respectively to the maps of (2) provides canonical isomorphisms*

$$H_n\big(N_*(A)\big) \xrightarrow{\sim} \pi_n(A) \qquad H_n(C_*) \xrightarrow{\sim} \pi_n(\mathrm{DK}(C_*))$$

for all $n \geq 0$*, natural in* A *and* C_**.*

Proof In (2) and (3), the versions involving DK are immediate from the N_* versions and the Dold-Kan correspondence. The map in (2) is the composite

$$\tau_{\geq 0}(N_*(A)[-n]) \xrightarrow{\sim} N_*(M^n(A)) \to N_*(\Omega^n(A))$$

where the first map is from Lemma 2.A.9, but we don't yet know that the second map is a quasi-isomorphism. In any case, if we apply H_0 and use the

identification $H_0 \circ N_* = \pi_0$, we obtain

$$H_n(N_*(A)) \xrightarrow{\sim} H_0 N_* M^n A = \pi_0 M^n A \xrightarrow{\sim} \pi_0 \Omega^n A = \pi_n(A),$$

because $M^n(A) \to \Omega^n(A)$ is a homotopy equivalence. This gives (3). Now if $A \to B$ is a map in **sAb** which is a weak homotopy equivalence, then by naturality of (3) we deduce that $H_n * (N_* A) \to H_n(N_* B)$ is an isomorphism for all N, i.e. $N_* A \to N_* B$ is a quasi-isomorphism. This retro-actively proves that $N_* M^n A \xrightarrow{\sim} N_* \Omega^n A$ is a quasi-isomorphism, and hence completes (2).

Since the natural map $N_* \to (-)_*$ is a component-wise chain homotopy equivalence (Example 2.A.2), this also completes the first part of (1). Quasi-isomorphisms, by naturality in (3), are sent to maps inducing isomorphisms on all homotopy groups, hence the middle part of (1). Finally if $X \to Y$ is a weak homotopy equivalence of simplicial sets, then the induced map $C_*(X) \to C_*(Y)$ is a quasi-isomorphism by Prop. 2.A.4, and applying DK shows that $\mathbb{Z}[X] \to \mathbb{Z}[Y]$ is a weak homotopy equivalence. □

Remark 2.A.11 When we're working with a simplicial abelian group A, base-points essentially don't matter: indeed, the multiplication map $A \times A \to A$ is a map of simplicial sets, and hence for each $a_0 \in A$ we have a canonical isomorphism $(-) + a_0 \colon A \to A$ (of simplicial sets). In particular this sends $\Omega^n(A) \xrightarrow{\sim} \Omega_n(A, a_0)$, and hence yields canonical isomorphism $\pi_n(A) \xrightarrow{\sim} \pi_n(A, a_0)$.

Example 2.A.12 In particular, for a simplicial set X we have $H_n(X) \cong \pi_n(\mathbb{Z}[X])$, so the homology is the homotopy of the 'abelianization', in some sense. If G is an abelian group and $n \geq 0$, we write $K(G, n) \coloneqq \mathrm{DK}(G[-n])$, which defines a functor $K(-, n) \colon \mathbf{Ab} \to \mathbf{sAb}$. Then $K(G, n)$ is a Kan complex with only one nontrivial homotopy group, namely G in degree n. More generally, we can use DK to make a space with any homotopy groups we like.

Example 2.A.13 Let A be a simplicial abelian group, and define the functor $B(-) \colon \mathbf{sAb} \to \mathbf{sAb}$ by sending $A \mapsto \mathrm{DK}(N_*(A)[1])$. Thus by Lemma 2.A.9 we have a canonical isomorphism $M^1(BA) \cong A$, and hence a canonical equivalence $A \xrightarrow{\sim} \Omega(BA)$. If we write $EA \coloneqq P(BA, 0)$, the canonical fiber sequence $\Omega(BA) \to P(BA) \to BA$ provides a homotopy fiber sequence $A \to EA \to BA$ with $EA \to BA$ a Kan fibration and EA contractible, whose long exact sequence records the shift in homotopy groups. If $f \colon A \to B$ is any map of simplicial abelian groups, we can take the cone sequence

$$0 \to N_* B \to \mathrm{Cone}(N_* f) \to N_* A[1] \to 0$$

which gives an exact sequence $0 \to B \to E_f \to BA \to 0$ of simplicial abelian groups. Since the latter map is degree-wise surjective, $E_f \to BA$ is a Kan fibration with fiber B.

To say more about this, it is useful to have a more concrete construction of the homotopy groups, precisely the group structure, than via loop spaces. We discuss this now: Generalizing our notation before, if (X, x_0) is a pointed Kan

complex we define $M(X, x_0) = \big((X/x_0)_{x_0}, 1_{x_0}\big)$, which is again a pointed Kan complex, and then $M^n(X, x_0) = M(M^{n-1}(X, x_0))$ inductively for $n \geq 2$. By convention we set $M^0(X, x_0) = (X, x_0)$. This construction provides functors $\mathbf{Kan}_* \to \mathbf{Kan}_*$. Of course, we could also define this just as well for arbitrary simplicial sets.

