Appendix: Obvious Topological Facts

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This is a notes document meant for proving 'obvious' facts in algebraic topology: things like 'canonical homeomorphisms' between different objects like spheres and projective spaces, facts about quotients, CW complexes, and other related notions. Our purpose is to supplement almost every standard algebraic topology text by explicitly writing out some of these isomorphisms which are usually intuitively clear.

1 Level One

This *level* will focus mostly on very explicit objects: n-spheres, disks, etc. It will also serve as the general introduction to this document, defining notations for its remainder (which, for this work, is crucial). We will use the notational conventions set out in the following list:

1. For $n \geq 0$, $\mathbb{R}^n$ is the space of n-tuples of real numbers with the product topology (= open ball topology = metric topology, etc.). Likewise $\mathbb{C}^n$ for the complex numbers.
2. D^n is the *(closed) unit disk* in $\mathbb{R}^n$, namely the subset
$$D^n \coloneqq \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum\nolimits_i x_i^2 \leq 1\}.$$
S^n is the *unit sphere* in $\mathbb{R}^{n+1}$, i.e. the subset
$$S^n \coloneqq \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum\nolimits_i x_i^2 = 1\},$$
so evidently $S^n \subset D^{n+1}$ is a closed subset. We will also write $\partial D^n \subset D^n$ for S^{n-1}. $B^n \coloneqq D^n - \partial D^n$ is the open complement. $I = [0, 1] \subset \mathbb{R}$.
3. A *pair* means a pair of spaces (X, A) where $A \subset X$ is a topological subspace. A morphism $f\colon (X, A) \to (Y, B)$ of pairs is a map $f\colon X \to Y$ such that $f(A) \subset B$.
4. Throughout this document, the sign '=' will often be used to 'identify' two topological spaces (or pairs, triples, or more data). This is usually done when a particular isomorphism is in some way canonical, and identification is exactly the data of a fixed isomorphism between two objects. For example, when we write $[0, 1] = [-1, 1]$ we are referring to the underlying fixed isomorphism $t \mapsto 2t - 1$. This will be useful.

1.1 Spheres, Cubes, and Disks

Along with the general notation, we will also write ∂I^n for the *boundary of* $I^n \subset \mathbb{R}^n$, which means the closed subspace of points for which at least one coordinate is equal to 1. Though $[-1,1]$ is identified with I as before, we will sometimes consider it separately and in those cases write $I_\pm = [-1,1]$.

Let us now discuss some canonical isomorphisms. The open interval $(0,1)$ is isomorphic to $\mathbb{R}$ by a variety of standard isomorphisms; we will choose to first define

$$u\colon (-1,1) \to \mathbb{R} \qquad x \mapsto x/(1-x^2).$$

Since the endpoints are excluded this is well-defined and continuous (and smooth). We also have

$$u'(x) = (1-x^2)^{-2}(1+x^2).$$

First of all, $u(0) = 0$ and clearly there are no other zeros. Also $\pm u(x) > 0$ whenever $\pm x > 0$. $\lim_{x\to 1^-} u(x) = +\infty$ and $\lim_{x\to -1^+} u(x) = -\infty$, so u is surjective. The expression for u shows also that u is increasing on its entire domain, so it has to be injective too. Because it is continuously differentiable and has everywhere positive derivative, it is a homeomorphism too. From this we will make the identification $(-1,1) = \mathbb{R}$. Note that it fixes 0 and preserves positivity and negativity, and inequalities. Because of $(0,1) = (-1,1)$, we can compose to get an identification $(0,1) = \mathbb{R}$ given by

$$x \mapsto \frac{2x-1}{4x(x-1)}.$$

There is another isomorphism in cases where we want to write down the inverse explicitly. Namely, $[0,1) \xrightarrow{\sim} \mathbb{R}_{\geq 0}$ by $x \mapsto x/(1-x)$ because it has inverse $y \mapsto y/(1+y)$. The inverse also provides an isomorphim $(-1,0] \xrightarrow{\sim} \mathbb{R}_{\leq 0}$ sending $0 \mapsto 0$, so gluing these together gives a homeomorphism to $\mathbb{R}$.

Both of these are everywhere increasing, and with $(0,1) = \mathbb{R}$ they make $1/2$ correspond to 0, and are symmetric, but the first is smooth, and the second has an easily writable inverse. These provide isomorphisms $(0,1)^n = \mathbb{R}^n$ also.

Next let us discuss an isomorphism $I^n \cong D^n$. Recall the following norms:

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2} \qquad |x|_\infty = \max\left\{ |x_1|, \ldots, |x_n| \right\}$$

for $x \in \mathbb{R}^n$. We know that $x \mapsto |x|$ is a continuous map $\mathbb{R}^n \to \mathbb{R}$. To see the same about $x \mapsto |x|_\infty$, observe that the open cube of radius r (so sidelength $2r$) about $x \in \mathbb{R}^n$ is

$$\{y \in \mathbb{R}^n : |y_i - x_i| < r\} = \{y \in \mathbb{R}^n : |y-x|_\infty < r\}$$

from the definition, and similarly for the closed cube. But the ball of radius r is incribed in the cube of radius r, so $B_r(x) \subset \{y : |y-x|_\infty < r\}$. In particular, if $\epsilon > 0$ and $y \in B_{\epsilon/\sqrt{2}}(x)$, then $|y-x|_\infty < \epsilon/\sqrt{2}$ so y is in the cube of radius

$\epsilon/\sqrt{2}$ about x. All points in this cube have distance from x strictly less than $\sqrt{2}\epsilon/\sqrt{2} = \epsilon$, proving continuity.

Now, consider the maps

$$F\colon D^n \to [-1,1]^n \qquad x \mapsto \frac{|x|}{|x|_\infty} x$$

and

$$G\colon [-1,1]^n \to D^n \qquad y \mapsto \frac{|x|_\infty}{|x|} x,$$

and additionally define $F(0) = G(0) = 0$. For $a \in \mathbb{R}$, note that $|ax| = |a|\,|x|$, and similarly $|ax|_\infty = |a|\,|x|_\infty$ by the definition. It follows that

$$|F(x)|_\infty = |x| \qquad |G(x)| = |x|_\infty .$$

These show that F and G are well-defined and mutually inverse to each other. They also imply that $\partial D^n \leftrightarrow \partial[-1,1]^n$ under this bijection, and the correspondence is given by sending a point on the sphere ∂D^n to the unique intersection point between $\partial[-1,1]^n$ and the ray from 0 to the point. From what we have said, these maps are both continuous except at zero, and it will suffice by compactness to show continuity of one of them, say G, at zero.

But indeed, for $\epsilon > 0$, whenever $|x| < \epsilon$ we have $|G(x)| = |x|_\infty < \epsilon$ because the cube of radius ϵ contains the ball of radius ϵ. So F, G are inverse homeomorphisms. We have proved the following:

Proposition 1.1.1 *Let $n \geq 1$.*

1. *By the maps F and G defined above, $(D^n, \partial D^n) = ([-1,1]^n, \partial[-1,1]^n)$.*
2. *$(D^n, \partial D^n) = (I^n, \partial I^n)$ by $x \mapsto (F(x)+1)/2 = |x|\,x/(2\,|x|_\infty) + 1/2$. As this map is radial, it preserves quadrants, hemispheres, etc.*
3. *From the same map $D^n/\partial D^n = I^n/\partial I^n$, fixing boundary basepoints.*
4. *This identifies $B^n = (0,1)^n = \mathbb{R}^n$.*

The following application is important e.g. when studying product CW structures.

Corollary 1.1.2 *Let $m, n \geq 1$. We have canonical identifications*

$$\begin{aligned}(D^m \times D^n, \partial D^m \times D^n \cup D^m \times \partial D^n) &= (D^m, \partial D^m) \times (D^n, \partial D^n) \\ &= (I^m, \partial I^m) \times (I^n, \partial I^n) \\ &= (I^{m+n}, \partial I^{m+n}) = (D^{m+n}, \partial D^{m+n}).\end{aligned}$$

Proof The first line just comes from tracking where the boundaries go in the product. The second line and the second equality in the last line is by

Prop. 1.1.1. The last line remains: indeed, $I^m \times I^n = I^{m+n}$, and the boundaries go to

$$I^m \times \partial I^n \cup \partial I^m \times I^n \subset I^{m+n} \subset \partial I^{m+n},$$

the containment because each element of the LHS has at least one coordinate equal to 0 or 1. But this is actually an equality since any $x \in \partial I^{m+n}$ has some $x_i \in \{0, 1\}$ and either $i \leq m$ or $m + 1 \leq i \leq m + n$, so in the former case $x \in \partial I^m \times I^n$, and in the latter $x \in I^m \times \partial I^n$. □

We conclude this section by discussing what is, after the identification of Prop. 1.1.1, the next most important homeomorphism: stereographic projection, which is an isomorphism between S^n minus a point and $\mathbb{R}^n$. For our purposes, the sphere is always implicitly a pointed space $(S^n, 1)$ where

$$1 = (1, 0, \ldots, 0) \in S^n.$$

We always have a canonical inclusion $S^{n-1} \subset S^n$ which is given by

$$S^{n-1} = \{(x_0, \ldots, x_n) \in S^n : x_n = 0\},$$

and this preserves the basepoint $(S^{n-1}, 1) \subset (S^n, 1)$. We define the upper and lower hemispheres of the n-sphere as

$$D^n_\pm \coloneqq \{(x_0, \ldots, x_n) \in S^n : \pm x_n \geq 0\},$$

so that clearly $D^n_+ \cap D^n_- = S^{n-1}$. Writing $D^n_+ \cup_{S^{n-1}} D^n_-$ for the quotient of the disjoint union by the the equivalence relation identifying the inclusions $S^{n-1} \hookrightarrow D^n_\pm$, the inclusions induce a continuous bijection

$$D^n_+ \cup_{S^{n-1}} D^n_- \to S^n$$

which is a homeomorphism by compactness. So we identify $D^n_+ \cup_{S^{n-1}} D^n_- = S^n$. The '$D$' notation is because of the natural map

$$i_\pm \colon D^n \to D^n_\pm \mapsto (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, \pm\sqrt{1 - \textstyle\sum_j x_j^2}),$$

which is a homeomorphism because it has the inverse

$$D^n_\pm \to D^n \qquad (x_0, \ldots, x_n) \mapsto (x_0, \ldots, x_{n-1}).$$

This means that the following diagram

$$\begin{array}{ccc} \partial D^n & \hookrightarrow & D^n \\ \downarrow & & \downarrow{\scriptstyle i_-} \\ D^n & \xrightarrow{i_+} & S^n \end{array}$$

is a pushout, and we also write $D^n \cup_{\partial D^n} D^n = S^n$ using the maps i_+ and i_-. We mention for now that this gives S^n the structure of a CW complex with two cells in each dimension $i = 0, 1, \ldots, n$, where the attaching maps are $S^{i-1} \hookrightarrow D^i_+$ and $S^{i-1} \hookrightarrow D^i_-$ in each dimension.

Now let us define the stereographic projection (from $1 \in S^n$). We will give a map $S^n - 1 \to \mathbb{R}^n$ as follows: Consider the hyperplane $\{x_0 = 0\}$ in $\mathbb{R}^{n+1}$, and for any point $x \in S^n - 1$ the line from 1 to x intersects $\{x_0 = 0\}$ in a unique point, becoming the image of the stereographic projection.

Here's how this looks explicitly.

$$\sigma_n \colon S^n - \{1\} \to \mathbb{R}^n \qquad (x_0, \ldots, x_n) \mapsto \frac{(x_1, \ldots, x_n)}{1 - x_0}.$$

This is well-defined, so clearly a continuous (and smooth) map. Observe that

$$|\sigma_n(x)|^2 = \frac{1 - x_0}{(1 - x_0)^2} = \frac{1 + x_0}{1 - x_0}.$$

So we see that $\sigma^n(x)$ is 0 if and only if $x_0 = -1$, i.e. iff $x = (-1, 0, \ldots, 0)$. Now define

$$\sigma_n^{-1} \colon \mathbb{R}^n \to S^n - \{1\} \qquad y \mapsto \frac{(|y|^2 - 1, 2y_1, \ldots, 2y_n)}{|y|^2 + 1}.$$

This is well-defined because the length of the numerator squared is

$$(|y|^2 - 1)^2 + 4\,|y|^2 = |y|^4 + 2\,|y|^2 + 1 = (|y|^2 + 1)^2,$$

so indeed the image is on the sphere. Moreover, $|y|^2 - 1 = |y|^2 + 1$ leads to a contradiction, so 1 is not in the image. Let us see these maps are actually mutually inverse. First, $\sigma_n^{-1} \circ \sigma_n$: Note

$$\frac{|y|^2 - 1}{|y|^2 + 1} = \frac{2x_0/(1 - x_0)}{2/(1 - x_0)} = x_0,$$

and since $|y|^2 + 1 = 2/(1 - x_0)$, the rest of the coordinates follow. The other direction is trivial since

$$1 - \frac{|y|^2 - 1}{|y|^2 + 1} = \frac{2}{|y|^2 + 1},$$

so it follows that σ_n and σ_n^{-1} are mutually inverse homeomorphisms.

Let us make some preliminary observations about the stereographic projection. Firstly, σ_n carries the $n - 1$-dimensional sphere $S^n \cap \{x_0 = 0\}$ to the unit $(n-1)$-sphere inside $\mathbb{R}^n$. It also has the following obvious naturality property, because the map doesn't affect coordinates beyond zero:

$$\begin{array}{ccc} S^{n-1} - 1 & \xrightarrow{\sigma_{n-1}} & \mathbb{R}^{n-1} \\ \downarrow & & \downarrow \\ S^n - 1 & \xrightarrow{\sigma_n} & \mathbb{R}^n \end{array}$$

We see that σ_n maps the upper hemisphere $D_+^n - 1$ into the upper half-plane of $\mathbb{R}^n$ because $1 - x_0$ is never negative, and likewise σ_n^{-1} maps the upper half-plane into D_+^n, so it restricts to a homeomorphism there (and likewise for D_-^n). The embeddings of $D_\pm^n$ for n and $n - 1$ also commute with σ.

