Notes: Math 252

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1 Lecture 2/26

Let k be a field and A a finite k-algebra. Then we have filtrations

$$V \supset \operatorname{Rad} V \supset \operatorname{Rad}^2 V \supset \cdots \supset \operatorname{Rad}^l V = 0$$

and similarly with the socle. The radical (socle) filtration is the shortest descending (ascending) filtration such that all quotients are semisimple. The length ℓ is the same.

$M \twoheadrightarrow N$, $M \hookrightarrow N$, radicals (resp. socles) surject (resp inject) onto. $\operatorname{Soc}^k(V^*) = (\operatorname{Rad}^k(V))^\perp = \langle \varphi \in V^* \mid \varphi(\operatorname{Rad}^k V) \rangle$. If $\operatorname{Soc} V$ or $V/\operatorname{Rad} V$ is simple, then V is indecomposable.

Example 1.0.1 $\operatorname{char} k = p > 0$, and $|G| = p^n$, $\operatorname{Rad} kG = IG$ augmentation ideal, then $G/IG = k$, kG is indecomposable.

Remark 1.0.2 If A, B are finite dimensional k-algebras, $A \otimes_k B$ is a new finite dimensional k-algebra. Then $kG \otimes_k kH = k(G \times H)$. Then the radical of the tensor product is:

$$\operatorname{Rad}(A \otimes_k B) = \operatorname{Rad}(A) \otimes_k B + A \otimes_k \operatorname{Rad}(B).$$

Taking the quotient, we see

$$(A \otimes_k B)/(\operatorname{Rad}(A) \otimes_k B + A \otimes_k \operatorname{Rad}(B)) = (A/\operatorname{Rad} A) \otimes_k (B/\operatorname{Rad} B).$$

$|G| = p^n$, and $\operatorname{char} k = p$, we want to describe the radical filtration of kG.

Final project topic: Quillen's proof of Jenning's theorem via Lie theory.

1.1 Lecture 3/3

If P is a projective cyclic A-module, then $P \cong Ae$ where $e \in A$ is an idempotent. If P is indecomposable, then e is *primitive* in the decomposition.

Definition 1.1.1 An idempotent $e \in A$ is primitive if whenever $e = e_1 + e_2$ is a sum of orthogonal idempotents, then e_1 or e_2 is 0.

Suppose $e_1, \dots, e_n \in A$ are primitive mutually orthogonal idempotents, such that $\sum_i e_i = 1$. Then we may write

$$_AA = Ae_1 \oplus \cdots \oplus Ae_n.$$

If A is f.d. k-algebra, then we always have such a decomposition by induction.

Example 1.1.2 D a division ring, $M_n(D)$ matrix algebraic is s/s, so a set of orthogonal idempotents are $\{E_{ii} : i = 1, \dots, n\}$. Then AE_{ii} just is just a column vector.

Example 1.1.3 k a field, $\operatorname{char} k = 3$, then kS_3 has two simple modules: trivial and sign. Then $e_1 = (1 + (12))/2$, and $1 - e_1 = e_2 = (1 - (12))/2$, $\{e_1, e_2\}$ are nontrivial orthogonal idempotents. Then

$$kS_3 = S_3e_1 \oplus S_3e_2.$$

If $\operatorname{char} k = 2$, then $kS_3 = \dfrac{\text{triv}}{\text{triv}} \oplus V \oplus V$. Then

$$e_1 = 1+(123)+(123)^2 \quad e_2 = 1+(123)\omega+(123)^2\omega^2 \quad e_3 = 1+(123)\omega^2+(123)^2\omega$$

gives an idempotent decomposition.

Theorem 1.1.4 (lifting of idempotents) *Let A be a f.d. k-algebra, $I \subset A$ nilpotent ideal. If $e \in A/I$ is an idempotent, then there exists $f \in A$ idempotent such that $e = f + I$. If e is primitive then f is too.*

Proof By induction on n such that $I^n = 0$, may assume $n = 2$ by replacing $I \subset A$ by $I \subset A/I^{n-1}$. $a + I = e$, then $a^2 - a \in I$ so take $f = 3a^2 - 2a^3 \in a + I$, and

$$f^2 - f = (3a^2 - 2a^3)(3a^2 - 2a^3 - 1) = -(3 - 2a)(1 + 2a)(a^2 - a) \in I^2 = 0.$$

Suppose for contradiction that e is primitive but f isn't; so let $f = f_1 + f_2$ so under the projection $e = \bar{f}_1 + \bar{f}_2$, so f_1 or f_2 is in I. This is impossible because $1 = f_i^2 \in I^2 = 0$. □

Corollary 1.1.5 *If in A/I we have a primitive decomposition $1 = e_1 + \cdots e_n$, these lift to $1 = f_1 + \cdots + f_n$ where $f_j + I = e_j$.*

An important special case: $I = \operatorname{Rad} A$, $A/\operatorname{Rad} A = \bigoplus_i S_i^{n_i}$ and the number of primitive idempotents in A is the same as in $A/\operatorname{Rad} A$.