The canonical pointed equivalence $(X/x_0)_{x_0} \xrightarrow{\sim} X(x_0, x_0) = \Omega(X, x_0)$ provides a canonical equivalence $M^n(X, x_0) \xrightarrow{\sim} \Omega(M^{n-1}(X, x_0))$, and then inductively we obtain an equivalence $M^n(X, x_0) \xrightarrow{\sim} \Omega^n(X, x_0)$ by composing. In particular this induces an isomorphism $\pi_0 M^n(X, x_0) \cong \pi_n(X, x_0)$. After we get a good understanding of the structure of π_n through the LHS, we will treat this as an identification.

Lemma 2.A.14 *Let (X, x_0) be a pointed Kan complex and $n \geq 1$.*

1. *$M^n(X, x_0)$ is canonically a subcomplex of $X/(\Delta^{n-1}, x_0)$ sliced over the constant map $\Delta^{n-1} \to X$ at x_0. As such, a k-simplex $\Delta^k * \Delta^{n-1} \to X$ belongs to $M^n(X, x_0)$ if and only if its restriction along*

$$\Delta^k * \partial\Delta^{n-1} \cup \varnothing * \Delta^{n-1} \hookrightarrow \Delta^k * \Delta^{n-1}$$

   is constant at x_0.

2. *In particular a point of $M^n(X, x_0)$ is a map $\alpha\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)$ of pairs. There is an edge between any two such pairs α, β if and only if there is a map*

$$h\colon (\Delta^1 \times \Delta^n, \Delta^1 \times \partial\Delta^n) \to (X, x_0)$$

   of pairs such that $h|_{0\times\Delta^n} \equiv \alpha$ and $h|_{1\times\Delta^n} \equiv \beta$.

Proof (1) For $n = 1$, we have $M(X, x_0) \subset X/x_0$, and inductively if $M^{n-1}(X, x_0) \subset X/(\Delta^{n-2}, x_0)$ then

$$M^n(X, x_0) \subset \Big(\big(X/(\Delta^{n-2}, x_0)\big)/c_{x_0}\Big)_{c_{x_0}} \subset X/(\Delta^{n-1}, x_0)$$

by associativity of joins. For the second part, this is by induction on n. For $n = 1$ the statement is tautological. Suppose the formula holds for $M^{n-1}(X, x_0) \subset X/(\Delta^{n-2}, x_0)$. Then a k-simplex of $M^n(X, x_0)$ is firstly a map $\Delta^k * \Delta^0 \to M^{n-1}(X, x_0)$, i.e. a map $\Delta^k * \Delta^0 * \Delta^{n-2} \to X$ such that the restriction to

$$\Delta^k * \Delta^0 * \partial\Delta^{n-2} \cup \varnothing * \varnothing * \Delta^{n-2} \subset \Delta^k * \Delta^0 * \Delta^{n-2}$$

is constant at x_0; the further conditions imposed are that we are slicing over c_{x_0}, so it is constant at x_0 over $\varnothing * \Delta^0 * \Delta^{n-2}$, and then we take the fiber over x_0, which requires the restriction over $\Delta^k * \varnothing * \Delta^{n-2}$ to be constant. Taking the union of the three subcomplexes of $\Delta^k * \Delta^0 * \Delta^{n-2} = \Delta^k * \Delta^{n-1}$ gives

$$\begin{aligned}&\Delta^k * \big(\Delta^0 * \partial\Delta^{n-2} \cup \varnothing * \Delta^{n-2}\big) \cup \varnothing * \Delta^{n-1}\\ &= \Delta^k * \partial\Delta^{n-1} \cup \varnothing * \Delta^{n-1}\end{aligned}$$

as desired.

(2) The first sentence holds because, taking $k = 0$ in (1) gives

$$\Delta^0 * \partial\Delta^{n-1} \cup \varnothing * \Delta^{n-1} = \partial\Delta^n \subset \Delta^0 * \Delta^{n-1} = \Delta^n,$$

providing the description as maps of pairs. Let $L = \Delta^1 * \partial\Delta^{n-1} \cup \varnothing * \Delta^{n-1} \subset \Delta^1 * \Delta^{n-1}$, and consider the inclusion $j\colon \Delta^{n+1} \hookrightarrow \Delta^1 \times \Delta^n$ corresponding to

$$(0,0) \to (0,1) \to (1,1) \to \cdots \to (1,n).$$

We claim that $j^{-1}(\Delta^1 \times \partial\Delta^n) = L$. Indeed, as a subcomplex of Δ^{n+1}, L coincides with the union:

$$\Delta^{\{2,\ldots,n+1\}} \cup \bigcup_{i=2}^{n+1} \Delta^{[n+1]\setminus i}$$

Further viewing this in $\Delta^1 \times \Delta^n$, the left component becomes $\Delta^{10,\ldots,1n}$, and the right one becomes:

$$\bigcup_{i=1}^{n} \Delta^{\{00,01,11,\ldots,\widehat{1i},\ldots,1n\}}$$

Thus the intersection of $\Delta^1 \times \partial\Delta^n$ with this copy of Δ^{n+1} is precisely L. Moreover, there is a retract $r\colon \Delta^1 \times \Delta^n \to \Delta^{n+1}$ of j defined by

$$r(\varepsilon, i) = \begin{cases} \epsilon & i = 0 \\ i+1 & i \geq 1 \end{cases}.$$

with $rj = 1_{\Delta^{n+1}}$, and $r(\Delta^1 \times \partial\Delta^n) \subset L$. Both j and r respect the endpoints, and allow us to go between maps from $(\Delta^1 \times \Delta^n, \Delta^1 \times \partial\Delta^n)$ and maps of pairs from (Δ^{n+1}, L), which is what we wanted. □