Now, let us use our order-preserving identification $\mathbb{R} = (-1,1)$, giving a composite $S^n - 1 \to \mathbb{R}^n = (-1,1)^n \hookrightarrow [-1,1]^n \twoheadrightarrow [-1,1]^n/\partial[-1,1]^n$. We claim that, by extending this to all of S^n by sending $1 \mapsto *$ the basepoint in the quotient, this produces a homeomorphism. Let us consider an inverse: define

$$[-1,1]^n \to S^n$$

by sending $\partial[-1,1]^n \mapsto 1 \in S^n$, and on $(-1,1)^n$ using the identification with $\mathbb{R}^n$ and then σ_n^{-1}. We need to show this is continuous on the boundary points. Write $\nu\colon (-1,1) = \mathbb{R}$, so that the map above becomes

$$t = (t_1, \ldots, t_n) \mapsto \frac{(|y|^2 - 1, 2\nu(t_1), \ldots, 2\nu(t_n))}{|y|^2 + 1}$$

where $y = (\nu(t_1), \ldots, \nu(t_n))$. As t approaches a boundary point, some coordinate t_i approches ± 1, so $\nu(t_i) \to \pm\infty$, hence $|y| \to \infty$. The first coordinate of the image approaches 1, as for the others, we observe that

$$0 \leq |2\nu(t_k)| \leq 2\,|y| \Rightarrow 0 \leq \frac{|2\nu(t_k)|}{|y|^2 + 1} \leq \frac{2\,|y|}{|y|^2 + 1}.$$

As $|y| \to \infty$, the RHS goes to 0, so the middle goes to 0. Hence we have shown that the image goes to $(1, 0 \ldots, 0)$ as desired. This induces a map

$$([-1,1]^n/\partial[-1,1]^n, *) \to (S^n, 1)$$

which is an isomorphism subtracting the basepoints, hence a continuous bijection and thus a homeomorphism by compactness.

Theorem 1.1.3 *Let $n \geq 1$.*

1. *We have an identification $D^n/\partial D^n = I^n/\partial I^n = S^n$ given by*

$$I^n/\partial I^n = [-1,1]^n/\partial[-1,1]^n \to S^n$$

extending the stereographic projection σ_n^{-1} map.

As we saw, the identification extends the composite

$$S^n - 1 \xrightarrow{\sigma_n} \mathbb{R}^n = (-1,1)^n \hookrightarrow [-1,1]/\partial[-1,1] = I^n/\partial I^n.$$

Here are some consequences of that. The first map carries $S^{n-1} - 1$, $D^n_\pm$ to the hyperplane $\mathbb{R}^{n-1} \times 0$, and $\mathbb{R}^{n-1} \times \mathbb{R}_{\pm \geq 0}$. Hence, in $[-1,1]^n/\partial[-1,1]^n$, S^{n-1} is taken homeomorphically to

$$[-1,1]^{n-1} \times \{0\} \cup \{*\}$$

where $*$ is the basepoint, and similarly D^n_+ is taken to the same with $\{0\}$ replaced by $\{x \geq 0\}$, likewise for D^n_-. This works the same way as with $I^n/\partial I^n$, except that 0 is replaced by $1/2$. Then, because of our choice of identification $D^n = [-1,1]^n$, in $D^n/\partial D^n$ the top (bottom) hemisphere of S^n is mapped homeomorphically onto the top (bottom) half of $D^n \cup *$ sending $1 \mapsto *$.

It will be important to keep track of these facts about our isomorphism, but since all of the maps involved are relatively simple (at least in one direction), we will just refer back to the definition when needed. Our next goal is to construct and define the *pinching map*.

Let $n \geq 1$; the presentation $S^n = D^n_+ \cup_{S^{n-1}} D^n$ allows us to take the quotient by S^{n-1}, which collapses to

$$D^n_+/S^{n-1} \cup_* D^n_-/S^{n-1} = D^n/\partial D^n \vee D^n/\partial D^n.$$

Explicitly, the quotient map

$$c\colon S^n \to D^n_1/\partial D^n_2 \vee D^n_2/\partial D^n_2$$

is then given explicitly as follows:

$$c(x_1, \ldots, x_{n+1}) = \begin{cases} (x_1, \ldots, x_n)_1 & x_{n+1} \geq 0 \\ * & x_{n+1} = 0 \\ (x_1, \ldots, x_n)_2 & x_{n+1} \leq 0 \end{cases}$$

where $*$ is again the joining basepoint, and where $1, 2$ refer to the corresponding wedge summand. Let us consider the identification $S^n = D^n/\partial D^n$. On one hand, this gives us a pinch map

$$D^n/\partial D^n \to D^n/\partial D^n \vee D^n/\partial D^n,$$

and from our explicit description we know this takes the top half of D^n plus the basepoint homeomorphically onto the first wedge summand, and vice versa for the bottom half. Alternatively, we can consider this as a map

$$c\colon S^n \to S^n \vee S^n,$$

which will be important when we discuss homotopy group constructions starting in the next section.

1.2 Some important homotopies

In this section we will further study constructions in the previous one but focus on homotopies between certain maps. Recall that given maps $f, g\colon X \to Y$, a *homotopy* between them is a map $X \times I \to Y$ which on $X = X \times 0$ is f and on $X = X \times 1$ is g. In this case we write $f \sim g$. Note that we can always replace I with $[a, b]$ for any $a < b$ because the parameterization homeomorphism

$$I \to [a, b] \qquad t \mapsto (1 - t)a + tb$$

takes one to the other. We most often use this with $[a, b] = [-1, 1]$. $\sim$ is an equivalence relation because $f \sim f$ by the homotopy $(x, t) \mapsto f(x)$; a homotopy $f \sim g$ can be converted to one $g \sim f$ by precomposing with $(x, t) \mapsto (x, 1 - t)$; finally if f_t is a homotopy $f_0 \sim f_1$, and g_t is a homotopy $f_1 = g_0 \sim g_1$, then

the map

$$(x,t) \mapsto \begin{cases} f_{2t}(x) & t \le 1/2 \\ g_{2t-1}(x) & t \ge 1/2 \end{cases}$$

is a homotopy $f_0 \sim g_1$.

If $A \subset X$ and $f|_A = g|_A$, then a homotopy $F\colon f \sim g$ is a *homotopy rel* A if $F(x,t) = F(x,0)$ for all $t \in I$, that is the homotopy stays fixed on A. If f, g are maps of pairs $(X,A) \to (Y,B)$, then usually a homotopy $F\colon f \sim g$ means a homotopy *through* maps $(X,A) \to (Y,B)$, so that $F(a,t) \in B$ for all $a \in A$. These are also equivalence relations as above.

Homotopy respects composition in both directions. If $F\colon f \sim g$ and $h\colon W \to X$ is a map, then $F(h(x),t)$ is a homotopy $f \circ h \sim g \circ h$, and if $h\colon Y \to Z$ is a map then $h \circ F$ is a homotopy $h \circ f \sim g \circ f$.

A map $f\colon X \to Y$ is a *homotopy equivalence* if there is a map $g\colon Y \to X$ such that $gf \sim 1_X$ and $fg \sim 1_Y$. This is easily seen to be an equivalence relation as homotopy respects composition (and for the same reason, an isomorphism is a homotopy equivalence). A map $f\colon X \to Y$ is *nullhomotopic* if there is $y_0 \in Y$ such that f is homtopic to the constant map at y_0. A space X is *contractible* if it is homotopy equivalent to a point, that is $X \to *$ is a homotopy equivalence. In other words, 1_X is nullhomotopic.

Observe that any map $f\colon X \to Y$ with either X contractible or Y contractible must be nullhomotopic. Namely, just compose $1_X \sim x_0$ or $1_Y \sim y_0$ with f in the right direction. Also, recall that a subset $A \subset X$ is a *retract* if there is $r\colon X \to A$ such that $r|_A = 1_A$. It is a *deformation retract* if there is a continuous homotopy $r_t\colon (X,A) \to (X,A)$ where $r_0 = 1_X$, $r_t|_A = 1_A$, and $r_1(X) \subset A$. So, a deformation retract gives a homotopy equivalence.

Let $X \subset \mathbb{R}^n$ be a convex subset of $\mathbb{R}^n$ and suppose $A \subset X$. A *reparameterization* is a map $\varphi\colon (X,A) \to (X,A)$ such that $\varphi|_A = 1_A$. The map

$$\varphi_t(x) = (1-t)\varphi(x) + tx$$

defines a continuous homotopy $\varphi_0 = \varphi$ and $\varphi_1 = 1_X$, and if $x \in A$ then $\varphi_t(x) = (1-t)(x) + tx = x$, so φ_t remains fixed on A. Hence φ is homotopic rel A to 1_X. For example, if $x \in X$ and X is a convex subset of $\mathbb{R}^n$, this shows that $X \to \{x\}$ is a deformation retract, hence X is contractible. So, if φ is a reparameterization then $f \circ \varphi \sim f$ or the other way around, whenever it's defined.

Definition 1.2.1 A pair (X,A) has the *homotopy extension property* (HEP) if for every space Y, every homotopy $f_t\colon A \to Y$, and an extension of f_0 to $g\colon X \to Y$, there is an extension $g_t\colon X \to Y$ of the homotopy f_t.

We also say a pair (X,A) has the *extension property* if for any space Y, every map $A \to Y$ extends to X. Assuming $A \subset X$ is closed, the HEP for (X,A) is

a special case of the extension property for the pair

$$(X \times I, A \times I \cup X \times 0),$$

because $A \times I$ and $X \times 0$ are closed subsets of $A \times I \cup X \times 0$ and we are given maps on them which agree on the intersection.

Lemma 1.2.2 *A pair (X, A) has the extension property if and only if A is a retract of X. Hence, if $A \subset X$ is closed, then (X, A) has the HEP if and only if $A \times I \cup X \times 0$ is a retract of $X \times I$.*

Proof The implication is immediate from the above observation. For the first part, the identity $A \to A$ extends to X giving a retraction, and conversely any map $A \to Y$ can be composed with a retraction $X \to A$ to obtain an extension. □

In fact this is true (but more difficult) without the assumption that A is closed, though we won't need it. Let us make the following easy observation:

Proposition 1.2.3 *Let $X \subset \mathbb{R}^n$ be a convex subset and (X, A) a pair with the HEP. Then a map $\varphi\colon (X, A) \to (X, A)$ of pairs for which $\varphi|_A\colon A \to A$ is homotopic to 1_A is homotopic through maps $(X, A) \to (X, A)$ to 1_X.*

Proof The homotopy of $\varphi|_A\colon A \to A \hookrightarrow X$ to $A \hookrightarrow X$ extends to a homotopy of φ to a map $X \to X$ restricting to $A \hookrightarrow X$ on A, and the homotopy restricts to that of $\varphi|_A$ hence it sends A to A at each time t. So φ is homotopic through the pairs to a map fixing A, and by convexity this is homotopic (stationary on A) to the identity. □

Our primary example of this will be the pair $(I^n, \partial I^n)$, used to define homotopy groups. Namely, if $f\colon (I^n, \partial I^n) \to (X, x_0)$ is some map and $\varphi\colon (I^n, \partial I^n) \to (I^n, \partial I^n)$ is a map whose restriction to the boudnary is homotopic to the identity, then then $[f] = [f \circ \varphi] \in \pi_n(X, x_0)$. To show that this is an HEP, we will prove some auxiliary facts and take a short detour into cones and suspensions.

For a space X, the *cone over* X is the space $CX \coloneqq X \times I / X \times 1$. We usually regard the space as a pair (CX, X), where X is the subspace $X \times 0 \subset CX$. This is homeomorphic to X by the emebdding $x \mapsto (x, 0)$ with inverse the projection. This construction is functorial: if $f\colon X \to Y$ is a map, then we get a map of pairs

$$\widetilde{f}\colon (CX, X) \to (CY, Y)$$

by extending to $X \times I \to Y \times I$ as the identity on I, well-defined since $X \times 1 \mapsto f(X) \times 1 \subset Y \times 1$. This induces a functorial quotient map

$$Sf\colon SX \coloneqq CX/X \to SY \coloneqq CY/Y$$

referred to as the *suspension* of f, X, and Y. So $S(-)$ is also functorial, and we usually consider it to be a pair (SX, X) where this time X is embedded as $x \mapsto (x, 1/2)$.

When the space in question is D^n or S^n, then the cones in question are canonically identified with other spaces. For instance, let $C^{n+1} \subset \mathbb{R}^{n+1}$ be the union of the set of lines from $D^n \times 0$ to $(0,1) \in D^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$. In other words, it is the closed subset

$$C^{n+1} := \{((1-t)x, t) \in D^n \times I : x \in D^n, t \in I\} = \{(y,t) : |y| \leq 1 - t\}.$$

Then the map $D^n \times I \to C^{n+1}$ given by $(x,t) \mapsto ((1-t)x, t)$ sends $(x,1) \mapsto (0,1)$ hence induces $CD^n \to C^{n+1}$, and this is bijective because there is the (set) inverse

$$(y,t) \mapsto (y/(1-t), t)$$

when $t < 1$ and (y,t) maps to the apex point $D^n \times 1$ when $t = 1$. Therefore by compactness, it is a homeomorphism and this is a continuous inverse. One application is that the map

$$C^{n+1} - (0,1) \xrightarrow{\sim} CD^n - D^n \times 1 = D^n \times [0,1) \to D^n$$

corresponding to projcetion onto the first factor is the (continuous) projection

$$(y,t) \mapsto y/(1-t).$$

Note that $((1-t)x, t) \mapsto x$ under the map, which means that it is the radial projection from the apex of the cone to the copy of $D^n = D^n \times 0$ at the base; in particular it is also a retraction.

Another application is the suspension of S^n. Under the identification $CD^n = C^{n+1}$ above, the subpair $(CS^{n-1}, S^{n-1}) \subset (CD^n, D^n)$ is sent to the outside of the cone:

$$\partial C^{n+1} = \{((1-t)x, t) \in D^n \times I : x \in S^{n-1}\} = \{(y,t) : |y| = 1 - t\}.$$

Now the map $\partial C^{n+1} \to D^n$ given by $(x,t) \mapsto x$ is in fact a homeomorphism, because it has continuous inverse $y \mapsto (y, 1 - |y|)$ as $t = 1 - |y|$. So we identify $(\partial C^{n+1}, S^{n-1}) = (D^n, S^{n-1})$.

Now recall the identification $[0,1] = [-1,1]$ given by $t \mapsto 2t - 1$, which induces an isomorphism $CS^{n-1} \cong S^{n-1} \times [-1,1]/S^{n-1} \times 1$. This sends $S^{n-1} \times 1/2 \mapsto S^{n-1} \times 0$. If we apply this instead to the suspension SS^{n-1} we get an isomorphism

$$CS^{n-1} \cup_{S^{n-1}} CS^{n-1} \cong SS^{n-1}$$

which on the first summand is $(x,t) \mapsto (x, (1+t)/2)$ and on the second it is $(x,t) \mapsto (x, (1-t)/2)$. The identifications $CS^{n-1} = \partial C^{n+1} = D^n$ preserve the boundary S^{n-1}, giving an isomorphism

$$S^n = D^n \cup_{S^{n-1}} D^n = CS^{n-1} \cup_{S^{n-1}} CS^{n-1} = SS^{n-1}.$$

This is an important identification, and it is easy to write down the formula for it. The importance is that a map $f\colon S^n \to X$ can be extended to $Sf\colon S^{n+1} \to SX$ using functoriality. This will be used in the case of maps $f\colon S^n \to S^n$, where we will always regard Sf as a map $S^{n+1} \to S^{n+1}$ using this identification. We will see later that these preserve degree.