Definition 1.1.6 Let A be an arbitrary ring. A map $f\colon V \to W$ of A-modules is *essentially surjective* if:

1. f is surjective.
2. Whenever $U \subsetneq V$ is a proper submodule, then $f(U) \neq W$ (i.e. f is not surjective on any proper submodule).

Proposition 1.1.7 *Let V be a Noetherian A-module. Then $f\colon V \to V/\operatorname{Rad} V$ is essentially surjective.*

Proof Any proper $U \subsetneq V$ is contained in a maximal submodule $M \subset V$, so we may consider only maximal submodules M. If $f(M) = V$, then $M + \operatorname{Rad} V = V$, which is a contradiction because $\operatorname{Rad} V \subset M$ for any maximal submodule so $M = V$, and so M is not proper. □

Exercise 1.1.8

1. *$V \xrightarrow{f} W \xrightarrow{g} U$. Then if any two of $\{f, g, gf\}$ are essentially surjective then all three are essentially surjective.*
2. *Given $\{f_i\colon V_i \to W_i\}$, the induced map $f\colon \bigoplus_i V_i \to \bigoplus_i W_i$ is essentially surjective iff f_i is for all i.*

Let $f\colon V \to W$; for any $M \subset W$ maximal submodule, $f^{-1}(M)$ is either V or a maximal submodule of V (clear). Therefore

$$f^{-1}(\operatorname{Rad} W) \supset \operatorname{Rad} V \Rightarrow f(\operatorname{Rad} V) \subset \operatorname{Rad} W.$$

In particular, f induces a map

$$\bar{f}\colon V/\operatorname{Rad} V \to W/\operatorname{Rad} W.$$

Proposition 1.1.9 *Let V and W be Noetherian A-modules, and $f\colon V \to W$ a map. f is essentially surjective if and only if*

$$\bar{f}\colon V/\operatorname{Rad} V \xrightarrow{\sim} W/\operatorname{Rad} W$$

is an isomorphism.

Proof Considering

$$\begin{array}{ccc} V & \xrightarrow{\quad f \quad} & W \\ \downarrow & & \downarrow \\ V/\operatorname{Rad} V & \xrightarrow{\bar{f}} & W/\operatorname{Rad} W \end{array}$$

draw the diagonal map across. Reduce to the claim that $\operatorname{Rad} V = \operatorname{Rad} W = 0$, and $f\colon V \to W$ is essentially surjective, then f is an isomorphism. Considering $\ker f$, the radical being zero means there is a maximal submodule M which doesn't contain $\ker f$; if it is nonzero, then $M + \ker f = V \Rightarrow f(M) = W$, but then we have found a proper submodule on which restricting surjective. □

Definition 1.1.10 A projective module P over A equipped with an essentially surjective morphism $\varphi\colon P \to V$ is a called a *projective cover* of V.

Lemma 1.1.11 *Observe that if V admits a projective cover (P, φ), then it is unique up to isomorphism.*

Proof Use projectivity to extend to a map $P \to P'$,then do the reverse, then the kernel splits as a direct summand. □

Note If A is a f.d. k-algebra, then projective covers exist for all modules.

2 Terminological review of later chapters

The goal of this part of the document is to help catch up on some missed material in the later chapters of the textbook. I will begin with Chapter 5, on Mackey's theorem and Clifford's theorem. Unless otherwise stated, G is always a finite group, p is always a prime greater than 0.

2.1 Notation

Here I will keep a running list of notation that is used. R is generally a commutative ring, A is usually a (finitely generated) algebra over R. Most often $R = k$ is a field and A is a finite dimensional k-algebra, and more often $A = kG$, the group algebra over the field k.

1. If U is an RG-module, and $H \subset G$ is a subgroup, then U is naturally an RH module by *restriction* under the map $RH \hookrightarrow RG$. This is written $U\downarrow_H^G$ and called the restriction to H.
2. If U is an RH-module and $H \leq G$, then $U \otimes_R RG$, the base change under the map $RH \hookrightarrow RG$ is an RG-module, written $U\uparrow_H^G$. In the context of group algebras, there are a bunch of formulas relating these two concepts that can be found in Chapter 4 (e.g. 'Frobenius reciprocity').
3. $[G/H]$ means a set $\{g_1, \ldots, g_k\} \subset G$ of left coset representatives, so that $k = [G : H]$ is the index of H in G. $\mathrm{cc}(G)$ means the set of conjugacy classes of G, or (using context) a set of conjugacy class representatives for G.

The list above will continually be amended. Also, most of what we need about induction and restriction is the following.

1. First, If $H \trianglelefteq G$, then any representation $\rho\colon G/H \to \mathrm{GL}(V)$ *lifts* to one of G, simply by composition. So, $g \cdot v \coloneqq [g]v$ for $g \in G$. If V is simple over G/H, then it is simple over G by surjectivity. Since abelian groups are usually easier to compute with, an initial step in finding representations will be to consider those of $G/[G, G]$, the maximal abelian quotient of G.
2. If $H \leq G$, we can also compose $G \to \mathrm{GL}(V)$ with the inclusion to get a map $H \to \mathrm{GL}(V)$, and this is the same as $V\downarrow_H^G$.
3. In general,
$$V\uparrow_H^G = \bigoplus_{i=1}^{[G:H]} g_i \otimes V \qquad \text{over } R,$$

where $g_i \otimes V = \{g_i \otimes v : v \in V\}$. For $x \in G$, x acts on $g_i \otimes V$ by finding the unique j such that $xg_i = g_j h$ for some $h \in H$, and then $x(g_i \otimes v) = (xg_i) \otimes v = (g_j h) \otimes v = g_j \otimes (hv)$. In the free case (e.g. if $R = k$ is a field), the rank of $V\uparrow_H^G$ is the rank of V times $[G : H]$.