The lemma means that under $\pi_0 M^n(X, x_0) \xrightarrow{\sim} \pi_n(X, x_0)$, we can describe an element of $\pi_n(X, x_0)$ as a class $[\alpha]$ with $\alpha\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)$, where any α, β satisfy $[\alpha] = [\beta]$ iff there is a relative homotopy $(\Delta^1 \times \Delta^n, \Delta^1 \times \partial\Delta^n) \to (X, x_0)$ between them. This is clearly functorial in (X, x_0). As a convention, when we talk about specific elements of π_n as classes, we will mean this.

To complete the identification, we want to explicitly identify the group structure on $\pi_0 M^n(X, x_0)$. Because of the equivalence $M^n(X, x_0) \xrightarrow{\sim} \Omega^n(X, x_0)$, it is precisely given by the group structure in the automorphism group

$$\mathrm{Hom}_{\mathrm{h}M^{n-1}(X,x_0)}(c_x, c_x) \cong \pi_0 M^n(X, x_0).$$

In what follows, we will write this as addition even though for $n = 1$ the group may not be abelian.

Lemma 2.A.15 *Let (X, x_0) be a pointed Kan complex. Under the canonical identification of sets for $n \geq 1$*

$$\{\textit{maps } (\Delta^n, \partial\Delta^n) \to (X, x_0)\}/\sim \xrightarrow{\sim} \pi_n(X, x_0)$$

described above, the group operation is determined as follows: If $[a], [b], [c]$ *are classes on the left-hand side, then* $[b] = [a] + [c]$ *iff there is an* $(n+1)$*-simplex* $\sigma \in X_{n+1}$ *such that*

$$d_0\sigma = a, \qquad d_1\sigma = b, \qquad d_2\sigma = c,$$

and $d_i\sigma = c_{x_0}$ *for* $3 \le i \le n+1$.

Proof This is essentially by definition. Indeed, we view $[a], [b], [c]$ as morphisms $c_x \to c_x$ in the ∞-groupoid $M^{n-1}(X, x_0)$, then we know that $[a] + [c] \equiv [a] \circ [c] = [b]$ iff there is a 2-simplex $\sigma\colon \Delta^2 \to M_{n-1}$ whose boundary represents $a \circ c \sim b$. For $n = 1$ this is tautological, so assume $n > 1$.

By Lemma 2.A.14, any 2-simplex in M_{n-1} can be described as an $(n+1)$-simplex $\sigma\colon \Delta^{n+1} \to X$ such that

$$\sigma|_{\Delta^{\{3,\dots,n+1\}}} = x_0 \qquad d_i\sigma = x_0 \quad 3 \le i \le n+1.$$

The additional conditions that make it a witness to $[a] \circ [c] \sim [b]$ are precisely that $d_0\sigma = a, d_1\sigma = b, d_2\sigma = c$. Then the first condition above becomes vacuous, since e.g.

$$\Delta^{\{3,\dots,n+1\}} \subset \Delta^{\{2,\dots,n+1\}} \subset \partial\Delta^{\{1,\dots,n+1\}},$$

and the restriction to $\Delta^{\{1,\dots,n+1\}}$ is already required to be σ, and $\partial\sigma$ is required to be x_0 by definition. □

Remark 2.A.16 Here's another useful version of the previous description connecting this with simplicial abelian groups.

1. Let A be a simplicial abelian group, and consider $\pi_n(A)$. It follows from our observations in Remark 2.A.7 that, if we view the elements as representatives $(\Delta^n, \partial\Delta^n) \to (A, 0)$, the addition in $\pi_n(A)$ coincides with the addition induced by A, namely by addition in A_n.

2. More precisely, consider the isomorphism of Lemma 2.A.9; in degree 0 this sends a map $(\Delta^n, \partial\Delta^n) \to (A, 0)$ to an n-cycle $Z_n(N_*A) = \bigcap_i \ker d_i$, which are the same thing. In particular the isomorphism $\pi_n(A) \xrightarrow{\sim} H_n(N_*A)$ simply sends $[\alpha] \mapsto [\alpha]$.

3. It is sometimes useful to view (2) using the quotient-by-degeneracies model of N_*A, especially when $A = \mathbb{Z}[X]$. Namely, let X be a pointed simplicial set and consider the composite

$$M^n(X, x_0) \to M^n(\mathbb{Z}[X], x_0) \xrightarrow{\sim} M^n(\mathbb{Z}[X]) \xrightarrow{\sim} \mathrm{DK}\big(\tau_{\ge 0}\big(C_*(X)[-n]\big)\big)$$

   where the middle map is subtraction $(-) - x_0$. This sends $\alpha\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)$ first to $\alpha - x_0$, and then to the corresponding cycle. However if we use $N_*A = A_*/DA_*$, then $[\alpha - x_0] = [\alpha]$ because x_0 is degenerate (for $n \ge 1$). In particular by taking π_0 we obtain the natural group homomorphism

$$h_n\colon \pi_n(X, x_0) \to H_n(X)$$

sending $[\alpha] \mapsto [\alpha - x_0] = [\alpha]$ for $n \geq 1$. For $n = 0$ it just sends a 0-cell x to $[x - x_0] \in H_0(X)$. This is called the *Hurewicz map*, and is natural with respect to pointed maps for all n.