Theorem 1.2.4 *For all $n \geq 1$, the subspace $D^n \times 0 \cup \partial D^n \times I$ is a deformation retract of $D^n \times I$. In particular, $(D^n, \partial D^n)$ and $(I^n, \partial I^n)$ have the HEP.*

Proof The consequences are obvious because $\partial D^n \subset D^n$ is a closed subspace and because $(D^n, \partial D^n)$ and $(I^n, \partial I^n)$ are homeomorphic pairs. For the proof, we will view $D^n \times I \subset D^n \times \mathbb{R}$, and construct the radial projection from $(0, 2) \in D^n \times \mathbb{R}$ of $D^n \times I$ onto $D^n \times 0 \cup \partial D^n \times I$.

By identifying $[0, 1] = [0, 2]$ via $x \mapsto 2x$, we can consider the cone C^{n+1} as constructed before except with apex $(0, 2)$. Let C be its intersection with $D^n \times I$. We already showed previously that the radial projection from the apex onto D^n is a continuous retraction, and we wrote an explicit formula for it. So, $C = \{(x, t) : 2\,|x| \leq 2 - t\}$, and let

$$D = \{(x, t) \in D^n \times I : 2\,|x| \geq 2 - t\}.$$

Then these are both closed and $C \cup D = D^n \times I$. Now, Observe that D retracts onto ∂D^n by the radial projection sending the pair (x, t) to have D^n-coordinate $x \mapsto x/\,|x|$ at height $2 + (t - 2)/\,|x|$. Overall, our map is given as follows:

$$r(x, t) = \begin{cases} (2x/(2 - t), 0) & (x, t) \in C \\ (x/\,|x|\,, 2 - (2 - t)/\,|x|) & (x, t) \in D \end{cases}$$

We already know that r retracts C onto $D^n \times 0$ and D onto ∂D^n, so it suffices to check that r is well-defined and continuous. Indeed,

$$C \cap D = \{(x, t) : 2\,|x| = 2 - t\},$$

so $2 - (2 - t)/\,|x| = 0$ and $2x/(2 - t) = x/\,|x|$, therefore the two maps agree on the intersection.

Lastly, let us observe that this is extends to a deformation retraction: defining $((x, t), s) \mapsto sr(x, t) + (1 - s)(x, t)$ gives a continuous homotopy from the identity to $r(x, t)$. This works because of convexity of $D^n \times I$. □

Corollary 1.2.5 *Let $\varphi\colon (I^n, \partial I^n) \to (I^n, \partial I^n)$ be such that $\varphi|_{\partial I^n}$ is homotopic to the identity $\partial I^n \to \partial I^n$. Then φ is homotopic through such maps to the identity $(I^n, \partial I^n) \to (I^n, \partial I^n)$.*

Here is another useful homotopy technique which often helps when analyzing maps to spheres:

Lemma 1.2.6 *Let X be a space and $f, g\colon X \to S^n$ two maps such that $f+g$ is nowhere-zero, i.e. f and g are never antipodal. Then $f \sim g$. If $f|_A \equiv g|_A$ for some $A \subset X$, then the $f \sim g$ rel A,*

Proof Consider the homotopy

$$(x,t) \mapsto \frac{(1-t)f(x)+tg(x)}{|(1-t)f(x)+tg(x)|}.$$

It is obvious that this is a continuous homotopy from f to g as long as the denominator is never zero. If for some x it is, then $(1-t)f(x) = -tg(x)$, so $1-t = t$ as f, g lie on the sphere, so $t = 1/2$. But $f(x)+g(x)$ is nonzero for all x, so we are done.

For the second statement, note that when $x \in A$, $y = f(x) = g(x)$ so the numerator just becomes $(1-t)y + ty = y$, and the denominator is one. □

This is useful in practice. For example, it gives another proof that non-surjective maps to S^n are nullhomotopic. The first proof is that a non-surjective map can be homotoped by a rotation to miss 1, and hence factors through $\mathbb{R}^n$ which is contractible. The easier proof from the lemma is just that if a map misses $x \in S^n$, then it is homotopic to the constant map at $-x$.

Another application is that a map $f\colon S^n \to S^n$ without any fixed points is homotopic to the antipodal map $x \mapsto -x$. Indeed, $f(x)+(-x)$ is nowhere zero because otherwise x would be fixed.

This simple lemma also justifies the intuition that if two maps to S^n stay somewhat close to one another, then they are homotopic. Namely, two points $x, y \in S^n$ are antipodal iff $|x-y| = 2$, iff $|x-y| \geq 2$. Hence, if $f, g\colon X \to S^n$ and $|f-g| < 2$, then f and g are homotopic.

1.3 Homotopy group constructions

One often uses two homotopy-equivalent methods of defining $\pi_n(X, x_0)$ for a space X, and the goal is to (rigorously) show that these are equivalent using the previous section. More importantly, we will be concerned with the addition operation in π_n.

Let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be maps. Recall that $[f] = [g]$ in π_n iff f is homotopic to g through maps $(I^n, \partial I^n) \to (X, x_0)$. Then one defines the sum of f and g as follows

$$(f+g)(x) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \leq 1/2 \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \geq 1/2 \end{cases}$$

giving a new map $f+g\colon (I^n, \partial I^n) \to (X, x_0)$ by the gluing lemma. We will momentarily prove various facts about the group $\pi_n(X, x_0)$ using this definition, but first we discuss the alternative definition, which is defined using a variant of the pinch map from before.

Let $S_1^{n-1} \subset S^n$ be the subsphere $\{(x_0, x_1, x_2, \ldots, x_n) : x_1 = 0\}$, so that in particular $1 \in S_1^{n-1}$. By transferring results from the first subsection to the second coordinate instead of the last, we observe that

$$S^n = D^n \cup_{S_1^{n-1}} D^n$$

which maps the region where $x_1 \geq 0$ to the first D^n by forgetting the x_1-coordinate, and likewise for $x_1 \leq 0$ to the second coordinate. This gives a quotient map as follows:

$$\omega \colon S^n \to S^n/S_1^{n-1} = D^n/S_1^{n-1} \cup_* D^n/S_1^{n-1} = D^n/\partial D^n \vee D^n/\partial D^n.$$

Using the identification $D^n/\partial D^n = S^n$, this gives us a map $S^n \to S^n \vee S^n$. So, we can define $\pi_n(X, x_0)$ by based homotopy classes of maps $(S^n, 1) \to (X, x_0)$ where the addition is defined by

$$f + g = S^n \xrightarrow{\omega} S^n \vee S^n \xrightarrow{(f,g)} X,$$

using the universal property of wedge sums. We want to show that these two definitions coincide. First, we use the identification $I^n/\partial I^n = S^n$ to convert between maps out of $(I^n, \partial I^n)$ and those out of $(S^n, 1)$. Identifying the latter S^n's with $I^n/\partial I^n$ factors through the inverse of the identification with $D^n/\partial D^n$; so, we claim it suffices to show the map

$$I^n/\partial I^n \to S^n \xrightarrow{\omega} D^n/\partial D^n \vee D^n/\partial D^n = I^n/\partial I^n \vee I^n/\partial I^n \qquad (*)$$

is based homotopic to the map

$$I^n/\partial I^n \to I_1^n/\partial I_1^n \vee I_2^n/\partial I_2^n$$

given by

$$(x_1, \ldots, x_n) \mapsto \begin{cases} (2x_1, x_2, \ldots, x_n)_1 & x_1 \leq 1/2 \\ (2x_1 - 1, x_2, \ldots, x_n)_2 & x_1 \geq 1/2 \end{cases}.$$

Indeed, the second map is the cubical addition formula and composition preserves homotopy.

Proposition 1.3.1 *The map* $(*)$ *is based homotopic to cubical addition.*

To prove this, let us first replace all I's with $[-1, 1]$ (while keeping the notation). The addition formula becomes $(2x_1 \mp 1, x_2, \ldots, x_n)$ when $\pm x_1 \geq 0$. Now we examine $(*)$: it maps the left half $\{x_1 \leq 0\}$ of the cube to the subspace $\{(x_0, \ldots, x_n) : x_1 \leq 0\}$ of S^n, i.e. the upper D^n above S_1^{n-1}. So, by precopmosing $(*)$ with a contraction map

$$(x_1, \ldots, x_n) \mapsto ((x_1 - 1)/2, x_2, \ldots, x_n)$$

we can extend $(*)$ to the entire I^n and obtain a map

$$I^n \to D^n \to I^n$$

which sends $x_1 = -1$ to $x_1 = 0$.

Remark 1.3.2 NOTE TO SELF: WE WANT TO CHANGE MODIFY THIS BY CHANGING THE DEFINITION OF THE ISOMORPHISM BETWEEN CUBICAL AND SPHERICAL ADDITION.

Let (X, A) be a pair; then the following sequence is exact:

$$\cdots \to \pi_n(A, x_0) \to \pi_n(X, A, x_0) \to \pi_n(X, x_0) \to \pi_{n-1}(A, x_0) \to \cdots$$

Here, the middle map is the boundary map.

1.4 V2: Homotopy group constructions

In this section we will define the homotopy groups $\pi_n(X, x_0)$ and relative homotopy groups $\pi_n(X, A, x_0)$ for all n (and all $n \geq 1$, respectively). Here, we will frequently go between pairs like $(I^n, \partial I^n)$ and (D^n, S^{n-1}) using the isomorphisms from the previous sections, and we will often identify I and $D^1 = [-1, 1]$.

Let $n \geq 0$, taking the convention that $I^0 = \{*\}$ and $\partial I^0 = \varnothing$, and fix a based space (X, x_0). We define the set $\pi_n(X, x_0)$ to be classes of maps

$$(I^n, \partial I^n) \to (X, x_0)$$

up to *based homotopy*, i.e. homotopy rel ∂I^n, which is equivalent to saying that homotopies must be through maps $(I^n, \partial I^n) \to (X, x_0)$. This quotient is well-defined since we proved before that homotopy rel A is an equivalence relation. $\pi_n(X, x_0)$ are all pointed sets in a natural way: the specified point is the homotopy class of the constant map $[\Delta_{x_0}]$ at x_0.

Observe that $\pi_0(X, x_0)$ is identified with the set of path components of X, with basepoint the path component of x_0, by identifying a class $[* \to X]$ with the path component of the image. Note that a homotopy between two such classes is a path between the images, so this is well-defined, and conversely such a path gives a homotopy, so it's injective and clearly surjective.

Now let $n \geq 1$; we will make $\pi_n(X, x_0)$ into a group with identity the class of Δ_{x_0}. By convention, we will denote the operation (both on maps and on homotopy classes) as addition, and write Δ_{x_0} as $0 \in \pi_n(X, x_0)$. Given $f, g \colon (I^n, \partial I^n) \to (X, x_0)$, we define the map $f + g$ by

$$(f+g)(x) = \begin{cases} f(2x_1, x_2, \ldots, x_n) & x_1 \leq 1/2 \\ g(2x_1 - 1, x_2, \ldots, x_n) & x_1 \geq 1/2 \end{cases}.$$

This is well-defined and continuous because $(1, x_2, \ldots, x_n) \in \partial I^n$, so f and g share an image there. Let us also make the following observation: if f and g are replaced by based homotopies f_t and g_t, then $f_t + g_t$ is a continuous based homotopy between $f_0 + g_0$ and $f_1 + g_1$. This is because $\{x_1 \leq 1/2\} \times I$ and $\{x_1 \geq 1/2\} \times I$ are closed subsets of $I^n \times I$, so we can still apply the gluing lemma. It follows that the relation

$$[f] + [g] \coloneqq [f + g] \in \pi_n(X, x_0)$$

gives a well-defined addition. Our goal is the following theorem.

Theorem 1.4.1 *For all $n \geq 1$, $\pi_n(X, x_0)$ is a group with respect to $+$.*

To prove this, we will first focus on the special case of $n = 1$, and package it slightly differently into a different object called the *fundamental groupoid*, $\Pi_1(X)$ of a space. Suppose we have arbitrary paths $f, g \colon I \to X$, such that

$f(1) = g(0)$. Then we define

$$(f * g)(s) = \begin{cases} f(2s) & t \leq 1/2 \\ g(2s-1) & t \geq 1/2 \end{cases}$$

to be the concatenated path, clearly well-defined and continuous beacuse of the assumption. If f is a path from x_0 to x_1, then we define the *inverse path* $\bar{f}$ to be $\bar{f}(t) = f(1-t)$, which is from x_1 to x_0. We construct a category $\Pi_1(X)$ as follows: its objects are the points of X, and the set of morphisms $x \to y$ are the homotopy rel ∂I classes of maps $I \to X$ with $0 \mapsto x$ and $1 \mapsto y$.

The identity morphism 1_x for each point is the constant path $[\Delta_x]$, and composition in $\Pi_1(X)$ is given by concatenation: $[g] \circ [f] \coloneqq [f * g]$. Observe that this is well-defined for the same reason as earlier: if f_t is a homotopy of paths from x_0 to x_1, and g_t from x_1 to x_2, then $f_t * g_t$ gives a homotopy rel ∂I between $f_0 * g_0$ and $f_1 * g_1$, continuous by the gluing lemma.

Lemma 1.4.2 $\Pi_1(X)$ *as defined forms a category, which moreover is a groupoid.*

Proof (1) Identity If $f\colon I \to X$ is a path from x_0 to x_1, then we claim $f * \Delta_{x_1}$ and $\Delta_{x_0} * f$ are both homotopic to f. Indeed, each one respectively is given by composing f with the one of the following two reparameterizations:

$$t \mapsto \begin{cases} 2t & t \leq 1/2 \\ 1 & t \geq 1/2 \end{cases} \qquad t \mapsto \begin{cases} 0 & t \leq 1/2 \\ 2t-1 & t \geq 1/2 \end{cases}$$

Both of these are maps $I \to I$ restricting to the identity on ∂I, hence they are homotopic rel ∂I to the identity, which proves what we wanted.

(2) Associativity To see that composition is associative, let f, g, h be paths $x_0 \to x_1 \to x_2 \to x_3$. We will provide a homotopy $f * (g * h) \sim_{\partial I} (f * g) * h$. By inspecting the formulas for each, we can see that the second is obtained from the first by pre-composing with the following map $I \to I$:

$$t \mapsto \begin{cases} 2t & t \leq 1/4 \\ t + 1/2 & 1/4 \leq t \leq 1/2 \\ t/2 + 1/2 & 1/2 \leq t \leq 1 \end{cases} \tag{1}$$

This is the identity on ∂I, hence a reparameterization, which shows associativity. In particular, $\Pi_1(X)$ is a category.