2.2 Chapter 5

Let $g \in G$ and $H \leq G$, then for any $h \in H$, ${}^g h$ means ghg^{-1}, so e.g. if $H \trianglelefteq G$ then ${}^g h \in H$. In general, we write ${}^g H$ for the conjugate of H. If V is a representation of H, then ${}^g V$ is the representation of ${}^g H$ by setting ${}^g V = V$ as vector spaces, and defining the action to be $ghg^{-1} \cdot v = hv$.

Theorem 2.2.1 (5.2.1, Mackey decomposition formula) *Let $H, K \leq G$ and V a representation of K over R. Then*

$$(V\uparrow_K^G)\downarrow_H^G \cong \bigoplus_{g \in H/GK} ({}^g(V\downarrow_{H^g \cap K}^K))\uparrow_{H \cap {}^g K}^H$$

as RH-modules.

We next move on to Clifford's theorem; the point is to relate representations and their restrictions to a normal subgroup.

Theorem 2.2.2 (5.3.1) *Let k be a field, and U a simple kG-module, and $N \trianglelefteq G$. Then $U\downarrow_N^G$ is semisimple over kN.*

Theorem 2.2.3 (5.3.2; Clifford's theorem) *Let k be a field, $N \trianglelefteq G$, and U a simple kG-module. By Thm. 2.2.2, write*

$$U\downarrow_N^G = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$$

for the decomposition of U into non-isomorphic simple modules, where each summand is called a homogeneous component. Then:

1. *G permutes the homogeneous components transitively (i.e. for all $g \in G$ and i, there's a unique j such that $g(S_i^{a_i}) = S_j^{a_j}$).*
2. *The muliplicities $a_1 = \cdots = a_r$ are all equal, and $\dim(S_1) = \cdots \dim(S_r)$.*
3. *Under of G on the homogeneous components from (1), let H be the stabilizer $Stab_G(S_1^{a_1})$. Then $U \cong S_1^{a_1}\uparrow_H^G$ over kG.*

Lemma 2.2.4 *Let $|G|$ be invertible in k, or if not assume k is algebraically closed. Then if G is abelian, all simple kG-modules are 1-dimensional.*

Corollary 2.2.5 *Let k be algebraically closed and G be a p-group. Then every simple module for G has the form $U\uparrow_H^G$ for U a 1-dimensional for some subgroup $H \leq G$.*

2.3 Chapter 6

Proposition 2.3.1 (6.1.1) *Let* $\operatorname{char} k = p > 0$ *and let* $G = \langle g \rangle$ *be cyclic of order* p^n*. Then* $kG \cong k[x]/(x^{p^n})$*. In particular,* kG *is a local ring.*

Proof The isomorphism is given by the unique map $kG \to k[x]/(x^{p^n})$ sending $g \mapsto x + 1$. □

Definition 2.3.2 An A-module U is *indecomposable* if whenever $U = V \oplus W$, then V or W is 0.

If A is semisimple, then U is indecomposable iff it is simple. In general, there are non-simple indecomposable modules which we now aim to study; the point is that when kG isn't semisimple, then to get all representations we cannot just find the simple representations.

Theorem 2.3.3 *Let* $\operatorname{char} k = p > 0$.

1. *Every f.g.* $k[x]/(x^{p^n})$*-module is a direct sum of cyclic modules* $U_r = k[x]/(x^r)$ *for* $1 \leq r \leq p^n$.
2. *There is only one simple module,* $U_1 = k[x]/(x)$*, up to isomorphism, corresponding to the unique maximal ideal.*
3. *Each module* U_r *has a unique composition series, and hence is indecomposable.*

In particular, if G *is cyclic of order* p^n*, then* kG *has exactly* p^n *indecomposable modules, one of each dimension* i *with* $1 \leq i \leq p^n$*, with each having a unique composition series.*

In the above theorem, a *composition series* for an A-module V means a sequence

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

of submodules such that V_{i+1}/V_i is a simple module for all i. If a module has a composition series, then its length, the *(composition) length of the module*, is unique by the Jordan-Holder theorem. The module is *uniserial* if it has a unique composition series. This is equivalent to all of its submodules being linearly ordered, and the unique series just being the list of all submodules. Note that if we could write $V = U \oplus W$ with both summands nonzero, then neither of V and W are contained in the other, so V cannot be uniserial. In particular, uniserial modules are indecomposable as claimed in the theorem.

Remark 2.3.4 Consider the r-dimensional indecomposable module U_r as in the theorem. It has basis $1, x, x^2, \ldots, x^{r-1}$, so x acts on it by a matrix all zero except with ones just below the diagonal. From the proof of 6.1.1 we know that the generator g of C_{p^n} corresponds to $x + 1$, so it has matrix with 1s on the diagonal and just below the diagonal. These are just Jordan blocks, whose sizes determine the module U_r.