We are next going to prove a much stronger condition which will be useful in establishing a few non-trivial homotopy theoretic results. First of all we use the elementary notion of connectedness and reducedness.

Proposition 2.A.17 *Let X be a Kan complex and $n \geq 0$. The following conditions are equivalent:*

(*i*) $\pi_0(X) = \{*\}$, *and for each* x_0 *and* $1 \leq k \leq n$ *we have* $\pi_n(X, x_0) = 0$.

(*ii*) *The n-truncation $\tau_{\leq n}X$ is contractible.*

(*iii*) *There exists a Kan complex X' such that $X'_k = \{*\}$ for $0 \leq k \leq n$ and an equivalence $X' \simeq X$.*

(*iv*) *The minimal model $X' \subset X$ satisfies the condition that $X'_k = \{*\}$ for all $0 \leq k \leq n$.*

Proof Indeed, (4) $\Rightarrow$ (3) because the minimal model is equivalent to X. (3) $\Rightarrow$ (2) In this case clearly $\pi_k(X'_k, *) = 0$ for all $0 \leq k \leq n$, so the composite $\Delta^0 \hookrightarrow X' \xrightarrow{\sim} X \to \tau_{\leq n}X$ induces isomorphisms on all homotopy groups. (2) $\Rightarrow$ (1) This is immediately clear using $X \to \tau_{\leq n}X$.

(1) $\Rightarrow$ (4) The trivial cofibration $X' \hookrightarrow X$ inherits the property (1). We prove the condition by induction on $n \geq 0$. If $n = 0$, any two points of X' are ∂-equivalent iff they are connected by a path, so $\pi_0(X') = \{*\}$ implies $X'_0 = \{*\}$. For general $n \geq 1$, X' also satisfies the same condition for $(n-1)$ and thus by induction has only 1 $(n-1)$-simplex. In particular all elements of X'_n represent classes of $\pi_n(X', *)$, and it follows from Lemma 2.A.14 and the definition ([HCHA, Defn. 5.1.3]) that any two n-simplices are ∂-equivalent. Thus there is only one. □

Definition 2.A.18 Let X be a Kan complex and $n \geq 0$.

1. X is *n-reduced* if for $0 \leq k \leq n$, $|X_k| = 1$.
2. X is *n-connected* if it satisfies the equivalent conditions of Prop. 2.A.17.

More generally, a simplicial set X is n-connected if for all $x_0 \in X$, $\pi_k(X, x_0) = 0$ for $0 \leq k \leq n$; equivalently its Kan fibrant replacements are n-connected Kan complexes.

Lemma 2.A.19 *Let $n \geq 1$ and X be an $(n-1)$-reduced Kan complex. Assume $f\colon X_n \to G$ is a function (of sets) where G is an abelian group, and that f satisfies the following conditions:*

1. $f(c_*) = 0$.

2. *Whenever* $u \in X_{n+1}$ *and* $d_i u = c_*$ *for all* $i \geq 3$*, we have*

$$f(d_0 u) - f(d_1 u) + f(d_2 u) = 0$$

in G*.*

Then for all $\tau \in X_{n+1}$*, we have* $\sum_{i=0}^{n+1} (-1)^i f(d_i \tau) = 0$*.*

The main use is the following result, called the *homotopy addition lemma*.

Corollary 2.A.20 *Let* $n \geq 2$ *and* (X, x_0) *be a pointed Kan complex with a sequence* $\tau_i \colon (\Delta^n, \partial\Delta^n) \to (X, x_0)$ *for* $i = 0, \dots, n+1$*. Then* $(\tau_0, \dots, \tau_{n+1})$ *is the boundary of an* $(n+1)$*-simplex iff the relation*

$$\sum_{i=0}^{n+1} (-1)^n [\tau_i] = 0$$

holds in $\pi_n(X, x_0)$*.*

Proof ($\Rightarrow$) If X is $(n-1)$-reduced, then in light of Lemma 2.A.15, this follows from applying Lemma 2.A.19 to the function $X_n \to \pi_n(X, *)$ sending $\sigma \mapsto [\sigma]$, which is well-defined since there's only one $(n-1)$-simplex.

For general X, take the fiber X' of $X \to \operatorname{cosk}_{n-1} X$ over x_0. Then $X'_m = \{\sigma : \sigma|_{\operatorname{Sk}_{n-1}\Delta^m} = x_0\}$, so $X'_m = \{x_0\}$ for $m \leq n-1$. In particular X'_k admits trivial horn fillers in dimensions $\leq n$. Given $\Lambda^m_k \to X'$ for $m > n$, we have $\operatorname{Sk}_{n-1}\Delta^m \subset \Lambda^m_k$, so a filler in X already lies in X'. In particular $X' \subset X$ is an $(n-1)$-reduced Kan complex containing the $\{\tau_i\}$, so the previous paragraph applies to it.