(3) Inverses To see that $\Pi_1(X)$ is a groupoid, we need to show that each path f from x_0 to x_1 has an inverse; we claim it is $\bar{f}$. Indeed, let f_t be the path

$$f_t(s) = \begin{cases} f(s) & s \leq t \\ f(t) & s \geq t \end{cases}$$

from x_0 to $f(t)$. Then $\bar{f}_t$ is a path from $f(t)$ to x_0, and by the gluing lemma $f_t * \bar{f}_t$ is a continuous homotopy which starts at $f_0 * \bar{f}_0 = \Delta_{x_0}$ and ends at

$f * \bar{f}$. Note that at each t, 0 and 1 map to $f(0) = x_0$, so the homotopy is rel ∂I. Since applying $(\bar{-})$ twice is the identity, we get inverse in the other direction by replacing f with $\bar{f}$. This shows that $[\bar{f}]$ is a two sided inverse to $[f]$, hence $\Pi_1(X)$ is a groupoid. □

Remark 1.4.3 Since $\Pi_1(X)$ is a groupoid, the automorphism group of each object $x \in X$ is a group structure on the set of endomorphisms on x, i.e. it is the set of based homotopy classes of paths at x with concatenation. This group is written $\pi_1(X, x)$, and called the fundamental group of (X, x). It will be fruitful to study functors out of $\Pi_1(X)$, because such a functor F must induce a group homomorphism $F\colon \pi_1(X, x_0) \to \mathrm{Aut}(F(x_0))$ for each $x_0 \in X$, i.e. an action of π_1 on the object $F(x_0)$ by automorphisms.

Proof (of Thm. 1.4.1) The work for the proof has already been done in the fundamental groupoid construction. Let f be a map $(I^n, \partial I^n) \to (X, x_0)$, and let us show that $[f]+0 = [f] = 0+[f]$. $f+0$ is obtained from f by precomposing f with a map $\alpha \times \mathrm{Id}_{I^{n-1}}$, where α is the path $\mathrm{Id}_I * \Delta_1$ in $\Pi_1(I)$. But then

$$[\alpha] = 1_1 \circ [\mathrm{Id}_I] = [\mathrm{Id}_I] \qquad \text{in } \Pi_1(I),$$

where the homotopies are rel ∂I, hence

$$[f] + 0 = [f + 0] = [f \circ (\alpha \times \mathrm{Id}_{I^{n-1}})] = [f \circ \mathrm{Id}_{I^n}] = [f].$$

The same proof works to show $0+[f] = [f]$. Associativity is along similar lines: given f, g, h, $(f + g) + h$ is obtained from $f + (g + h)$ by precomposing with a map $\alpha \times \mathrm{Id}_{I^{n-1}}$, where $\alpha\colon I \to I$ is the map of Eq. (1). Thus $\alpha \sim_{\partial I} \mathrm{Id}_I$, so this map is homotopic to the identity.

Finally, let us show that inverses exist in $\pi_n(X, x_0)$. Given f, set $-f$ to

$$-f(x_1, \ldots, x_n) = f(1 - x_1, x_2, \ldots, x_n).$$

We claim $f + (-f) \sim 0 \sim -f + f$. Indeed, both are obtained from f by precomposing with a map $\alpha \times I^{n-1}$, where α is one of the following maps:

$$\mathrm{Id}_I * \bar{\mathrm{Id}}_I \qquad \text{or} \qquad \bar{\mathrm{Id}}_I * \mathrm{Id}_I \qquad \text{in } \Pi_1(I).$$

Thus α is based homotopic to I, and we are done as before. □

Usually when we talk about the 'groups π_n', we implicitly include the *set* π_0 of path components. Let $f\colon (X, x_0) \to (Y, y_0)$ be a pointed map. There is an induced map

$$f_*\colon \pi_n(X, x_0) \to (Y, y_0) \qquad \forall n$$

given by the formula: $f_*([\alpha]) = [f \circ \alpha]$, using the composite map

$$(I^n, \partial I^n) \xrightarrow{\alpha} (X, x_0) \xrightarrow{f} (Y, y_0).$$

Since $\alpha \sim \beta$ implies $f \circ \alpha \sim f \circ \beta$, this is well-defined on homotopy classes for all n. If $n \geq 1$, this is a group homomorphism: $f_*([\alpha] + [\beta])$ is represented by

$f \circ (\alpha + \beta)$, but

$$f \circ (\alpha+\beta)(x) = \begin{cases} f(\alpha(2x_1, x_2, \ldots, x_n)) & x_1 \leq 1/2 \\ f(\beta(2x_1 - 1, x_2, \ldots, x_n)) & x_1 \geq 1/2 \end{cases} = ((f \circ \alpha) + (f \circ \beta))(x),$$

hence we have

$$f_*([\alpha + \beta]) = [f \circ (\alpha + \beta)] = [f \circ \alpha + f \circ \beta] = f_*([\alpha]) + f_*([\beta]).$$

It is clear that this is functorial in f, hence we have functors

$$\pi_0 \colon \mathbf{Top}_* \to \mathbf{Set} \qquad \pi_n \colon \mathbf{Top}_* \to \mathbf{Grp} \qquad n \geq 1$$

Hence the groups π_n are topological invariants of pointed spaces. Suppose two maps $f, g \colon (X, x_0) \to (Y, y_0)$ are *based* homotopic, then clearly

$$[f \circ \alpha] = [g \circ \alpha] \implies f_* = g_* \colon \pi_n(X, x_0) \to \pi_n(Y, y_0),$$

so these functors factor through $\mathbf{hTop}_*$, and hence are invariants of based homotopy type too. We will see soon that they are in fact regular homotopy invariants. First, let us discuss an alternative perspective on the homotopy groups $\pi_n(X, x_0)$.

Let $*$ denote the baspoint $\partial I^n / \partial I^n$ of $I^n / \partial I^n$. Observe that $\pi_n(X, x_0)$ could just as well be described by based homotopy classes of maps $I^n / \partial I^n \to X$ using the universal property of quotient to convert back and forth between the two. On one hand, tt is clear that given based homotopic maps $I^n / \partial I^n \to X$, precomosing with the quotient map gives a homotopy rel ∂I^n for the corresponding map out of $(I^n, \partial I^n)$.

On the other hand, suppose we're given a homotopy $(I^n, \partial I^n) \to (X, x_0)$. Consider the following equivalence relation on $I^n \times I$: declare $(x, t) \sim (x', t')$ iff $t = t'$ and $x, x' \in A$ (clearly equivalence). The map

$$I^n \times I \xrightarrow{q \times \mathrm{Id}_I} (I^n / \partial I^n) \times I$$

obviously has the property that (x, t) and (x', t') share an image iff they are $\sim$-related, hence this induces an injective map

$$(I^n \times I) / \sim \to (I^n / \partial I^n) \times I,$$

and it is clearly surjective as the preimage of $([x], t)$ is just $[x, t]$. But the LHS is compact since I^{n+1} is, and the RHS is Hausdorff because $I^n / \partial I^n = S^n$, hence this is a homeomorphism. Composing with the inverse map

$$(I^n / \partial I^n) \times I \to (I^n \times I) / \sim \to (X, x_0)$$

gives us the desired homotopy, proving the claim. It follows that we can view $\pi_n(X, x_0)$ as based homotopy classes of maps $S^n \to X$ using the identification $S^n = I^n / \partial I^n$ from the previous section.

This transports an addition structure to based classes of maps $S^n \to X$, which we also write as $+$; we can write it down explicitly. To do that, first observe

that the cubical addition we defined earlier was done by precomposing (f, g) with the 'collapsing' map $I^n/\partial I^n \to I^n/\partial I^n \vee I^n/\partial I^n$ defined by

$$(x_1, \ldots, x_n) \mapsto \begin{cases} i_{\leq}(2x_1, x_2, \ldots, x_n) & x_1 \leq 1/2 \\ i_{\geq}(2x_1 - 1, x_2, \ldots, x_n) & x_1 \geq 1/2 \end{cases}$$

where $i_{\geq}, i_{\leq}$ are the canonical embeddings to each wedge summand. So, by virtue of the canonical identification this induces the bottom map in the diagram:

$$\begin{array}{ccc} I^n/\partial I^n & \longrightarrow & I^n/\partial I^n \vee I^n/\partial I^n \\ \| & & \| \\ S^n & \xrightarrow{\omega} & S^n \vee S^n \end{array}$$

It immediately follows that addition $f + g$ maps $f, g \colon S^n \to X$ sending $1 \mapsto x_0$ is done by precomposing ω with $S^n \vee S^n \xrightarrow{(f,g)} X$. It is also helpful to view both collapse maps as (isomorphic to) quotient maps by subspaces. Namely, let H be the subspace $\{x \in I^n/\partial I^n : x_1 = 1/2\}$. The collapsing map sends H to zero, and hence clearly induces a continuous bijection

$$(I^n/\partial I^n)/H \xrightarrow{\sim} I^n/\partial I^n \vee I^n/\partial I^n,$$

which is a homeomorphism by compactness. Everything commutes, so the collapse map is isomorphic to the quotient by H. From the formula of the identification, the subspace H maps homeomorphically onto the subsphere $S_1^{n-1} \coloneqq \{(x_0, \ldots, x_n) \in S^n : x_1 = 0\}$, so ω is isomorphic to collapsing the subspace S_1^{n-1}.

1.4.1 Relative homotopy groups

Now we move on to defining $\pi_n(X, A, x_0)$ for a triple (X, A, x_0). Set $J^0 = \{1\}$, and for $n \geq 2$ let J^{n-1} be the following (closed) subspace of ∂I^n:

$$J^{n-1} \coloneqq \{x \in \partial I^n : \exists i \neq n, x_i \in \{0, 1\}\}.$$

In other words, J^{n-1} is the union of all the faces of ∂I^n except for $I^{n-1} \subset I^n$. For each $n \geq 1$, define $\pi_n(X, A, x_0)$ to be the set of classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ up to homotopy through maps of the same form. For $n \geq 2$, there is a natural addition structure on $\pi_n(X, A, x_0)$, namely with the same

1.5 Properties and Change of Basepoint

1.6 Fibrations and Fiber Bundles

2 Appendix: Basic Topology

This section will be focused on general topology, and basic constructions and facts about them which are used often in the prequel.

2.1 General topology

Let X be a space. Let us recall some basic facts about topology which we use throughout this document. Remember that the continuous image of a compact space is compact. Also (obviously) a closed subspace of a compact space is compact. Moreover:

Lemma 2.1.1 *A compact subspace $X \subset Y$ of a Hausdorff space is closed.*

Proof Fix $y \notin X$, then for each $x \in X$, let $U_x \ni x$ and $V_x \ni y$ be disjoint open nbhds of those points, so $\{U_x : x \in X\}$ forms an open cover of X, hence has a finite subcover indexed by some $x_1, \ldots, x_n$. Therefore the set

$$V_{x_1} \cap \cdots \cap V_{x_n}$$

is an open nbhd of y which is disjoint from X. □

Corollary 2.1.2 *Let X be compact, Y be Hausdorff, and $f\colon X \to Y$ a map.*

1. *f is closed.*
2. *If f is injective, then f is a closed embedding.*
3. *If f is bijective, then f is a homeomorphism.*

Proof Everything follows from f being closed, which itself follows from Lemma 2.1.1 because a closed subspace of X is compact, so its image in Y is compact and hence closed. □

Proposition 2.1.3 *Let X, Y be spaces.*

1. *Let $X_1, \ldots, X_n \subset X$ be closed subsets such that $\bigcup_i X_i = X$, and suppose we have maps $f_i\colon X_i \to Y$ satisfying $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$ for all i, j. Then there is a unique map $f\colon X \to Y$ such that $f|_{X_i} = f_i$ for all i.*
2. *Let $U_\alpha \subset X$ for $\alpha \in I$ be an open cover of X, and suppose we have maps $f_\alpha\colon U_\alpha \to Y$ satisfying $f_\alpha|_{X_\alpha \cap X_\beta} = f_\beta|_{X_\alpha \cap X_\beta}$ for all α, β. Then there is a unique map $f\colon X \to Y$ such that $f|_{X_\alpha} = f_\alpha$ for all α.*

Proof Let us prove these simultaneously. In each case, the uniqueness is obvious because maps $X \to Y$ are determined by their values on a cover (closed or open). Moreover, it is clear that this unique set function is well-defined by taking the common values on pairwise intersections. Finally, for continuity, in case (1) the preimage of a closed set becomes a finite union of closed subsets of X hence closed, and in case (2) the preimage of an open set is a union of open sets. □

Corollary 2.1.4 *If $X_1, \dots, X_n$ is a closed cover of X then*

$$X = \varinjlim_{i,j} \Big(X_i \leftarrow X_i \cap X_j \rightarrow X_j \Big).$$

The same conclusion holds if $\{X_i\}$ is an open cover by arbitrarily many opens.

We can easily extend this to a more general circumstance with no further work. A cover $\{X_\alpha\}$ of X is *locally finite* if each point of x has an open nbhd intersecting only finitely many of the $\{X_\alpha\}$. For example, the closed cover $\{[k, k+1] : k \in \mathbb{Z}\}$ of $\mathbb{R}$ is locally finite, because the open interval $(x - 1/2, x + 1/2)$ around any $x \in \mathbb{R}$ intersects at most two covering sets.

Corollary 2.1.5 *Let $\{X_\alpha : \alpha \in I\}$ be a closed cover of X which is locally finite. Then*

$$X = \varinjlim_{\alpha,\beta} \Big(X_\alpha \leftarrow X_\alpha \cap X_\beta \rightarrow X_\beta \Big).$$

Equivalently, given maps $f_\alpha \colon X_\alpha \to Y$ such that $f_\alpha \equiv f_\beta$ on $X_\alpha \cap X_\beta$, there is a unique $f \colon X \to Y$ restricting to the f_α.

Proof We need only check continuity of the collated map $f \colon X \to Y$, for which it suffices (by the open gluing lemma) to check continuity of the restriction $f|_U \colon U \to Y$ for U an open nbhd of an arbitrary x intersecting finitely many of the X_α. Indeed, $\{X_\alpha \cap U\}$ forms a finite closed cover of U, hence the maps $f_\alpha|_{X_\alpha \cap U}$ collate to a continuos map $f|_U$, agreeing with f on U.

A slicker way to prove this is to observe that (where U_x is the appropriate nbhd of x intersecting finitely many X_α's):

$$\begin{aligned} X &= \varinjlim_{x \in X} U_x = \varinjlim_{x \in X} \varinjlim_{\alpha \in I} (U_x \cap X_\alpha) \\ &= \varinjlim_{\alpha \in I} \varinjlim_{x \in X} (U_x \cap X_\alpha) = \varinjlim_{\alpha \in I} X_\alpha. \end{aligned}$$

We used the following facts in the following order: (1) the open gluing lemma, (2) the closed gluing lemma for U_x, (3) that colimits commute with colimits, and (4) the open gluin lemma for the open cover $U_x \cap X_\alpha$ of X_α. □

2.2 Specific constructions I

Here is the fundamental construction which we rely on:

Definition 2.2.1

1. Let X be a set and $\{Y_i\}$ be spaces. The *initial (or strong) topology* on X with respect to a family of maps $\{f_i \colon X \to Y_i\}$ is the coarsest topology making these maps continuous.
2. Let Y be a set and $\{X_i\}$ be spaces. The *final (or weak) topology* on Y with respect to a family of maps $\{f_i \colon X_i \to Y\}$ is the finest topology on Y making these maps continuous.