Observe also that, in the theorem, the only simple module was $k[x]/(x) = k$, the trivial representation. This holds for all p-groups, not just cyclic ones.

Theorem 2.3.5 *Let* $\operatorname{char} k = p > 0$ *and* G *be a* p*-group. Then the only simple* kG*-module is* k*.*

Proof We proceed by induction on $|G|$ so assume $G \neq 1$, and let N be an index p-subgroup which is therefore normal. Let S be simple over kG, then by Thm. 2.2.2, $S\downarrow_N^G$ is semisimple. By induction, $S\downarrow_N^G = k^n$ so kN acts trivially on S, and thus S is lifted from $G/N \cong C_p$, so we're done. □

If $p \mid |G|$, there's a unique largest normal p-subgroup. Indeed, if H and K are two maximal p-subgroups, then either $HK = H$, so $K = H$, or HK is a larger p-subgroup, which would be a contradiction. This subgroup is called $O_p(G) \trianglelefteq G$. If G is a group, then the common kernel on the simple modules means the subgroup

$$H = \{g \in G : gs = s \qquad \text{for all } s \in \mathrm{S}, \text{ all simple } S\}.$$

That is, it is the subgroup of elements which act trivially on the simple modules.

Corollary 2.3.6 *Let* $\operatorname{char} k = p$*. Then the common kernel on the simple modules is exactly* $O_p(G)$*, so the simple* kG*-modules are exactly those of* $k(G/O_p(G))$ *lifted from the quotient map.*

The conclusion comes from the fact that whenever a normal subgroup $N \trianglelefteq G$ acts trivially on a kG-module, then it is induced from G/N. This is most often useful when G has a normal Sylow p-subgroup, i.e. when $G = H \rtimes K$ with $p \nmid |K|$ and H is p-group. In this case obviously $O_p(G) = H$, so all simple modules are induced from K. This trivializes the problem because kK is semisimple as $|K|$ is invertible in k.

Example 2.3.7 S_3 is the semidirect product $C_3 \rtimes C_2$, so over any field of characteristic 3 there are only two simple modules, coming from the two 1-dimensional representations of C_2. They are the trivial rep and the sign rep.

Let us recall now some stuff from earlier.

Definition 2.3.8 Let A be a ring and U an A-module.

1. The *socle* of U, written $\operatorname{Soc}(U)$, is the sum of all simple submodules of U.
2. The *radical* of U, written $\operatorname{Rad}(U)$, the intersection of all simple submodules U.

By maximal submodule we generally mean a *proper* submodule contained in no other proper submodules. If the intersection is empty then the radical is U itself, but in our case (Noetherian) it is always $\operatorname{Rad}(U) \subsetneq U$.

Let $M \subset U$ be a submodule, and recall that U/M is simple if and only if M is maximal. If $M_1, \dots, M_n$ are maximal submodules, then there is some $I \subset \{1, \dots, n\}$ such that

$$U/(M_1 \cap \cdots \cap M_n) \cong \bigoplus_{i \in I} U/M_i.$$

So this is semisimple. If U has the DCC (is Artinian), then $U/\operatorname{Rad}(U)$ is a semisimple module, and $\operatorname{Rad} U$ is the unique smallest submodule of U with semisimple quotient.

Theorem 2.3.9 *Let A be a ring.*

1. $\operatorname{Rad} A = \{a \in A : a \cdot S = 0 \qquad \text{for all simple } S\}$.
2. $\operatorname{Rad} A$ *is a 2-sided ideal of A.*

Suppose further that A is a finite dimensional k-algebra.

3. $\operatorname{Rad} A$ *is the smallest left ideal of A such that $A/\operatorname{Rad} A$ is a semisimple A-module.*
4. *A is semisimple if and only if* $\operatorname{Rad} A = 0$.
5. $\operatorname{Rad} A$ *is nilpotent, and is the largest nilpotent ideal of A (i.e. it contains all nilpotent ideals).*
6. $\operatorname{Rad} A$ *is the unique nilpotent ideal with semisimple quotient.*

We will use this structure theorem all the time, especially the nilpotent one; it is especially useful because nilpotence carries over through ring maps, tensor products, and other things.

Definition 2.3.10 Let R be a commutative ring.

1. The ring homomorphism $\epsilon\colon RG \to G$ sending $g \mapsto 1$ is called the augmentation map.
2. We set $IG \coloneqq \ker \epsilon$ and call it the *augmentation ideal*.

Proposition 2.3.11 *Let R be a commutative ring.*

1. *Writing R for the trivial RG-module, $IG = \{x \in RG : x \cdot R = 0\}$.*
2. *IG is free over R with basis $\{g - 1 : 1 \neq g \in G\}$, so has codimension 1 in RG.*
3. *If $R = k$ and* $\operatorname{char} k = p > 0$, *and G is a p-group, then $IG = \operatorname{Rad} kG$. Hence it is nilpotent in this case.*

Proposition 2.3.12 *Let A be a finite dimensional k-algebra and U a finite dimensional A-module.*

1. *Then* $\operatorname{Rad} U$ *is the smallest submodule of* U *with semisimple quotient, and is equal to* $\operatorname{Rad} A \cdot U$.
2. *Then* $\operatorname{Soc} U$ *is the largest semisimple submodule of* U*, and is equal to* $\{u \in U : \operatorname{Rad} A \cdot u = 0\}$.