($\Leftarrow$) Since the $\{\tau_i\}$ have constant boundary x_0, we may fill the horn $\Lambda^{n+1}_{n+1} \to \Delta^{n+1}$ to some τ' prescribing $d_i \tau' = \tau_i$ for $i \leq n$. Applying the forward direction to τ' and cancelling terms shows that $[d_{n+1}\tau'] = [\tau_{n+1}]$ in $\pi_n(X, x_0)$. Hence there is a homotopy $h \colon \Delta^1 \times \Delta^n \to X$ between the two n-simplices fixed on $\Delta^1 \times \partial\Delta^n$. Moreover the inclusion

$$\{0\} \times \Delta^{n+1} \cup \Delta^1 \times \partial\Delta^{n+1} \hookrightarrow \Delta^1 \times \Delta^{n+1}$$

is anodyne because $\{0\} \to \Delta^1$ is anodyne. On $0 \times \Delta^{n+1}$ we prescribe τ'; next use the decomposition $\partial\Delta^{n+1} = \Lambda^{n+1}_{n+1} \cup \Delta^{[n+1]\setminus n+1}$, then on $\Delta^1 \times \Lambda^{n+1}_{n+1}$ use the $(n+1)$-horn we started with (constant along Δ^1), and on $\Delta^1 \times \Delta^{[n+1]\setminus n+1}$ use h. On the intersection $\Delta^1 \times \partial\Delta^n$ both are constant.

Our maps out of $0 \times \Delta^{n+1}$ and $\Delta^1 \times \partial\Delta^{n+1}$ are compatible because on the intersection $0 \times \partial\Delta^{n+1}$ both are $\partial\tau'$. Thus there is a lift, whose restriction to $1 \times \Delta^{n+1}$ we call $\tau \in X_{n+1}$. It has the properties that

$$\tau|_{\Lambda^{n+1}_{n+1}} = (\tau_0, \dots, \tau_n) \qquad \tau_{\Delta^{[n+1]\setminus n+1}} = h(1, -) = \tau_{n+1},$$

which completes the proof. □

We remark that the argument at the end of the proof is just the following simple fact: if we have a simplex in a Kan complex and a homotopy of one of its faces fixed on the boundary, we can extend to a homotopy of the simplex fixing the other faces and modifying the one with the homotopy.

Proof (of Lemma 2.A.19) This is actually a pretty easy but slightly confusing proof, so here are the steps we'll follow:

1. Note that the statement is tautological when $n = 1$, so we proceed by induction on n. We first form the Kan complex $X' = (*\backslash X)_*$, which by definition satisfies
$$X'_k = \{\sigma \in X_{k+1} : d_0\sigma = *\}$$
since there's only one vertex $*$, so in particular X' is $(n-2)$-reduced and $X'_{n-1} = X_n$. The main part will be showing that f, viewed as $f\colon X'_{n-1} \to G$, satisfies the listed conditions for $n-1$, thus allowing us to apply the inductive hypothesis.

2. Assuming this verification, take $\tau \in X_{n+1}$. We can't apply (1) yet because $d_0\tau$ may not be $*$, so instead we first construct an n-simplex b such that $f(b) = f(d_1\tau) - f(d_0\tau)$.

3. Then we build a new $\tau' \in X_{n+1}$ such that $d_0\tau' = *$, $d_1\tau' = b$, and $d_i\tau' = d_i\tau$ for $i \geq 2$. Now τ' *does* form an n-simplex of X', so (1) gives
$$0 = f(b) - f(d_2\tau') + \cdots = -f(d_0\tau) + f(d_1\tau) - f(d_2\tau) + \cdots$$
as desired.

Let's finish steps (2) and (3). We can construct $u \in X_{n+1}$ by filling along $\Lambda^{n+1}_2 \hookrightarrow \Delta^{n+1}$ with the requirements that
$$d_0u = d_0\tau, \qquad d_1u = d_1\tau, \qquad d_iu = *, i \geq 3,$$
because all restrictions to an $(n-1)$-face is necessarily constant. We take $b \coloneqq d_2u$, then our assumption implies the desired relation $f(b) = f(d_1\tau) - f(d_0\tau)$. For (3), we can construct $U \in X_{n+2}$ by filling along $\Lambda^{n+2}_3 \hookrightarrow \Delta^{n+2}$ where
$$U_0 = s_0(d_0\tau) \qquad U_1 = u \qquad U_2 = \tau \qquad U_i = s_2(d_{i-1}\tau), i \geq 4,$$
with $d_iU = U_i$. Indeed, first note that
$$d_{\ell-1}U_0 = d_{\ell-1}s_0d_0\tau = \begin{cases} d_0\tau & \ell = 1, 2 \\ * & \ell > 2 \end{cases}$$
by simplicial identities, which agrees with the other d_0U_ℓ, and in the case of $\ell \geq 4$ this is because $d_0U_\ell = s_1d_0d_{\ell-1}\tau = *$. The same shows that $d_1U_\ell = *$ for $\ell \geq 4$, and so we have $d_{\ell-1}u = d_1U_\ell$ for $\ell > 1$. Third, for $d_{\ell-1}U_2 = d_{\ell-1}\tau$ for $\ell > 2$, which agrees with
$$d_2U_\ell = d_2s_2(d_{\ell-1}\tau) = d_{\ell-1}\tau.$$
Finally, the $\{U_4, U_5, \dots\}$ are all compatible amongst each other, because applying d_k for $k \geq 4$ produces a degeneracy of an $(n-1)$-simplex, i.e. $*$.