Recall that 'coarsest' means 'with the least open sets' and 'finest' means 'with the most open sets'. So, the initial topology on X has a subbase consisting of sets $f_i^{-1}(U)$ for $U \subset Y_i$ an open set. On the other hand, in the final topology on Y a subset $U \subset Y$ is open if and only if $f_i^{-1}(U) \subset X_i$ is open for each i. Indeed, this forms a topology on Y and for any topology making the f_i continuous, an open U has open preimage in all the X_i hence is also open in our candidate topology.

These topologies are very categorical in nature: the weak topology has the universal property that for any Z, a map $Y \to Z$ is the same as a function $Y \to Z$ such that the compositions $X_i \to Z$ are continuous. Indeed, these compositions must be continuous and conversely if they all are, then the preimage in Y of any open set is open because it has open preimage in each X_i.

Similarly, if X has the initial topology, then a map $g\colon W \to X$ is just a set function $W \to X$ making each composite $W \to Y_i$ continuous. Indeed, each composite must be continuous, and conversely since the subsets $f_i^{-1}(U)$ form a subbase, each of their preimages is open in W by continuity of $f_i g$, so g is continuous.

Here is the consequence: if $Y_\bullet\colon I \to \mathbf{Top}$ is a small diagram, then $\varprojlim_i Y_i$ can be constructed as follows: just construct the limit in **Set**, and then take the initial topology with respect to the maps $\varprojlim_i Y_i \to Y_j$ for each j. Likewise, the same works for the weak topology. So, **Top** has all small (co)limits, and the forgetful functor to **Set** preserves them.

Now let us examine specific examples. Let $\{X_i\}$ be a collection of spaces. The above says that the categorical product is just the cartesian product $X \coloneqq \prod_i X_i$ with topology generated by products where all factors but one are X_i, and one X_j is replaced by an open subset of X_j. Finite intersections of these are just the product $\prod_i X_i$ where finitely many factors are replaced by open subsets, so that is a basis for the product topology. In particular, on $X \times Y$ a basis for the topology are the sets $U \times V$ where $U \subset X$ and $V \subset Y$ are open.

By category theory the flipping map $X \times Y \xrightarrow{\sim} Y \times X$ is a homeomorphism. Also, for any $y_0 \in Y$ the map $X \to X \times Y$ given by $x \mapsto (x, y_0)$ is continuous by the universal property and continuity of the constant map at y_0. This is an embedding, because the image $X \times \{y_0\}$ has inverse given by the restriction of the projection.

In the general case $X = \prod_i X_i$, the closed subsets are the intersections of complements of the products with finitely many factors open and not X_i. But in the finite case, especially the case $X \times Y$, the subsets $A \times B$ are closed where A and B are closed, because

$$\begin{aligned} X \times Y - A \times B &= (X - A) \times B \cup X \times (Y - B) \\ &= (X \times Y - X \times B) \cup (X \times Y - A \times Y). \end{aligned}$$

In particular, if e.g. Y is Hausdorff, then $X = X \times y_0 \subset X \times Y$ is a closed subset. Let us also briefly mention that the subspace topology $A \hookrightarrow X$ is the initial topology with respect to the inclusion map, clear from the definitions.

Here are some examples of the weak topology. Firstly, we know from above that the coproduct of spaces X_i is just the disjoint union $X \coloneqq \bigsqcup_i X_i$ equipped with the topology that a set is open if and only if its preimage in each X_i is open. Equivalently, a set is closed if each preimage is closed. Note that the inclusions $X_i \hookrightarrow X$ are open and closed embeddings, because they are continuous injections and an open set $U \subset X_i \subset X$ has preimage either U or empty (and same for closed). Therefore we regard $X_i \subset X$ as clopen subspaces.

Let X be a space and $\sim$ an equivalence relation on X. Then the *quotient topology* on $X/\sim$ is the weak topology with respect to the quotient map of sets $q\colon X \to X/\sim$, so a set A is open/closed iff $q^{-1}(A)$ is open/closed in X. A special case of this construction is when there is a subspace $A \subset X$ and $\sim$ is the equivalence relation that $x \sim y$ iff $x, y \in A$. Then we write $X/A \coloneqq X/\sim$.

Most often, we will consider quotients by subspaces when $A \subset X$ is a closed subspace, in which case it is much more well-behaved. Firstly, writing $* \coloneqq A/A \subset X/A$, its preimage in X is A, hence $*$ is a closed point. Moreover in that case, $X - A \hookrightarrow X \to X/A$ is an open embedding. Indeed, it is a continuous injection, and the image of an open subset $U \subset X - A$ in X/A has (open) preimage $U \subset X$, so it is an open map.

Here is another example of the weak topology: Let $A \subset X$ and $\varphi\colon A \to Y$ be a map. Then we define $X \cup_\varphi Y$ to be the pushout of φ along the inclusion $A \hookrightarrow X$. The description above shows that this is equal to $(X \sqcup Y)/\sim$ where $\sim$ is the equivalence relation generated by $a \sim \varphi(a)$ for $a \in A$.

Lemma 2.2.2 *The pushout of a topological (resp. closed, open) embedding is a topological (resp. closed, open). In particular, if $A \subset X$ is a closed subspace and $\varphi\colon A \to Y$ is an attaching map, then $Y \subset X \cup_\varphi Y$ is a closed subspace.*

Proof First we show that the map $j\colon Y \to X \cup_\varphi Y$ is injective. We claim that the equivalence relation generated by $a \sim \varphi(a)$ for $a \in A$ is the following: $x \sim y$ iff either $x \in A$ and $y = \varphi(x) \in Y$ (or flipped), or $x, y \in A$ and $\varphi(x) = \varphi(y)$, or $x = y$. This is symmetric and reflexive and contains the relation $a \sim \varphi(a)$, and by transitivity any equivalence relation containing that must contain this one too. This one is obviously transitive from the definitions, proving our claim.

So, if $j(y) = j(y')$, then neither of these are in A, hence $y = y'$, making j injective. Consider the restriction $j\colon Y \to j(Y)$, and let $U \subset Y$ be open. We now claim

$$j(U) = j(Y) \cap [X \sqcup U], \tag{$*$}$$

where $[X \sqcup U]$ is the image of $X \sqcup U \subset X \sqcup Y$. The preimage of this subset in $X \sqcup Y$ is exactly $X \sqcup U$ because if $y \in \varnothing \sqcup Y$ has image in $[X \sqcup U]$ then $y \in U$ by the description of $\sim$. So the RHS of $(*)$ is open in the subspace topology.

The equality holds because $\subset$ is obvious, and if $y \in j(Y)$ and in $[X \sqcup U]$ then $y \in j(U)$. This proves that j is an embedding.

Finally, observe that the preimage of $j(Y)$ in $X \sqcup Y$ is $A \sqcup Y$. So, if A is open (resp. closed), then this subspace is also open (resp. closed). □

An important special case is the *wedge sum*. Let (X_i, x_i) be a collection of pointed spaces. The wedge sum $\bigvee_i X_i$ is the space X with the following universal property: maps $X \to Y$ are equivalent to maps $X_i \to Y$ which send x_i to the same point. It is then clear that we can construct

$$\bigvee_i X_i = (\bigsqcup_i X_i)/\{x_i\}$$

as a quotient like so. This comes equipped with pointed maps $X_i \to X$, and by the universal property these are embeddings. Note that when the basepoints are closed (e.g. the spaces are Hausdorff), they are closed embeddings, and so the spaces X_i form a closed cover of X. When there are only finitely many, this is a finite cover. This can sometimes make proofs much easier.

Lemma 2.2.3 *The canonical map $X \vee Y \to X \times \{y_0\} \cup \{x_0\} \times Y \subset X \times Y$ is a homeomorphism for pointed spaces (X, x_0) and (Y, y_0).*

Proof We have already described (pointed) embeddings $X \to X \times y_0 \to X \times Y$ and likewise for Y, so these jointly induce a pointed map $X \vee Y \to X \times Y$ which induces a continuous bijection $\varphi\colon X \vee Y \to X \times y_0 \cup x_0 \times Y$. We claim this is an open map. If U is an open set, its open preimage in $X \sqcup Y$ is of the form $U_X \sqcup U_Y$, so that we have

$$\varphi(U) = U_X \times y_0 \cup x_0 \times U_Y.$$

If $* \notin U$, then $x_0 \notin U_X$ and $y_0 \notin U_Y$, which means that

$$U_X \times X \cap (X \times y_0 \cup x_0 \times Y) = U_X \times X \cap X \times y_0 = U_X \times y_0,$$

and vice versa for $x_0 \times Y$, so $\varphi(U)$ is open in the subspace topology. If $* \in U$, then $x_0 \in U_X$ and $y_0 \in U_Y$, hence

$$(U_X \times U_Y) \cap (X \times y_0 \cup x_0 \times X) = U_X \times y_0 \cup x_0 \times U_Y = \varphi(U)$$

is open once again. □

Corollary 2.2.4 *The wedge sum of pointed spaces (X, x_0) and (Y, y_0) has the following universal properties for pointed spaces W or Z:*

1. *Pointed maps $X \vee Y \to Z$ canonically correspond to pairs of pointed maps $X \to Z$ and $Y \to Z$.*
2. *Pointed maps $W \to X \vee Y$ canonically correspond to pairs of pointed maps $f_X\colon W \to X$ and $f_Y\colon W \to Y$ such that for each $w \in W$, either $f_X(w) = x_0$ or $f_Y(w) = y_0$.*

Proof (1) is just the universal property of wedge sums as defined and constructed. For (2) use the identification $X \vee Y = X \times y_0 \cup x_0 \times Y$ of Lemma 2.2.3, the universal property of the product $X \times Y$, and the universal property of the subspace topology from the initial topology. □

Here is a last example of the weak topology. Let $X_0 \subset X_1 \subset \cdots$ be an increasing sequence of spaces and write X for the union of these sets. Our description shows that the filtered colimit X has the weak topology with respect to the inclusions $X_i \hookrightarrow X$, which means that a set is open (or closed) in X iff its intersection with each X_i is open (or closed). This is crucial to studying CW complexes.

2.3 Basic homotopies

Suppose we have a homotopy $f_0 \sim f_1 \colon X \to Y$ and one $g_0 \sim g_1 \colon X' \to Y'$. Then the following two maps are homotopic:

$$f_0 \times g_0 \sim f_1 \times g_1 \colon X \times X' \to Y \times Y'$$

The homotopy is given explicitly by

$$X \times X' \times I \xrightarrow{(x,x',t) \mapsto (x,t,x',t)} X \times I \times X' \times I \to Y \times Y',$$

so ultimately this comes down to the continuity of the diagonal map $I \to I \times I$. A similar thing happens in the case of $\sqcup$: Namely,

$$f_0 \sqcup g_0 \sim f_1 \sqcup g_1 \colon X \sqcup X' \to Y \sqcup Y',$$

because the homotopy is obtained using the map

$$(X \sqcup Y) \times I \to (X \times I) \sqcup (Y \times I),$$

and continuity is because $X \times I$ and $Y \times I$ form a (disjoint) open cover of the domain. (In fact, it is a homeomorphism because the map in the opposite direction induced by $X \times I \to (X \sqcup Y) \times I$ and likewise is its inverse.)

Homotopy generally behaves especially well with respect to limits. In particular, if we have homotopies $f_0 \sim f_1 \colon X \to Y$ and $g_0 \sim g_1 \colon X \to Y'$, then the induced maps

$$(f_0, g_0) \sim (f_1, g_1) \colon X \to Y \times Y'$$

are also homotopic since we can take the induced map out of $X \times I$. The same works for disjoint unions because of the homeomorphism we just described. But for general colimits it doesn't work.

The following is another important situation:

Proposition 2.3.1 *Let X, X', Y, Y' be pointed spaces and suppose we have*

$$f_0 \sim f_1 \colon X \to Y \qquad g_0 \sim g_1 \colon X' \to Y'$$

which are based homotopies between pointed maps. Then the maps

$$f_0 \vee g_0 \sim f_1 \vee g_1 \colon X \vee X' \to Y \vee Y'$$

are based homotopic.

Proof Observe that a based homotopy $X \times I \to Y$ must send $x_0 \times I$ to y_0, so it defines a pointed map $X \times I/x_0 \times I \to Y$ where the basepoint of the LHS is $* = x_0 \times I/x_0 \times I$. So, we can construct a natural map

$$(X \vee X') \times I \to (X \times I/x_0 \times I) \vee (X' \times I/x_0' \times I) \qquad (*)$$

using the universal property (2) of Cor. 2.2.4: Namely, use the pair of maps

$$(X \vee X') \times I \to X \times I \to X \times I/x_0 \times I$$

(and likewise for X') where the first map is induced by the identity and the constant pointed map $X' \to I$ times I. It is well-defined and continuous because the LHS of $(*)$ is covered by the sets $X \times I$ and $X' \times I$, and on $X \times I$ the map to $X' \times I/x_0' \times I$ remains at the basepoint.

To finish, by precomposing the wedge sum of our homotopies with $(*)$ we get a map

$$(X \vee X') \times I \to Y \vee Y'.$$

On $(X \vee X') \times 0$, this composite is just $f_0 \vee g_0$, and similarly for 1. Moreover, this is fixed on the common basepoint $*$, which factors through the basepoint of the RHS of $(*)$. □

Another case we would like to handle is the following: since maps $(X, A) \to (Y, y_0)$ correspond to pointed maps $X/A \to Y$, do homotopies rel A correspond to based homotopies? The answer is related to the commutativity of a certain limit (the product $(-) \times I$) with a certain colimit (taking the quotient). For fun, we will prove this in two different ways; the cases that we need only rely on an easier fact, so we use that instead.

Let (X, A) be a pair, and consider the following relation $\sim$ on $X \times I$, where $(x, t) \sim (x', t')$ if and only if $t = t'$ and either $x = x'$ or $x, x' \in A$. This is clearly symmetric and reflexive, and it is transitive since equality and membership in A are both transitive. The map

$$q \times 1_I \colon X \times I \to (X/A) \times I$$

has the property that $q \times 1_I(x, t) = q \times 1_I(x', t')$ if and only if $(x, t) \sim (x', t')$, since it's equivalence to $t = t'$ and $q(x) = q(x')$. Therefore this induces a continuous injection

$$\varphi \colon (X \times I)/\sim \to (X/A) \times I,$$

and of course an element $([x], t)$ has preimage $[x, t]$, so this is a continuous bijection. The relevant question is whether φ^{-1} is continuous.

Lemma 2.3.2 *If, for a given pair (X, A), φ is bicontinuous, then for any (Y, y_0), homotopies of maps off pairs $(X, A) \to (Y, y_0)$ correspond to based homotopies of maps $X/A \to Y$.*

Proof On one hand, given a based homotopy $X/A \times I \to Y$, we can precompose by the map $q \times 1_I$ to get a homotopy $X \times I \to Y$ sending A to the basepoint at all times t. On the other hand, such a homotopy $F\colon X \times I \to Y$ identifies any $\sim$-related points, hence it factors through $(X \times I)/\sim$, so by the presumed homeomorphism we get a composite map

$$(X/A) \times I \xrightarrow{\sim} (X \times I)/\sim \to Y,$$

which sends $([x], t) \mapsto F(x, t)$, giving the desired homotopy. □

A pair (X, A) is called *compact Hausdorff* if X is compact Hausdorff and $A \subset X$ is closed. This is a special case which does have the property of the lemma.