Definition 2.3.13 Let A be a ring and U an A-module.

1. For $n \geq 1$, write $\operatorname{Rad}^n(U) \subset U$ for the radical of $\operatorname{Rad}^{n-1}(U)$.
2. For $n \geq 1$, write $\operatorname{Soc}^n(U)$ for the submodule containing $\operatorname{Soc}^{n-1}(U)$ corresponding to the submodule $\operatorname{Soc}(U/\operatorname{Soc}^{n-1}(U)) \subset U/\operatorname{Soc}^{n-1}(U)$.

Then from the proposition, $\operatorname{Rad}^n(U)$ is $(\operatorname{Rad} A)^n \cdot U$, and $\operatorname{Soc}^n U$ is just the set of elements annihilated by $\operatorname{Rad}^n U$. Hence, we get the *radical series of* U, which increases to U as n decreases:

$$\cdots \subset \operatorname{Rad}^2(U) \subset \operatorname{Rad}(U) \subset U,$$

and the *socle series of* U, which increases from 0 as n increases:

$$0 \subset \operatorname{Soc}(U) \subset \operatorname{Soc}^2(U) \subset$$

The quotients $\operatorname{Rad}^{n-1}(U)/\operatorname{Rad}^n(U)$ are the *radical layers*, and the quotients $\operatorname{Soc}^n(U)/\operatorname{Soc}^{n-1}(U)$ are the *socle layers* of U. The *length* of each respective series is the first m for which $\operatorname{Rad}^m(U) = 0$, or $\operatorname{Soc}^m(U) = U$, and these are called the Loewy lengths. We will show momentarily that the two are equal.

Proposition 2.3.14 *Let* A *be an f.d.* k*-algebra and* U, V *f.d.* A*-modules. Then*

$$\operatorname{Rad}^n(V \oplus W) = \operatorname{Rad}^n(V) \oplus \operatorname{Rad}^n(W) \qquad \operatorname{Soc}^n(V \oplus W) = \operatorname{Soc}^n(V) \oplus \operatorname{Soc}^n(W)$$

Corollary 2.3.15 *Let* $\operatorname{char} k = p > 0$ *and* G *a* p*-group. Then the regular representation* kG *is indecomposable.*

Proof This is by the above proposition and the fact that $IG = \operatorname{Rad}(kG)$ has to have codimension 1 in kG. □

Proposition 2.3.16 *Let* A *be an f.d.* k*-algebra and* U *an* A*-module. Then:*

1. *The radical (socle) series is the fastest descending (ascending) series of* U *with semisimple quotients.*
2. *The two series terminate, and have the same length, called the Loewy length.*

Let us now briefly discuss but not prove or use *Jennings' theorem*. Let k have characteristic p and G be a p-group. Recall that a *central* series of G is a sequence

$$\{1\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_{n-1} \trianglelefteq G_n = G$$

of subgroups, each normal in the next, such that $[G, G_{i+1}] \leq G_i$ for all i, which implies for instance that the series has abelian quotients (and we assume $G_i \trianglelefteq G$).

The setup of the Jennings theorem is *Jennings series*, which is the fastest descending central series

$$\kappa_1 \supset \kappa_2 \cdots \kappa_n = \{1\}$$

such that $\kappa_1 = G$ and for each $g \in \kappa_i$, we have $g^p \in \kappa_{ip}$. This implies that

$$[\kappa_r, \kappa_s] \subset \kappa_{r+s},$$

and κ_r/κ_{2r} is an elementary abelian p-group (C_p^n for some n.) In fact, for each $i \geq 1$ let $d_i = \dim_{\mathbb{F}_p}(\kappa_i/\kappa_{i+1})$ of the corresponding elementary abelian p-group.

Theorem 2.3.17 (6.4.1; Jennings) *Let G be a p-group and* $\operatorname{char} k = p > 0$.

1. *The Jennings groups are $\kappa_r(G) = \{g \in G : g - 1 \in (IG)^r\}$.*
2. *The dimension of the rth radical layer of kG is the coefficient of t^r the expression*
$$\sum_{r \geq 0} \frac{\dim(IG)^r}{\dim(IG)^{r+1}} t^r = \prod_{i \geq 1} \left(\frac{1 - t^{ip}}{1 - t^i} \right)^{d_i}$$
known as the Poincaré polynomial.
3. *The radical and socle series of kG coincide.*

2.4 Chapter 7

First we need projective and injective modules as in algebra. Recall that $f\colon M \to N$ is a split epi if there is a section $g\colon N \to M$ such that $fg = 1_N$, and likewise if we are given a g like that then it is a split mono if there is such an f so that $fg = 1_N$. Recall that if

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$$

is an exact sequence, then it *splits* if β is a split epi, or α is a split mono, or the diagram induces an isomorphism $L \oplus N \cong M$. Two more equivalent conditions for the the sequence being split exact is that $\operatorname{Hom}_A(U, -)$ preserves it covariantly, or $\operatorname{Hom}_A(-, U)$ preserves it contravariantly.