Therefore there exists a lift U, and we set $\tau' \coloneqq d_3 U$. We have $d_i\tau' = d_2 U_i$ for $i \leq 2$, and $d_i\tau' = d_3 U_{i+1}$ for $i \geq 3$. Writing this out gives:

$$d_0\tau' = * \qquad d_1\tau' = d_2 u = b \qquad d_2\tau' = d_2\tau,$$

and $d_3\tau' = d_3(s_2(d_i\tau)) = d_i\tau$ for $i \geq 3$. This completes the proof of step (3).

It remains to finish *Step (1)*, namely showing that X' satisfies the $(n-1)$ hypothesis. Write d', s', f' for the corresponding versions for X' instead of X. First of all, $f'(*) = f(*) = 0$. Let $u' \in X'_n$ such that $d'_i u = *$ for $i \geq 3$. In other words $u' = u \in X_{n+1}$ satisfies

$$d_0 u = * \qquad d_4 u = \cdots = d_{n+1} u = *,$$

so we need to shift the $(1,2,3)$ faces to $(0,1,2)$-faces. Fill the horn $\Lambda^{n+1}_3 \hookrightarrow \Delta^{n+1}$ whose faces are

$$U_0 = s_2 d_3 u \qquad U_1 = u \qquad U_2 = s_1 a \qquad U_4 = s_0 d_3 u$$

with $U_i = *$ for $i \geq 5$. Here's why we can do this:

$$d_i U_0 = \begin{cases} d_3 u & i = 2,3 \\ * & \text{else} \end{cases}, d_i U_2 = \begin{cases} d_1 u & i = 1,2 \\ * & \text{else} \end{cases}, d_i U_4 = \begin{cases} d_3 u & i = 0,1 \\ * & \text{else} \end{cases}$$

The compatibilities between these three are

$$d_0 U_2 = * = d_1 U_0 \qquad d_0 U_4 = d_3 u = d_3 U_0 \qquad d_2 U_4 = * = d_3 U_2,$$

and it follows by definition that they are all compatible with $U_1 = u$. For $j \geq 5$, and $i = 0,1,2,3$, the higher compatibilities would be $d_i U_j = * = d_{j-1} U_i$, but $j - 1 \geq 4$, so $d_{j-1} U_i = *$ by the relations above (and the case for u).

Let U be a filler and write $w = d_3 U$ for the new face; then for $i = 0,1,2$, we have $d_i w = d_2 U_i$ which is $d_3 u, d_2 u, d_1 u$, respectively. For $i \geq 3$, $d_i w = d_3 U_{i+1}$, which are all $*$ either by assumption or by the equation for $d_i U_4$ above. Now applying the assumption to w shows that

$$f(d_3 u) - f(d_2 u) + f(d_1 u) = 0,$$

which is exactly what we wanted. □

The primary application of the homotopy addition lemma is to prove the simplicial Hurewicz theorem, whose proof is now easy.

Theorem 2.A.21 *Let X be an $(n-1)$-connected simplicial set for $n \geq 1$. Then $H_k(X) = 0$ for $1 \leq k \leq n-1$, and for each $x_0 \in X$, the Hurewicz map induces an isomorphism*

$$h_n \colon \pi_n(X, x_0)^{\mathrm{ab}} \xrightarrow{\sim} H_n(X)$$

of abelian groups; in particular $\pi_n(X, x_0) \xrightarrow{\sim} H_n(X)$ if $n \geq 2$.

Proof Both conclusions of the theorem are stable under weak homotopy equivalence, the second because the Hurewicz map (and hence the map from the

abelianization) is natural with respect to pointed maps. Hence we may assume X is a Kan complex, and therefore an $(n-1)$-reduced Kan complex. $H_k(X) = 0$ for $1 \leq k \leq n-1$ immediately follows because there are no non-degenerate cells in these dimensions.

Now, all n-cells of X are non-degnerate besides $*$, and as $C_{n-1}(X) = 0$, we have a cokernel in the top row of:

$$\begin{array}{ccccccc} C_{n+1}(X) & \xrightarrow{\partial} & C_n(X) & \twoheadrightarrow & H_n(X) & \longrightarrow & 0 \\ & & & \searrow & \downarrow & & \\ & & & & \pi_n(X,*)^{\mathrm{ab}} & & \end{array}$$

Here $C_n(X) = \mathbb{Z}[X_n \setminus \{*\}]$, so we have a canonical map $C_n(X) \to \pi_n(X)^{\mathrm{ab}}$ sending $\tau \mapsto [\tau]$, which makes sense because all boundaries must be $*$. The homotopy addition lemma (Cor. 2.A.20) ensures that its composite with ∂ is zero, where the $n = 1$ case is supplied directly from Lemma 2.A.15. Thus this descends to a unique map $H_n(X) \to \pi_n(X,*)^{\mathrm{ab}}$ sending $[\tau]_{H_n} \mapsto [\tau]_{\pi_n}$. Conversely the Hurewicz map sends $[\tau]_{\pi_n} \mapsto [\tau]_{H_n}$ by Remark 2.A.16. □

Remark 2.A.22 The Hurewicz theorem is one which has a much more pleasant and elementary proof for simplicial sets. It can also be reformulated in a much nicer way; if X is an $(n-1)$-connected Kan complex for $n \geq 2$, then the theorem says that $X \to \mathbb{Z}[X]$ induces an equivalence $\tau_{\leq n} X \to \tau_{\leq n}\mathbb{Z}[X]$ of n-truncations. This gives a basepoint-free formulation.