Proposition 2.3.3 *Compact Hausdorff pairs (X, A) satisfy the conclusion of lemma Lemma 2.3.2. In particular, it applies to the pairs $(I^n, \partial I^n)$, (I^n, J^{n-1}), and $(\partial I^n, J^{n-1})$.*

Proof It suffices to show that X/A is Hausdorff. This is because by $(X \times I)/\sim$ is the compact image of the compact space $X \times I$, and $(X/A) \times I$ will be Hausdorff, so φ is automatically a homeomorphism and Lemma 2.3.2 applies.

To prove this, recall that by closedness $X - A \to X/A$ is an open embedding with image $X/A - *$. So, for any $x \neq y \in X/A$ which aren't the basepoint, the required open nbhds separating the points exist in the open subspace $X - A$, and their images provide the one in X/A. So, w.l.o.g. assume $x = * = A/A$. Since compact Hausdorff spaces are regular, there is an open nbhd of U separating A from a nbhd V of y in X, and since U contains A it is saturated (because the preimage of $q(U)$ is $U \cup A = U$), hence the result. □

That is all we shall need for our purposes, but in fact this statement is true for all pairs (X, A) regardless of compact Hausdorff-ness or closedness of A. We will explore this in the next section about locally compact Hausdorff spaces, but let us first see that the condition of Lemma 2.3.2 is equivalent to

$$q \times 1_I\colon X \times I \to (X/A) \times I$$

being a quotient map. One direction is trivial by definition of φ. For the other, suppose it is one, and let the quotient $X \times I \to (X \times I)/\sim$ be denoted p. We claim φ is an open map. Indeed, for $U \subset (X \times I)/\sim$ an open subset, we have

$$p^{-1}(U) = p^{-1}(\varphi^{-1}(\varphi(U))) = (\varphi \circ p)^{-1}(\varphi(U)) = (q \times 1_I)^{-1}(\varphi(U))$$

being an open set by continuity of p and bijectivity of φ, so by our assumption of a quotient map $\varphi(U)$ is open. So, assuming the following proposition, we get the corollary for free:

Proposition 2.3.4 *Let (X, A) be any pair and write $q\colon X \to X/A$ for the quotient map. Then*

$$q \times 1_I\colon X \times I \to (X/A) \times I$$

is a quotient map.

Corollary 2.3.5 *For any pair (X, A) and pointed space (Y, y_0), homotopies $f_t\colon (X, A) \to (Y, y_0)$ correspond directly to based homotopies $\widetilde{f_t}\colon X/A \to Y$.*

2.4 Locally Compact Hausdorff Spaces

A space X is *locally compact* (LC) if each point $x \in X$ has a *compact nbhd*, by which we mean a compact set $K \subset X$ containing x which contains an open nbhd of x. X is LCH if it is locally compact and Hausdorff.

From the definition, it is obvious that compact Hausdorff spaces are LCH; note that in this case, the compact neighborhood is necessarily closed. A first important step is regularity.

Proposition 2.4.1 *An LCH space is regular.*

Lemma 2.4.2 *In a Hausdorff space, a point can be separated from a compact set by disjoint open nbhds. In particular, a compact Hausdorff space is regular.*

Proof Let $x \in X$ and $A \subset X$ a compact (so closed) set with $x \notin A$. For each $a \in A$, let U_a and V_a be disjoint open sets separating x and $a \in A$, then there are finitely many $a_1, \ldots, a_n$ such that

$$V \coloneqq V_{a_1} \cup \cdots \cup V_{a_n}$$

covers A. Then $U \coloneqq U_{a_1} \cap \cdots \cap U_{a_n}$ is disjoint from V and contains x. □

Proof (of Prop. 2.4.1) Let $x \in X$ and $A \subset X$ a closed set with $x \notin A$. Choose a compact nbhd $K \supset U \ni x$, so $A \cap K$ is a closed subset of K hence compact, and we may separate x from $A \cap K$ by disjoint open nbhds V and W by Lemma 2.4.2.

Finally, $U \cap V$ and $W \cup (X - K)$ are disjoint open nbhds of x and A respectively, where the latter is because $A \cap K \subset W$, and $A - A \cap K \subset X - K$. □

Lemma 2.4.3 *Let X be a regular space, $x \in X$, and U an open nbhd of x. Then there exists a smaller nbhd $V \subset U$ of x such that $\overline{V} \subset U$.*

Proof Choose V, W resp. separating the point x from the closed set $X - U$; then $X - W \subset U$, and

$$V \cap W = \varnothing \Rightarrow V \subset X - W \subset U.$$

Then V is a smaller open nbhd of x but it's contained in the closed subset $X - W$ of U, so its closure is in U. □

Corollary 2.4.4 *Let X be an LCH space, $x \in X$, and U an open nbhd of X. There exists a nbhd $V \subset U$ of x such that $\overline{V}$ is compact and contained in U.*

Proof First take a nbhd W of x contained in a compact set K. This must be closed, so $\overline{W} \subset K$ is also compact, and thus replacing W by $U \cap W$ we have $\overline{W}$ is a closed subset of a compact set hence compact, with $W \subset U$. Now, by the previous two results, intersect W with a smaller open nbhd whose closure is contained in U, then the closure of the intersection is compact and contained in U, completing the proof. □

Towards our main result of this section, we need one more general easy lemma about compact spaces, called the Tube Lemma; this result is usually used to prove that the product of compact sets is compact, which we have been assuming thus far.

Lemma 2.4.5 *Let X, Y be spaces with Y compact, $x_0 \in X$, and N an open nbhd of $x_0 \times Y \subset X \times Y$. Then there is an open nbhd W of x_0 such that $W \times Y \subset N$.*

Proof N can be covered by basis elements by openness, but $x_0 \times Y \cong Y$ is compact so finitely many such elements

$$\{U_1 \times V_1, \ldots, U_n \times V_n\}$$

cover $x_0 \times Y$, each of which is contained in N (and assumed nonempty intersection with $x_0 \times Y$). Taking $W \coloneqq U_1 \cap \cdots \cap U_n$, clearly an open nbhd of x_0, we claim that $W \times Y$ is in N. Indeed, for $(x, y) \in W \times Y$, observe that there exists i such that $(x_0, y) \in U_i \times V_i$, yet since $x \in W \subset U_i$ this means $(x, y) \in U_i \times V_i \subset N$. □

Finally, we are ready to prove the main result. Recall that a surjective map $q \colon X \to Y$ is a *quotient map* if a set $A \subset Y$ is open iff $q^{-1}(A)$ is open in X. Note that such a q is not necessarily open, since $q^{-1}q(U)$ might be strictly larger than U. A subset $U \subset X$ is *saturated* if $q^{-1}q(U) = U$, hence the image of a saturated open is open.

Note that for any map $q \colon X \to Y$ and subset $A \subset Y$, we have

$$\begin{aligned} qq^{-1}(A) &= \{q(x) : q(x) \in A\} \\ &= \{y \in Y : y \in A, y \in q(X)\} = q(X) \cap A. \end{aligned}$$

In particular, if q is a quotient map, then $qq^{-1}(A) = A$, which shows that each open $U \subset Y$ is the image of saturated open, namely $q^{-1}q(U)$. The following theorem is called Whitehead's theorem in general topology.

Theorem 2.4.6 *Let Z be an LCH space and $q \colon X \to Y$ a quotient map. Then*

$$q \times 1_Z \colon X \times Z \to Y \times Z$$

is a quotient map.

Since maps of the form $X \to X/A$ for any pair (X, A) are quotient maps, and since I is LCH (being compact Hausdorff), Whitehead's theorem implies Prop. 2.3.4, finishing the results form the last section.

Proof (of Thm. 2.4.6) The map is clearly surjective. Let $A \subset Y \times Z$; assuming $W := (q \times 1)^{-1}(A)$ is open in $X \times Z$, we must show A is open in $Y \times Z$. To do this, fix a point $(x_0, z_0) \in Z$, and we will show that there is a saturated nbhd $U \ni x_0$ and a nbhd $V \ni z_0$ such that $U \times V \subset W$. This will suffice because by surjectivity, any point (y_0, z_0) of A lifts to (x_0, z_0) in W, and then $q(U) \times V$ will be an open nbhd of the point contained in A.

To begin, by openness there exists an open nbhd $U_0 \times V_0 \subset W$ of (x_0, z_0). By Cor. 2.4.4, we can shrink V_0 around z_0 to V, so that $z_0 \in V \subset \overline{V} \subset V_0$ with $\overline{V}$ compact. Now we will construct a sequence $U_0 \subset U_1 \subset \cdots$ of open subsets with the following properties:

1. $q^{-1}q(U_i) \subset U_{i+1}$ for all $i \geq 0$.
2. $U_i \times \overline{V} \subset W$ for all $i \geq 0$.

Once this is done, we can set U to be the open nbhd $U = \bigcup_i U_i$ of x_0, then

$$U \subset q^{-1}q(U) = \bigcup_{i \geq 0} q^{-1}(U_i) \subset \bigcup_{i \geq 1} q^{-1}(U_i) \subset U,$$

so U is saturated, and moreover

$$U \times V \subset U \times \overline{V} = \bigcup_{i \geq 0} (U_i \times \overline{V}) \subset W,$$

which is what we wanted.

To construct the sequence $\{U_i\}$, start with keeping U_0 as before, and assume U_i has been constructed with the desired properties. First notice that $q^{-1}q(U_i) \times \overline{V} \subset W$, because

$$(q \times 1)(q^{-1}q(U_i) \times \overline{V}) = q(U_i) \times \overline{V} \subset (q \times 1)(q \times 1)^{-1}(A) = A$$

by property 2. Hence, for each $x \in q^{-1}q(U_i)$, this means that $x \times \overline{V} \subset W$, so there exists an open nbhd $U_{i,x}$ of x such that $U_{i,x} \times \overline{V} \subset W$ by the tube lemma. Letting $U_{i+1} = \bigcup_{x \in q^{-1}q(U_i)} U_{i,x}$ gives both properties. □

2.4.1 A category-theoretic proof

There is a more elegant proof of the homotopy-theoretic fact we used above with respect to LCH spaces, importantly the interval I. This involves the following useful construction:

Definition 2.4.7 Let X, Y be spaces. The *mapping space* $[X, Y]$ is the topological space with underlying set $\mathrm{Hom}_{\mathbf{Top}}(X, Y)$ and the *compact-open topology*, which has a subbasis consisting of open sets

$$\mathcal{B}(K, U) = \{f \colon X \to Y : f(K) \subset U\} \subset [X, Y]$$

for $K \subset X$ compact and $U \subset Y$ open.

This construction is functorial in both variables as one would expect, extending the functoriality of $\mathrm{Hom}(-,-)$, but we care about in the covariant variable: Given $h\colon Y \to Z$, the induced map

$$h_*\colon [X,Y] \to [X,Z] \qquad f \mapsto h \circ f$$

is continuous because

$$\begin{aligned} h_*^{-1}(\mathcal{B}(K,U)) &= \{f\colon X \to Y : (h \circ f)(K) \subset U\} \\ &= \{f\colon X \to Y : f(K) \subset h^{-1}(U)\} \\ &= \mathcal{B}(K, h^{-1}(U)). \end{aligned}$$

Let X, Y, Z be spaces; our goal is to extend the normal exponential law

$$\mathrm{Hom}(X \times Y, Z) \cong \mathrm{Hom}(X, [Y,Z])$$

which exists on the underlying sets (given by currying). Note that because the forgetful functor to sets is faithful, we just need to check that the bijection is well-defined (i.e. sends continuous maps to continuous maps in both directions), and the case in sets provides the exponential adjunction.

Theorem 2.4.8 *Let Y be an LCH space. Then the functor $- \times Y$ is left adjoint to the functor $[Y,-]$ from* **Top** $\to$ **Top**.

Proof As explained, we just need to show that both sides of the bijection above are well-defined. Given a map $g\colon X \times Y \to Z$, we claim that the corresponding map

$$g^\sharp\colon X \to [Y,Z] \qquad x \mapsto \big(y \mapsto f(x,y)\big)$$

is continuous. Indeed, given a subbasic open $\mathcal{B}(K,U)$, observe that

$$\begin{aligned} {g^\sharp}^{-1}(\mathcal{B}(K,U)) &= \{x \in X : g^\sharp(x)(K) \subset U\} = \{x \in X : g(x \times K) \subset U\} \\ &= \{x \in X : x \times K \subset g^{-1}(U)\}. \end{aligned}$$

This is open by the tube lemma: $g^{-1}(U)$ is open and K is compact, so we know that each such x has a nbhd W such that $W \times K \subset g^{-1}(U)$, i.e. $x \in W \subset {g^\sharp}^{-1}(\mathcal{B}(K,U))$.

In the other direction, given a map $g\colon X \to [Y,Z]$, we can obtain

$$X \times Y \xrightarrow{g \times 1_Y} [Y,Z] \times Y \xrightarrow{\varepsilon} Z,$$

so we really just need to check continuity of the counit (evaluation) map

$$\varepsilon\colon [Y,Z] \times Y \to Z \qquad (f,y) \mapsto f(y).$$

Letting $U \subset Z$ be an open, we have

$$\varepsilon^{-1}(U) = \{(f,y) : f(y) \in U\}$$

which we must show is open. Taking an arbitrary point (f, y) in this set, $f^{-1}(U)$ is an open nbhd of y, so by Cor. 2.4.4 there's a smaller open nbhd V whose closure is compact and inside $f^{-1}(U)$. Now, (f, y) is contained in $\mathcal{B}(\overline{V}, U) \times V$ because

$$f(\overline{V}) \subset f(f^{-1}(U)) = f(Y) \cap U \subset U,$$

and if $(f', y') \in \mathcal{B}(\overline{V}, U) \times V$, then since $y' \in V \subset \overline{V}$, we have

$$\varepsilon(f', y)' = f'(y') \in U \implies \mathcal{B}(\overline{V}, U) \subset \varepsilon^{-1}(U).$$

Hence $\mathcal{B}(\overline{V}, U)$ is an open nbhd of (f, y) contained in $\varepsilon^{-1}(U)$. □

Corollary 2.4.9 *The functor $- \times I$ preserves colimits, and more generally $- \times X$ preserves colimits for any LCH space X.*

For example, this trivially proves the fact that based homotopies of $X/A \to Y$ correspond to homotopies of $(X, A) \to (Y, y_0)$ rel A. One direction is trivial, and the defining pushout square of X/A gives

$$\begin{array}{ccc} A \times I & \xrightarrow{\mathrm{pr}_I} & I \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \Delta_* \times 1} \\ X \times I & \xrightarrow{q \times 1} & (X/A) \times I \end{array}$$

as a pushout. Then the homotopy $X \times I \to Y$ and the constant map $I \to Y$ at y_0 induce the desired homotopy $(X/A) \times I \to Y$.