Let P be a projective A-module. Recall that this means either that whenever we have a surjection $M \to N \to 0$, then any map $P \to N$ extends to M. Alternatively, it means that P is a direct summand of a free module, and if P is finite projective then it is the sumand of a finite free module. Alternatively, every exact sequence

$$0 \to M \to N \to P \to 0$$

splits, and finally, that $\operatorname{Hom}_A(P, -)$ is a right exact functor.

Recall that *injective modules* are those which split every injection: I is injective if whenever $0 \to M \to N$ is injective, any map $M \to I$ extends to a map $V \to I$. Equivalently, every monomorphism $I \to V$ is split, and $\mathrm{Hom}_A(-, I)$ is exact.

Recall that $e \in A$ is *idempotent* if $e^2 = e \in A$. An idempotent e is *primitive* if whenever $e = f + g$ with $f, g \in A$ idempotents, either $f = 0$ or $g = 0$. Note that if $e^2 = e$, then $e(1-e) = 0$ and as A-modules $Ae \oplus A(1-e)$, so Ae and $A(1-e)$ is always projective (though they might just be A and 0 themselves). Idempotents e, f are *orthogonal* if $ef = 0$. In particular e and $1-e$ are always orthogonal idempotents.

Proposition 2.4.1 *Decompositions of the regular rep A as a direct sum of submodules*

$$_AA = A_1 \oplus \cdots \oplus A_r$$

are in bijection with the the expressions $1 = e_1 + \cdots e_r$ as a sum of mutually orthogonal idempotents given by $A_i = Ae_i$. the summand A_i is indecomposable if and only if e_i is primitive.

Proof Here is most of the logic. If we have $1 = e_1 + \cdots e_r$, then obviously $Ae_1 + \cdots + Ae_r$ is A. In general, if $x \in Ae$ for e idempotent, then $x = ae$, so $xe = ae^2 = ae = x$. So, if $x \in Ae_i \cap \sum_{j \neq i} Ae_j$, then

$$xe_i = x = \sum_{j \neq i} a_j e_j,$$

so $xe_i = 0 = x$ by orthogonality. □

Note that the idempotents won't, nor will the subspaces Ae_i, in general be unique. Now, we move on to discussing the projective covers of modules.

Definition 2.4.2

1. A module map $f \colon V \to W$ is an *essential epimorphism* if it is surjective and whenever $U \subset V$ such that $f(U) = W$, $U = V$.
2. A map $P \to V$ with P projective is a *projective cover*.

Being an essential epimorphism is purely categorical: f is an essential epimorphism iff whenever g is another map such that fg is epi, then g is epi. If $U = W \oplus X$ and $f \colon U \to V$ is an essential epimorphism, then $f(W) = 0$ means that $f(X) = V$, so $W = 0$. In particular, f can never kill a nontrivial summand of V. We think of an essential epi as a minimal surjection.

Theorem 2.4.3 *If U is a Noetherian A-module (or A is an f.d. algebra over k), then $U \to U/\operatorname{Rad}(U)$ is an essential epimorphism. Equivalently, any submodule V such that $V + \operatorname{Rad}(U) = U$ is necessarily equal to U.*

Proof Our assumption means that any proper submodule is contained in a maximal one, so suppose $M \subset U$ is maximal and $M + \operatorname{Rad}(U) = U$. But then $\operatorname{Rad}(U) \subset M$, so we get $M = U$ a contradiction. This implies the first part, because if any submodule $V \subset U$ surjects onto $U/\operatorname{Rad}(U)$, then $V \equiv U$ modulo $\operatorname{Rad}(U)$, so $V + \operatorname{Rad}(U) = U$. □

Proposition 2.4.4 *Let A be any ring, but for (2) and (3) we assume either that the modules are Noetherian or that A is finite dimensional algebra over a field.*

1. *If f and g are composable, then if any two of f, g, and gf are essential epimorphisms, then the third is.*
2. *f is an essential epi iff the map $U/\operatorname{Rad}(U) \to V/\operatorname{Rad}(V)$ is an isomorphism.*
3. *$f_i \colon U_i \to V_i$ for $i = 1, \ldots, n$ maps, then f_i are all essential epis if and only if $\bigoplus_i U_i \to \bigoplus_i V_i$ is an essential epimorphism.*

Proof Let's do (2). Note that indeed $f(\operatorname{Rad}(U)) \subset \operatorname{Rad}(V)$. Indeed, if f is an epimorphism then it is true because then $f^{-1}(-)$ preserves maximal submodules, but even if not it is true in the f.d. algebra case because $f(\operatorname{Rad}(U))$ is nilpotent and $\operatorname{Rad}(V)$ contains all nilpotent submodules. The rest is just the above theorem. □

Now let us consider essential epimorphisms $P \to U$ where P is projective (i.e. projective covers).