Example 2.A.23 Let $n \geq 1$ and suppose that X is a simplicial set which is $(n-1)$-reduced. Since $\mathrm{Sd}(\Delta^k)$ is k-skeletal, we conclude that $\mathrm{Ex}(X)_k = *$ for each $k \leq n-1$. Iterating implies $\mathrm{Ex}^\infty(X)$ satisfies the same property, i.e. $\mathrm{Ex}^\infty(X)$ is an $(n-1)$-connected Kan complex, meaning that X is $(n-1)$-connected.

Let's apply this to $X = S^n_\Delta = \Delta^n/\partial\Delta^n$ for $n \geq 1$, so this is $(n-1)$-connected. Thus we deduce that the Hurewicz map $\pi_n(S^n_\Delta) \to H_n(S^n_\Delta) = \mathbb{Z}$ is an isomorphism, just as in topology. Hence the same also applies to $\partial\Delta^{n+1}$. Technically for $n = 1$, we have only shown that $\pi_1(S^1_\Delta)^{\mathrm{ab}} = \mathbb{Z}$; we will compute all of its homotopy groups momentarily.

Proposition 2.A.24 *Let (X, x_0) be a pointed connected simplicial set, and assume there exists $n \geq 1$ such that $\pi_i(X, x_0) = 0$ for all $i \neq n$. Assuming $\pi_n(X, x_0)$ is abelian, there is a weak homotopy equivalence $\pi_n(X, x_0) \xrightarrow{\sim} K\big(\pi_n(X, x_0), n\big)$.*

Proof First take a fibrant resolution, and then choose an equivalence to an $(n-1)$-reduced Kan complex; so we may assume X is $(n-1)$-reduced. Writing $G \coloneqq \pi_n(X, x_0)$, the Hurewicz map $G \to H_n(X)$ is an isomorphism by Thm. 2.A.21. Since $C_k(X) = 0$ for $1 \leq k \leq n$, we may define $C(X) \to G[-n]$ which is the quotient map in degree n (and the zero map in degree 0); this induces an isomorphism on H_n. By adjunction, this corresponds to a map

$X \to \mathrm{DK}(G[-n]) = K(G,n)$ inducing an isomorphism on π_n. We conclude since it trivially induces an isomorphism on all other homotopy groups. □

In light of this result, we call such a space satisfying the above property *a K(G,n)*, even if it is not literally the functorial $K(G,n)$ constructed above.

Example 2.A.25 Let G be a possibly non-abelian group, and consider the simplicial set BG, which is the nerve of the category with one object and whose morphisms are the elements of G. In particular it is a 0-reduced 1-truncated Kan complex. Moreover $\pi_1(BG,*) = \mathrm{Hom}_{\mathrm{h}(BG)}(*,*) \cong G$. Thus BG gives a non-abelian '$K(G,1)$'. The previous result also holds in this case: if X is connected and has all trivial homotopy groups except π_1, then its Kan model is 1-truncated so $X \to N(\tau X)$ is a weak homotopy equivalence, where $N(\tau X)$ is connected and 1-truncated, and τX is a groupoid. Connectedness implies that we may take τX to have 1 object, and hence $N(\tau X) = B\,\mathrm{Aut}_{\tau X}(*)$.

Proposition 2.A.26 *The canonical map $S^1_\Delta \to B\mathbb{Z}$ is a cofinal and final monomorphism. In particular S^1_Δ is a $K(\mathbb{Z},1)$.*

Proof Let $B\mathbb{N}$ be the nerve of the category with 1-object whose morphisms form the monoid $\mathbb{N} = \{0,1,\dots\}$. The inclusion $B\mathbb{N} \to B\mathbb{Z}$ is cofinal and final; we apply Quillen's Theorem A (Cor. 1.1.38) and its opposite. There is only one object, and the two relevant comma categories are simply the posets $(\mathbb{Z},\geq)$ and $(\mathbb{Z},\leq)$, which are isomorphic by $n \mapsto -n$. We claim they are weakly contractible: consider the inclusion $(\mathbb{Z}_{\geq 0},\leq) \hookrightarrow (\mathbb{Z},\leq)$, then for each $n \in \mathbb{Z}$ we have $n\backslash(\mathbb{Z}_{\geq 0},\leq) \cong (\mathbb{Z}_{\geq \max(0,n)},\leq)$. This has initial object $\max(0,n)$, hence is weakly contractible, so another application of Quillen's theorem A shows that the inclusion is final. Since $(\mathbb{Z}_{\geq 0},\leq)$ has initial object 0, it is also weakly contractible, and we conclude the original application of Quillen A.