Example 2.4.10 Another important case: suppose a space X has a filtration $X = \bigcup_{i \geq 0} X_i$ by closed subspaces X_i with $X_0 \subset X_1 \subset X_2 \subset \cdots$. Set-theoretically, we know that maps $X \to Y$ correspond to maps $f_i \colon X_i$ agreeing on all previous spaces by the filtered colimit in **Set**. For that to be the case topologically, continuity of the resulting map from $\{f_i\}$ requires X to have the *weak topology*: a subspace $Z \subset X$ is closed if and only if $Z \cap X_i$ is closed for all i. This is the case e.g. when X is a CW complex.

So, $X = \varinjlim_i X_i$ in **Top**, and thus $X \times I = \varinjlim_i (X_i \times I)$, which implies that a homotopy $X \times I \to Y$ is equivalent to a homotopy $X_i \times I \to Y$ for each i which agrees on all previous subspaces. Note that the same thing applies to any finite closed cover or infinite open cover.

2.4.2 More on Mapping Spaces

The adjunction above has many useful implications beyond just colimit preservation. First of all, the *free path space* Z^I is just the space $[I, Z]$ of paths $I \to Z$. By the adjunction, maps $X \to Z^I$ correspond to maps $X \times I \to Z$, i.e. free homotopies. Fix a basepoint $z_0 \in Z$. Then we define

$$P(Z, z_0) \equiv PZ \coloneqq \{f \in Z^I : f(0) = z_0\} \subset Z^I$$

with the subspace topology, called the (based) pathspace of (Z, z_0). Note that if $h\colon Z \to Z'$ is a pointed map, then $f(0) = z_0$ implies $(h \circ f)(0) = z_0'$, so the based pathspace is functorial.

Because of the universal property of subspaces, PZ has the following universal property: maps $X \to PZ$ correspond to maps $f\colon X \times I \to Z$ such that $X \times 0 \mapsto x_0$. Similarly, the pathspace $P(Z, z_0, z_1)$ is the subspace corresponding to paths starting at z_0 and ending at z_1, functorial in triples, and finally, the *loopspace* of (Z, z_0), called ΩZ, is the pathspace $P(Z, z_0, z_0)$.

So, this has the unviersal property that maps $X \to \Omega Z$ correspond to maps $f\colon X \times I \to Z$, such that for each x, the corresponding path $t \mapsto f(x, t)$ is a loop at z_0, in particular $f(X \times 0 \cup X \times 1) = x_0$. So, set $X = I$; a path $I \to Z$ corresponds a homotopy through maps $(I, \partial I) \to (Z, z_0)$.

2.5 CW Complexes

We will now go through, in somewhat excruciating detail, the definition and basics of CW complexes. Let us review some topological preliminaries from the previous section - we will make this discussion as categorical as possible.

First, let (X, A) be a pair and $\varphi\colon A \to Y$ a map. Then we can form the space $X \cup_\varphi Y$, which is the pushout of the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & Y \\ \downarrow & \ulcorner & \downarrow{\scriptstyle i} \\ X & \longrightarrow & X \cup_\varphi Y \end{array}$$

We gave an explicit construction of this space as $(X \sqcup Y)/\sim$, where $z_1 \sim z_2$ if and only if:

1. $z_1 \in A$ and $z_2 = \varphi(z_1) \in Y$, or $z_2 \in A$ and $z_1 = \varphi(z_2) \in Y$; or
2. $z_1, z_2 \in A$, and $\varphi(z_1) = \varphi(z_2)$; or
3. $z_1 = z_2$.

Throughout the discussion that follows, we will fix the quotient map $q\colon X \sqcup Y \to (X \cup_\varphi Y)$, so that e.g. $i = q|_X$. Our first observation was that $i\colon Y \to X \cup_\varphi Y$ is injective: Indeed, $i(y_1) = i(y_2)$ iff $y_1 \sim y_2$, and since neither is in A we must have $y_1 = y_2$. We will now show that it is always a topological embedding.

To begin let $U \subset Y$ be open, so that $\varphi^{-1}(U)$ is open in the subspace topology and there is some open $V \subset X$ such that $\varphi^{-1}(U) = V \cap A$. We claim firstly that the set $V \sqcup U$ is saturated under q, and secondly that the equation

$$i(U) = q(V \sqcup U) \cap i(Y) \tag{*}$$

holds; this will demonstrate that i is an embedding.

First we need to show $q^{-1}q(V \sqcup U) \subset= V \sqcup U$.

Case 1: Assume $x \in X$ and $q(x) \in q(V \sqcup U)$. If $q(x) \in q(U)$, then $x \in A$ and $\varphi(x) \in U$, hence $x \in \varphi^{-1}(U) \subset V$. Alternatively, if $q(x) \in q(V)$, then there is $x, x' \in A \cap V = \varphi^{-1}(U)$ such that $\varphi(x) = \varphi(x')$, so $x \in V$. Thus we have shown

$$X \cap q^{-1}q(V \sqcup U) \subset V.$$

Case 2: Now assume $y \in Y$ and $q(y) \in q(V \sqcup U)$. If $q(y) \in q(U)$ then $y \in U$ since q is injective on Y, and if $q(y) \in q(V)$ then there's an $x \in A \cap V = \varphi^{-1}(U)$ such that $\varphi(x) = y$, thus $y \in \varphi\varphi^{-1}(U) \subset U$. Thus we have

$$Y \cap q^{-1}q(V \sqcup U) \subset U. \tag{$**$}$$

Overall, the equations imply that

$$q^{-1}q(V \sqcup U) \subset X \cap q^{-1}q(V \sqcup U) \sqcup Y \cap q^{-1}q(V \sqcup U) \subset V \sqcup U,$$

so $q^{-1}q(V \sqcup U) = V \sqcup U$ and the set is saturated. To finish equation $(*)$, the $\subset$ direction is obvious, and if $q(y) \in i(Y)$ and $q(y) \in q(V \sqcup U)$, then $y \in Y \cap q^{-1}q(V \sqcup U) \subset U$ by equation $(**)$.

Proposition 2.5.1 *The pushout in* **Top** *of a topological (resp. open, resp. closed) embedding is a topological (resp. open, resp. closed) embedding.*

Proof We have already shown in all cases that it is an embedding. For the other cases, assume in the above that $A \subset X$ was open or closed; then

$$\begin{aligned} q^{-1}(i(Y)) &= \{z \in X \sqcup Y : q(z) \in q(\varnothing \sqcup Y)\} \\ &= \{x \in X : q(z) \in q(Y)\} \cup Y = A \sqcup Y. \end{aligned}$$

Since A is open/closed, $A \sqcup Y$ is open/closed in $X \sqcup Y$, so $i(Y)$ is open/closed in the quotient, and i is an open/closed embedding. □

Under good circumstances, the map $X \to X \cup_\varphi Y$ also behaves well: namely, it induces an embedding $X - A \hookrightarrow X \cup_\varphi Y$. Let us see when this is the case.

Lemma 2.5.2 *Let $A \subset X$ be a closed (resp. open) subspace of X. Then $q\colon X \to X/A$ restricts to a open (resp. closed) embedding $X - A \hookrightarrow X/A$.*

Proof If A is closed and $U \subset X - A$ is open, then it's open in X and saturated since $q^{-1}q(U) = U$ by disjointness. Thus $q|_{X-A}$ is an open map in this case. Similarly, if A is open and $Z \subset X - A$ is closed, then it's closed in X and saturated so $q(Z)$ is closed. □

Now, let us make the following purely categorical observation, given a pair (X, A) and an attaching map $\varphi\colon A \to Y$ like before. In the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & Y & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow & \ulcorner & \downarrow \\ X & \longrightarrow & X \cup_\varphi Y & \longrightarrow & (X \cup_\varphi Y)/Y \end{array}$$

the two inner squares are pushouts by definition, so the outer square is by the pasting lemma. But (by definition) the pushout of the outer span is X/A, so the map $X \to X \cup_\varphi Y$ sends A to $Y \subset X \cup_\varphi Y$, inducing an isomorphism out of X/A making the right square of the diagram

$$\begin{array}{ccccc} X - A & \hookrightarrow & X & \dashrightarrow & X/A \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \overline{\varphi}} & & \downarrow{\scriptstyle \cong} \\ X \cup_\varphi Y - Y & \hookrightarrow & X \cup_\varphi Y & \dashrightarrow & (X \cup_\varphi Y)/Y \end{array} \qquad (*)$$

commute. The left-most map is just the restriction of $\overline{\varphi}$, and commutes as long as $\overline{\varphi}$ sends the subspace $X - A$ to $X \cup_\varphi Y - Y$. Indeed, if $x \in X$ and $\overline{\varphi}(x) \in Y$, then $q(x) \in q(\varnothing \sqcup Y)$, so $x \in A$ and $q(x) = q(\varphi(x))$. In other words, if $x \notin A$ then $\overline{\varphi}(x) \notin Y$, which shows well-defined commutativity. Now:

Corollary 2.5.3 *Let $A \subset X$ be a closed (resp. open) subspace and $\varphi\colon A \to Y$ a map. Then the pushout $\overline{\varphi}\colon X \to X \cup_\varphi Y$ restricts to an open (resp. closed) embedding $X - A \hookrightarrow X \cup_\varphi Y$ with image $X \cup_\varphi Y - Y$.*

Proof Assume $A \subset X$ is closed (resp. open), and consider the left-most map j in the diagram $(*)$. We know that $Y \subset X \cup_\varphi Y$ is closed (open), so we just need to show j is a homeomorphism. To do this, Lemma 2.5.2 shows that the composite top and bottom rows of $(*)$ are open (closed) embeddings, and each one factors through the two subspaces below respectively:

$$X/A - * \qquad (X \cup_\varphi Y)/Y - *$$

Finally, since the right-most homeomorphism maps $* \mapsto *$, we get a modified diagram:

$$\begin{array}{ccccc} X - A & \xrightarrow{\cong} & X/A - * & \hookrightarrow & X/A \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ X \cup_\varphi Y - Y & \xrightarrow[\cong]{} & (X \cup_\varphi Y)/Y - * & \hookrightarrow & (X \cup_\varphi Y)/Y \end{array}$$

Commutativity is just because the map is the restricted form of the right-most isomorphism to isomorphic subspaces, and the left-hand square implies that j is a homeomorphism. □

We need one final bit of preparation before defining CW complexes: Let Y be a space and suppose we are given a filtration

$$Y_0 \subset Y_1 \subset Y_2 \subset \cdots \subset Y$$

of subspaces of Y such that $Y = \bigcup_i Y_i$. Following our terminology earlier, we will say that Y has the weak topology with respect to the $\{Y_i\}$ if a set $U \subset Y$ is open (or closed) iff $U \cap Y_i$ is open in Y_i for each i. The natural map $\varinjlim_i Y_i \to Y$ is a continuous bijection because of the explicit construction of the filtered colimit in **Set**. Based on our discussion of the weak/colimit/final topology, we know that the map is a homeomorphism if and only if Y has the weak topology with respect to this collection.

Definition 2.5.4 A *CW structure* on a space X is a filtration by subspaces

$$X^0 \subset X^1 \subset X^2 \subset \cdots \subset X$$

where $X = \bigcup_i X^i$ and for each $n \geq 1$ a (possibly empty) family of maps

$$\{\Phi_\alpha \colon (D^n, S^{n-1}) \to (X^n, X^{n-1}) : \alpha \in A_n\}$$

satisfying the following properties:

1. X^0 is a discrete subspace.
2. For each n, the commutative diagram

$$\begin{array}{ccc} \bigsqcup_{\alpha \in A_n} S^{n-1}_\alpha & \xrightarrow{(\varphi_\alpha)_\alpha} & X^{n-1} \\ \downarrow & & \downarrow \\ \bigsqcup_{\alpha \in A_n} D^n_\alpha & \xrightarrow{(\Phi_\alpha)_\alpha} & X^n \end{array}$$

   is a pushout square, where $\varphi_\alpha \coloneqq \Phi_\alpha|_{S^{n-1}} \colon S^{n-1} \to X^{n-1}$.
3. X has the weak topology with respect to the $\{X^i\}$, i.e. $X = \varinjlim_i X^i$.

A *CW complex* is a space which admits a CW structure. Often, we will use the term CW complex to refer to a space with a fixed CW structure.

Let us set some terminology; in what follows fix a CW complex X with a fixed CW structure. Since $S^{n-1} \subset D^n$ is closed, $\bigsqcup_\alpha S^{n-1}_\alpha \subset \bigsqcup_\alpha D^n_\alpha$ is a closed subspace, hence $X^{n-1} \subset X^n$ is closed for each n. Inductively, X^k is closed in X^n for all $n \geq k$, so by condition (3) the X^n are closed subspaces.

The space X^n is called the *n-skeleton* of X. By the earlier discussion, we know that the map $(\Phi_\alpha)_\alpha$ induces an open embedding

$$\bigsqcup_{\alpha \in A^n} D^n_\alpha - \bigsqcup_{\alpha \in A^n} S^{n-1}_\alpha = \bigsqcup_{\alpha \in A^n} B^n_\alpha \hookrightarrow X^n$$

with image $X^n - X^{n-1}$. Hence each Φ_α restricts to an open embedding $B^n_\alpha \to X^n$, and we write the open image $e^n_\alpha \coloneqq \Phi_\alpha(B^n_\alpha) \subset X^n$, called an n-cell of X. Φ_α is the *characteristic map* of e^n_α, and φ_α is the *attaching map* of the cell.

Therefore $X^n - X^{n-1} = \bigsqcup_\alpha e^n_\alpha$, and in the case of X^0 we identify the cells as discrete points $X^0 = \{e^0_\alpha : \alpha \in X^0\}$. In particular, we have

$$X = \bigcup_{n \geq 0} \bigcup_{\alpha \in A_n} e^n_\alpha,$$

so X is built up from its n-cells, each homeomorphic to an open ball. The integer n is the *dimension* of the cell e^n_α, so that the n-skeleton consists of cells of dimension less than or equal to n. X is called *finite-dimensional* if there exists n such that $X = X^n$, and *finite* if there are only finitely many cells in all of X.

Lemma 2.5.5 *A CW complex X has the weak topology with respect to its characteristic maps $\{\Phi_\alpha \colon D^n \to X : n \geq 0, \alpha \in A_n\}$.*

Proof Let $U \subset X$ be such that $\Phi_\alpha^{-1}(U)$ is open in D^n for all n, α. We show by induction on n that $U \cap X^n$ is open in X^n, where $n = 0$ is trivial by discreteness. Indeed, since we know that X^n is a pushout, it must have the colimit topology with respect to the maps

$$\bigsqcup_{\alpha \in A_n} D^n_\alpha \to X^n \qquad \text{and} \qquad X^{n-1} \hookrightarrow X^n.$$

Hence we need to show that $U \cap X^{n-1}$ and the preimage of U in $\bigsqcup_{\alpha \in A_n} D^n_\alpha$ are open, the first of which is done by induction. But $\bigsqcup_{\alpha \in A_n} D^n_\alpha$ has the weak topology with respect to each subspace D^n_α, and the preimage of U in these subspaces are open by assumption. □

Example 2.5.6 Points are closed in a CW complex X. Let $x \in X$, we show by induction that x is closed in X^n for all n, X^0 being trivial by discreteness. If $x \in X^n$, then either $x \in e^n_\alpha$ for one $\alpha \in A_n$ or $x \in X^{n-1}$. In the latter case x is closed in X^{n-1} hence in X^n, and in the former case x pulls back under Φ_α to a single point in $B^n \subset D^n$, which is closed.