Proposition 2.4.5

1. *Let $f \colon P \to U$ be a projective cover and $g \colon Q \to U$ any epimorphism with Q projective. Then $Q = Q_1 \oplus Q_2$, we can write $g = (g_1, 0)$, and there is an isomorphism $Q_1 \cong P$ identifying g_1 with f.*
2. *Projective covers, when they exist, are unique up to isomorphism commuting with the essential epi.*

Proof Part (1) says that P is a direct summand of Q in a way that g factors as the projection $Q \to P$ plus the projective cover f. The reason is that we can use projectivity to lift in both directions, and obtain a split epi. □

Corollary 2.4.6 *If projectives P, Q are Noetherian A-modules or A is an f.d. k-algebra, then $P \cong Q$ if and only if $P/\operatorname{Rad} P \cong Q/\operatorname{Rad} Q$.*

This is good because $P/\operatorname{Rad}(P)$ is a semisimple module over A, and moreover it is naturally an $A/\operatorname{Rad}(A)$-module, which is a semisimple ring. We know how to do semisimple stuff, so this means that we can identify isomorphic modules by passing to semisimple quotients. Let us analyze this more closely.

If $I \subset A$ is an ideal and $f \in A$ is nilpotent, then $[f] \in A/I$ is obviously nilpotent too. But if $[f]$ is nilpotent then all we know is that $f(1-f) \in I$, but it is not necessarily 0.

Theorem 2.4.7 *Let $I \subset A$ be nilpotent and $e \in A/I$ an idempotent of the quotient. Then e lifts to an idempotent of A, and if e was primitive then any of its lifts are also.*

Corollary 2.4.8 *Let $I \subset A$ be nilpotent and let $1 = e_1 + \cdots + e_n$ be a sum of orthogonal idempotents in the quotient A/I. Then this lifts to a sum $1 = f_1 + \cdots + f_n$ in A, where $e_i = [f_i]$ and the $\{f_i\}$ are orthogonal idempotents. If the e_i are primitive, then the f_i are too.*

Corollary 2.4.9 *Let $f \in A$ and $I \subset A$ a nilpotent ideal. Then f is primitive iff $[f] \in A/I$ is primitive.*

Theorem 2.4.10 *Let A be an f.d. k-algebra and S a simple A-module.*

1. *There is an indecomposable projective P_S such that $P_S/\operatorname{Rad}(P_S) \cong S$ and $P_S = Af$ for f a primitive idempotent in A. (So the quotient map $P_S \to S$ is a projective cover).*
2. *Furthermore, the idempotent f has the property that $fS \neq 0$, but for any simple module $T \ncong S$, we have $fT = 0$.*
3. *P_S is the projective cover of S, and is as such uniquely determined along with the map $P_S \to S$. Moreover, S is the unique simple quotient of P_S, and the kernel of $P_S \to S$ is always $\operatorname{Rad}(P_S)$.*
4. *It is also possible to find an idempotent $f_S \in A$ so that $f_S = S$ and $f_S T = 0$ for every (simple) module non-isomorphic to S.*

Proof (1) $1 \in A/\operatorname{Rad}(A)$ can always be written as a sum of primitive idempotents because it is semisimple, so there must be some primitive idempotent $e \in A/\operatorname{Rad}(A)$ such that $eS \neq 0$, which lifts to a primitive idempotent $f \in A$ such that $fS \neq 0$. Suppose $T \ncong S$. Since each element of the radical kills T and S, they are naturally $A/\operatorname{Rad}(A)$-modules so $T \ncong S$ in that ring too, and hence $eT = 0$ by semisimplicity. So $fT = 0$ (also for (4), we can take the idempotents to be central, showing that fS is actually a nonzero submodule of S).

Then, we already know $P_S = Af$ is an indecomposable projective. Now, in $A/\operatorname{Rad}(A)$, we know that S was a summand of the regular representation, so $(A/\operatorname{Rad}(A))e = S$. Therefore

$$\begin{aligned} P_S/\operatorname{Rad}(P_S) = P_S/(\operatorname{Rad}(A)P_S) &= (A/\operatorname{Rad}(A)(f + \operatorname{Rad}(A)) \\ &= (A/\operatorname{Rad}(A))e = S. \end{aligned}$$

This shows all the statements except unique simple quotient. But indeed if $U \subsetneq P_S$ has simple quotient then it has semisimple quotient, so $\operatorname{Rad}(P_S) \subset U \subset P_S$. But then $U = \operatorname{Rad}(P_S)$ by the simple quotient property. $\square$

Theorem 2.4.11 *Let A be an f.d. k-algebra; the indecomposable projectives are exactly the projective covers P_S of the simple modules S, and $P_S \cong P_T$ if and only if $S \cong T$. Each projective P_S appears as a summand of the regular representation as follows:*

$$ {}_AA \cong \bigoplus_{S \text{ simple}} (P_S)^{n_S}, $$

where n_S is the multiplicity of S in the decomposition of the semisimple module $A/\operatorname{Rad}(A)$, as in

$$ A/\operatorname{Rad}(A) = \bigoplus_{S \text{ simple}} (S)^{n_S}, $$

where we recall that $n_S = \dim_k(S)$ if k is algebraically closed, and in general letting the division algebra $D = \operatorname{End}_A(S)$ and S a module over it, $n_S = \dim_D(S)$.