We will finish the proof by showing that there is an inner anodyne extension $S^1_\Delta \hookrightarrow B\mathbb{N}$. Identifying $(B\mathbb{N})_k = \mathbb{N}^k$, for each $k \geq 0$ we define the subset

$$(Y_k)_n = \{(a_1,\dots,a_n) : \sum_i a_i \leq k\}.$$

Since the identity of $B\mathbb{N}$ is 0, degeneracies don't change the sum, and neither do inner faces (since they just perform adjacent addition). Outer faces remove either a_1 or a_n and hence can only decrease the sum, thus $Y_k \subset B\mathbb{N}$ is a subcomplex. Y_k has a canonical non-degenerate k-simplex $(1,\dots,1)$, we first compute the pullback:

$$\begin{array}{ccc} P & \longrightarrow & Y_{k-1} \\ \downarrow & \lrcorner & \downarrow \\ \Delta^k & \longrightarrow & Y_k \end{array}$$

An ℓ-face of Δ^k corresponds a sequence of $(k-\ell)$ contractions of the k-simplex $(1,\dots,1)$, all of which preserve the sum k unless one of the outer face operators

is involved, which reduces the sum into Y_{k-1}. In particular an ℓ-face maps into Y_{k-1} iff it excludes one of the vertices 0 or $k-1$, and therefore

$$P = \Delta^{\{0,\dots,k-1\}} \cup \Delta^{\{1,\dots,k\}} = \Lambda^n_{\{1,\dots,k-1\}}$$

Note that for $k \geq 2$ $P \to \Delta^k$ is inner anodyne by Lemma 1.A.15. Now we show that the diagram is a pushout, hence that $Y_{k-1} \to Y_k$ is inner anodyne for $k \geq 2$. Since $B\mathbb{N} = \bigcup_k Y_k$, this will imply that $Y_k \to B\mathbb{N}$ is inner anodyne for all $k \geq 1$.

We have to show that the diagram induces a bijection $\mathrm{ND}_\ell(\Delta^k) \setminus \mathrm{ND}_\ell(P) \xrightarrow{\sim} \mathrm{ND}_\ell(Y_k) \setminus \mathrm{ND}_\ell(Y_{k-1})$. On the right-hand side is the set of sequences $(a_1, \dots, a_\ell)$ of strictly positive numbers such that $\sum_i a_i = k$. On the left-hand side is the set of ℓ-faces of Δ^k containing both 0 and k, in other words increasing sequences

$$0 = a_0 < a_1 < \cdots < a_{\ell-1} < a_\ell = k.$$

The map sends such a sequence to the sequence of differences $(a_i - a_{i-1})_i$, which is evidently a bijection. This proves the diagram is a pushout, so in particular $Y_1 \hookrightarrow B\mathbb{N}$ is inner anodyne. But this pushout in the case $k = 1$ is precisely

$$\begin{array}{ccc} \{0\} \cup \{1\} & \longrightarrow & \Delta^0 \\ \downarrow & \lrcorner & \downarrow \\ \Delta^1 & \longrightarrow & Y_1 \end{array}$$

because $Y_0 = \Delta^0$, thus $\Delta^1/\partial\Delta^1 \cong Y_1$ completing the proof. □

Remark 2.A.27 This replaces the standard universal cover proof for S^1 in topology. It is interesting that the fundamental fact behind this proof is the correspondence between length-ℓ partitions of k and increasing sequences of length ℓ to from 0 to k. This proof also gives the computation $\tau S^1_\Delta = B\mathbb{N}$, and it is easy to check that the inclusion into $B\mathbb{Z}$ factors $S^1_\Delta \hookrightarrow \mathrm{Ex}^\infty(S^1_\Delta)$; the point is that the morphisms in $B\mathbb{Z}$ already represent all of the homotopy classes in $\pi_1(S^1_\Delta) = \mathbb{Z}$.

We conclude by explaining how to compute $\pi_3(S^2_\Delta) \cong \mathbb{Z}$ in a purely simplicial manner, which takes an extra step because we can't identify the Hopf bundle as easily. Nonetheless we will use the same Hopf bundle proof. We will emphasize being able to do this purely abstractly, i.e. without any 'explicit' computations.

Example 2.A.28 First we compute the homology of the $K(\mathbb{Z} \times \mathbb{Z}, 1)$ space $S^1_\Delta \times S^1_\Delta$. Since S^1_Δ is generated by its non-degenerate 1-cell e, we know that the product is generated by the distinct two non-degenerate cells

$$\Delta^2 \xrightarrow{i} \Delta^1 \times \Delta^1 \xrightarrow{(e,e)} S^1_\Delta \times S^1_\Delta$$

where i represents either $\Delta^{\{00,01,11\}}$ or $\Delta^{\{00,10,11\}}$. These are the two top non-degenerate simplices; on the other hand we know it has one 0-simplex, and 4 1-simplices $(e,*), (*,e), (e,e), (*,*)$, the first three of which are non-degenerate

and the last one is obviously degenerate. In particular the Euler characteristic of $C_*(S^1_\Delta \times S^1_\Delta)$ is $1-3+2=0$. On the other hand we know already know that

$$H_0(S^1_\Delta \times S^1_\Delta) = \mathbb{Z} \qquad H_1(S^1_\Delta \times S^1_\Delta) = \mathbb{Z} \times \mathbb{Z}$$

by Thm. 2.A.21, and that $H_2(S^1_\Delta \times S^1_\Delta)$ is a subgroup of the free abelian group $C_2(S^1_\Delta \times S^1_\Delta)$ in particular it is free. Hence its rank r satisfies $1-2+r=0$, so $r=1$. Thus $H_2(S^1_\Delta \times S^1_\Delta) \cong \mathbb{Z}$ is the top nonzero homology.

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