Proposition 2.5.7 *CW complexes are normal spaces.*

Proof By the easy direction of Urysohn's Lemma, we need to show that two disjoint closed subsets $Z, Z' \subset X$ can be separated by a continuous function $X \to \mathbb{R}$, for X a CW complex. Starting with X^0, we can clearly make a map $X^0 \to \mathbb{R}$ which sends $Z \cap X^0$ to 0 and $Z' \cap X^0$ to 1, as X^0 is discrete. Assume inductively that maps $f_0, \ldots, f_{n-1}$ have been constructed with $f_i \colon X^i \to \mathbb{R}$ separating $Z \cap X^i$ and $Z' \cap X^i$. It suffices to construct f_n in this way because $X = \varinjlim_i X^i$.

We may construct $f_n\colon X^n \to \mathbb{R}$ as follows: Let $\Phi_\alpha\colon D^n \to X^n$ be a characteristic map, and consider the two closed disjoint subsets

$$Y \coloneqq \Phi_\alpha^{-1}(Z \cap X^n) \qquad Y' \coloneqq \Phi_\alpha^{-1}(Z' \cap X^n)$$

of D^n, and use the gluing lemma to construct a map $g\colon Y \cup Y' \cup S^{n-1} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 0 & x \in Y \\ 1 & x \in Y' \\ f_{n-1}(\varphi_\alpha(x)) & x \in S^{n-1} \end{cases}$$

where $\varphi_\alpha = \Phi_\alpha|_{S^{n-1}}$ is the attaching map. $Y \cap Y" = \varnothing$, and if $x \in Y \cap S^{n-1}$, then $\Phi_\alpha(x) \in Z \cap X^n \subset Z$, and $\Phi_\alpha(x) \in X^{n-1}$, but $f_{n-1} \equiv 0$ on $Z \cap X^{n-1}$. Likewise on $Y' \cap S^{n-1}$, so g is continuous and well-defined from the closed subset $Y \cup Y' \cup S^{n-1} \subset D^n$.

To finish, since D^n is normal there is an extension $\widetilde{g}$ of g to all of D^n_α by the Tietze Extension Theorem. Repeating for all α and applying the universal property of pushouts, the corresonding map $\bigsqcup D^n \to \mathbb{R}$ and $X^{n-1} \to \mathbb{R}$ induce a map $f_n\colon X^n \to \mathbb{R}$ extending f_{n-1}. Lastly, if $y \in Z \cap X^n - Z \cap X^{n-1}$, then $y = \Phi_\alpha(x)$ for some $x \in B^n_\alpha$ and α, and by definition $x \in Y$ so it maps to 0 under f_n, and likewise for Y'. □

Corollary 2.5.8 *CW complexes are Hausdorff spaces.*

Proof Example 2.5.6 and Prop. 2.5.7. □

This means that the characteristic map $\Phi\colon (D^n, S^{n-1}) \to (X^n, X^{n-1})$ is from a compact space to a Hausdorff space, hence closed. In particular, $\Phi_\alpha(D^n)$ is a closed set of X containing e^n_α, so $\overline{e^n_\alpha} \subset \Phi_\alpha(D^n)$, yet on the other hand any closed subset $Z \subset X$ containing e^n_α has $\Phi_\alpha^{-1}(Z) = D^n$, thus $\Phi(D^n) \subset Z$, which implies $\Phi(D^n) = \overline{e^n_\alpha}$. Note that this means a set $Z \subset X$ is closed iff $Z \cap \overline{e_\alpha}$ is closed in $\overline{e_\alpha}$ for all cells e_α.

Lemma 2.5.9 *A compact subspace of a CW complex meets only finitely many cells.*

Proof If $Z \subset X$ were compact and met infinitely many cells, there's an infinite sequence $(z_i)_i \in Z$ such that each z_i lies in a different cell; the same applies to any subset $S \subset \{z_i\}$. We claim such an S is closed in X.

Indeed, we claim by induction that $S \cap X^n \subset X^n$ is closed for all n, the $n = 0$ case being trivial. If $S \cap X^{n-1}$ is closed in X^{n-1}, then to show $S \cap X^n$ is too we just need that the preimage of $(X^n - X^{n-1}) \cap S$ in D^n_α for each n-cell is closed. But this is obvious, beacuse it is either empty or a single point.

This argument shows $\{z_i\} \subset Z \subset X$ is discrete and closed in Z, hence discrete compact, hence necessarily finite, a contradiction. □

In particular, X is *closure-finite* in the sense that the closure of each cell meets only finitely many other cells by compactness. If X has a fixed CW structure, a *subcomplex* $A \subset X$ is a closed subspace of X that is a union of cells of X. In this case, we can treat A as a CW complex in its own right:

For each cell e^n_β in A, we observe that $\overline{e^n_\beta} = \Phi_\beta(D^n)$ is contained in A, so the characteristic maps for the cells in A all factor through A.

Since $\overline{e^n_\alpha} = \Phi_\alpha(D^n)$, we use the notation ∂e^n_α to mean the boundary of e^n inside X^n, which is $\operatorname{im} \varphi_\alpha \subset X^{n-1}$. (CONTINUE THE ABOVE DISCUSSION ON SUBCOMPLEXES).

Now let us discuss how to construct subspaces of arbitrary CW complexes. We will use the following 'construction lemma'.

Lemma 2.5.10 *Let X be a CW complex equipped with a CW structure, and suppose that for each $n \geq 0$, and each n-cell $e^n_\alpha \subset X^n$, we're given a subset $Z_\alpha \subset D^n$ satisfying the following property: For all $\alpha \in A_n$,*

$$Z_\alpha \cap S^{n-1} = \bigcup_{k<n} \bigcup_{\beta \in A_k} \Phi_\alpha^{-1} \Phi_\beta(Z_\beta).$$

Then there is a unique subset $Y \subset X$ such that

$$\Phi_\alpha^{-1}(Y) = \Phi_\alpha^{-1}(Y \cap X^n) = Z_\alpha$$

for each $\alpha \in A_n$. Furthermore $Y = \bigcup_n \bigcup_\alpha \Phi_\alpha(Z_\alpha)$.

Proof For $\alpha \in A_0$, we can identify Z_α with either singletons in X^0 or $\varnothing$; set $Y^0 = \bigcup_{\alpha \in A_0} Z_\alpha$, and inductively define $Y^n = Y^{n-1} \cup \bigcup_{\alpha \in A_n} \Phi_\alpha(Z_\alpha)$. Then $Y \coloneqq \bigcup_n Y^n = \bigcup_{n,\alpha} \Phi_\alpha(Z_\alpha)$, and we can see that the hypothesis means

$$Z_\alpha \cap S^{n-1} = \Phi_\alpha^{-1}\Big(Y^{n-2} \cup \bigcup_{\beta \in A_{n-1}} \Phi_\beta(Z_\beta)\Big) = \Phi_\alpha^{-1}(Y^{n-1}).$$

for $\alpha \in A_n$. As $\Phi_\alpha(Z_\alpha) = \Phi_\alpha(Z_\alpha \cap S^{n-1}) \cup \Phi_\alpha(Z_\alpha \cap B^n)$, and beacuse the assumption implies that $\Phi_\alpha(Z_\alpha \cap S^{n-1}) \subset Y^{n-1}$, we can alternatively write

$$Y^n = Y^{n-1} \cup \bigcup_{\alpha \in A_n} \Phi_\alpha(Z_\alpha \cap B^n).$$

Now, fixing some $\beta \in A_n$, we claim that

$$\Phi_\beta^{-1}\Big(\bigcup_\alpha \Phi_\alpha(Z_\alpha \cap B^n)\Big) = \Phi_\beta^{-1}\Phi_\beta(Z_\beta \cap B^n) = Z_\beta \cap B^n.$$

The first equality is just because, when $\alpha \neq \beta$, $e^n_\alpha \cap \overline{e^n_\beta} = \varnothing$, as $\overline{e^n_\beta} \subset X^{n-1} \cup e^n_\beta$ but $e^n_\alpha \subset X^n - X^{n-1}$ is disjoint from e^n_β. The second equality is beacuse if $x \in Z_\beta \cap D^n$ and $\Phi_\beta(x)$ is contained in $\Phi_\beta(Z_\alpha \cap B^n) \subset e^n_\beta$, then $x \notin S^{n-1}$ (since otherwise it would be in X^{n-1}), but $\Phi_\beta|_{B^n}$ is injective.

Putting everything together, this shows that

$$\Phi_\beta^{-1}(Y^n) = (Z_\beta \cap S^{n-1}) \cup (Z_\beta \cap B^n) = Z_\beta.$$

It remains to show that $Y \cap X^n = Y^n$. By induction, $Y^n \subset X^n$, so one direction is clear. For the other, we can show that $Y^k \cap X^n$ is Y^k for $k \leq n$ and Y^n for $k \geq n$. The former is obvious because $Y^k \subset Y^n \subset X^n$. We prove the latter by induction on $k \geq n$, the $k = n$ case having been done. Indeed, if $k > n$ and $Y^k \cap X^n = Y^n$, then

$$Y^{k+1} \cap X^n = (Y^k \cap X^n) \cup \bigcup_\alpha \Phi_\alpha(Z_\alpha \cap B^{k+1}) \cap X^n = Y^n,$$

because the first term is Y^n and $\Phi_\alpha(Z_\alpha \cap B^{k+1}) \subset e_\alpha^{k+1} \subset X^{k+1} - X^k$, yet $Y^n \subset Y^k \subset X^k$. This proves all the claims. □

This construction lemma is useful because we showed that a CW complex has the weak topology with respect to the characteristic maps, so if we choose all the Z_α to be open or closed, the resulting set is open or closed. Let us use this to construct canonical open nbhds inside a CW complex.

For setup, given a subset $A \subset D^n$ and an $\epsilon > 0$, write $U_\epsilon(A)$ for the *ϵ-nbhd in* B^n, defined by

$$U_\epsilon(A) \coloneqq \{x \in B^n : \exists y \in A\big(\, |x - y| < \epsilon\big)\}.$$

Of course, this is open in $B^n \subset D^n$. Next, consider the following continuous map:

$$\rho\colon I \times S^{n-1} \to D^n \qquad (t, x) \mapsto tx.$$

It is obviously surjective, and by compactness it's a closed map, hence is a quotient map. If $\rho(t, x) = \rho(t', x')$, then clearly $x = x'$ and $t = t'$, or else $t = t' = 0$. In other words, ρ is isomorphic to the quotient map $I \times S^{n-1} \to I \times S^{n-1}/(0 \times S^{n-1})$ by a closed subspace, and thus by Lemma 2.5.2 the restriction

$$\rho|_{I \times S^{n-1} - 0 \times S^{n-1}} : I \times S^{n-1} - 0 \times S^{n-1} \hookrightarrow D^n$$

is an open embedding onto $D^n - 0$. Hence the subspace $1 \times S^{n-1}$ maps homeomorphically onto ∂D^n.

Let ε be a function assigning a number $0 < \varepsilon_\alpha < 1$ to each cell e_α of a CW complex X. Let $A \subset X$ be an arbitrary subset. We will construct an open subset $N_\varepsilon(A) \subset X$ containing A as follows:

1. Set $N^0 = N_\varepsilon(A)^0 = A \cap X^0$, and define $N^k \subset X^k$ inductively below.
2. If N^{n-1} has been defined, define N^n via the construction lemma so that for each n-cell e_α^n, we have

$$Z_\alpha \coloneqq \Phi_\alpha^{-1}(N^n) = U_{\varepsilon_\alpha}(\Phi_\alpha^{-1}(A)) \cup \rho\big((1 - \varepsilon_\alpha, 1] \times \Phi_\alpha^{-1}(N^{n-1})\big)$$

Let us more closely unpack this: to be precise, we are defining N^n by using the construction lemma applied to the subcomplex X^n and remembering the

desired sets given to each cell. Of course, we must check the hypothesis of the lemma, done inductively. If N^{n-1} has been constructed, then the hypothesis is assumed checked for the $k < n$: we therefore have to show that

$$Z_\alpha \cap S^{n-1} = \Phi_\alpha^{-1}(N^{n-1})$$

Since U_ε is a subset of B^n, this is obvious by definition of ρ at $t = 1$, giving a well-defined open subset N^n. Finally, set $N_\varepsilon(A) = \bigcup_k N^k$.

2.5.1 Examples of CW complexes

Most familiar spaces are CW complexes. Note that, by the definition, to specify a CW structure on a space we need only check the first two axioms if the structure is finite-dimensional: The first two axioms imply inductively that $X^k \subset X^n = X$ is closed, and so the colimit is that of a closed cover.

The most basic example is the n-sphere S^n. Its CW structure is given with a single zero cell, usually $e^0 = 1 \in S^n$, and a single n-cell, $e^n = S^n - 1$; the defining pushout is

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{\Phi} & S^n \end{array}$$

where Φ is the quotient map $D^n \to D^n/S^{n-1} = S^n$. The next example is D^n; we make it by taking the structure for S^{n-1}, with one 0-cell and one $(n-1)$-cell, and attaching an n-cell via attaching map $S^{n-1} \to S^{n-1} = X^{n-1}$ and characteristic map $\mathrm{Id}\colon D^n \to D^n$. This is a pushout along the identity, and it makes $(D^n, \partial D^n)$ into a CW pair. These are both finite = compact complexes.

Of course, anything homeomorphc to these like $(I^n, \partial I^n)$ is also a CW pair. Spaces like $(0,1)^n = B^n = \mathbb{R}^n$ are also CW complexes, but since they're not compact their cell structures need infinitely many cells. Since $\mathbb{R}$ is locally compact, and $\mathbb{R}^n$ is an iterated product of $\mathbb{R}$, it suffices to see how $\mathbb{R}$ is a complex. Set $X^0 = \mathbb{Z} \subset \mathbb{R}$, and the one-cells will be $\{e_k^1 = (k-1, k+1) : k \in \mathbb{Z}\}$, with characteristic maps $\Phi_k\colon [-1,1] \to \mathbb{R}$ given by $\Phi_k(t) = t + k$, attaching e_k to the 0-cells $k-1$ and $k+1$.

The universal property is easily checked by the gluing lemma: given a map $\mathbb{Z} \to X$ (i.e. a sequence) and a sequence of maps $[-1,1] \to X$ agreeing on $\{-1,1\}$, we get a well-defined set function $\mathbb{R}^{\to}X$ collating the maps which agree on the overlap. Since the collection is locally finite we get continuity.