This is easy from what we have shown already. The main idea is that if P is any indecomposable projective module, then $P/\operatorname{Rad}(P)$ is semisimple, and we can use uniqueness of projective covers to show that it is the cover of only one simple module, $P/\operatorname{Rad}(P)$.

Theorem 2.4.12 *Let A be an f.d. k-algebra, and U an A-module. Then U has a projective cover, and it is the same one as $U/\operatorname{Rad}(U)$.*

Proof $U/\operatorname{Rad}(U) = S_1 \oplus \cdots \oplus S_n$ by semisimplicity, so we can take a projecive cover of each one, and then we can use the direct sum property of essential epimorphisms. □

Now we get to the most important point of this chapter, which is probably the reason that I created these notes.

Definition 2.4.13 Let U be a module possessing a composition series.

1. A *composition factor* of U is any of the simple modules appearing as successive quotients of the term in any composition series for U.
2. By the Jordan-Holder theorem, the composition factors coming from all series are isomorphic. So if S is a simple module, the *multiplicity of S in U* is the number of times a module isomorphic to S appears as a composition factor in a comopsition series for U.

Proposition 2.4.14 *Let A be an f.d. k-algebra. Let S be a simple module with projective cover P_S, and let U be any A-module.*

1. *If T is a simple A-module, then $\dim_k \operatorname{Hom}_A(P_S, T) = \dim_k \operatorname{End}_A(S)$ if and only if $S \cong T$, and the module $\dim_k \operatorname{Hom}_A(P_S, T) = 0$ otherwise.*
2. *The composition multiplicity of S in U is exactly*

$$ \dim_k \operatorname{Hom}_A(P_S, U)/\dim_l \operatorname{End}_A(S). $$

So if k is algebraically closed, then the it is just $\operatorname{Hom}_A(P_S, U)$ *by Schur's lemma.*

3. *If $e \in A$ is an idempotent,* $\dim_k \operatorname{Hom}_A(Ae, U) = \dim_k(eU)$.

Proof We provide some elements of the proof to be more precise than just the dimension. First, if $P_S \to T$ is a nonzero map, then by simplicity the kernel is maximal so contains $\operatorname{Rad}(P_S)$, but $\operatorname{Rad}(P_S)$ is also maximal because it has simple quotient, so $S \cong T$.

Now, there is a map over k induced by composition with the projective cover map $P_S \to S$

$$\operatorname{End}_A(S) \to \operatorname{Hom}_A(P_S, S).$$

Precomposition with an epimorphism is always a monomorphism, so this is injective. On the other hand, a map $P_S \to S$ must either be zero or surjective; if its surjective, them by projectivity it factors through $P_S \to S$ and an endomorphism (automorphism) $S \to S$. Hence all maps factor through P_S. The only other interesting part here is that if S is a composition factor, then there is an exact sequence

$$0 \to U_{n-1} \to U \to S \to 0,$$

and thus $\operatorname{Hom}_A(P_S, -)$ preserves it. So we can use some dimension stuff and maybe induction. □

Definition 2.4.15 (cartan matrix)

1. Let S, T be simple A-modules. Then the *Cartan invariant* c_{ST} is defined to be the composition multiplicity of S in P_T.
2. The square matrix $C = (c_{ST})$, with rows and columns indexed by isomorphism types of simple A-modules, is called the *Cartan matrix*.

So, by the proposition, c_{ST} is equal to $\dim_k \operatorname{Hom}_A(P_S, P_T)/\dim_k \operatorname{End}_A(S)$.

Proposition 2.4.16 *Let A be an f.d. k-algebra, and S and T simple modules so that $P_S = Ae_S$ and $P_T = Ae_T$ are the projective covers of S and T. Then*

$$c_{ST} = \frac{\dim_k(P_S, P_T)}{\dim_k \operatorname{End}_A(S)} = \frac{\dim_k(e_S A e_T)}{\dim_k \operatorname{End}_A(S)},$$

and if k is algebraically closed then the denominators go away.

2.5 Chapter 8

All of the results in the previous section held at least for A a finite dimensional algebra over a field. Now we look at the specific example $A = kG$ for G a finite group.

Theorem 2.5.1 *Let* $\operatorname{char} k = p > 0$ *and G be a p-group. Then kG is an indecomposable projective module which is the projective cover of the trivial representation k.*

Every finitely generated projective module over kG is free, and the only idempotents in kG are 0 or 1.

Proof This follows from a lot of things, but for example we know k is the only simple module, and $\operatorname{Rad}(kG) = IG$ has basis the non-identity elements of G, so $kG/\operatorname{Rad}(kG) = k$ Therefore $kG \to k$ is a projective cover of k, and so it is the only indecomposable projective. Every idempotent splits kG so this implies the last thing. □

Lemma 2.5.2 *Let $H \leq G$ and R be a commutative ring.*

1. *If P is projective over RG then $P{\downarrow}^G_H$ is projective over RH.*
2. *If Q is projective over RG then $Q{\uparrow}^G_H$ is projective over RG.*

Proof Since induction = coinduction, induction is both the left and right adjoint of restriction, and vice versa. This implies the result. □

Corollary 2.5.3 *Let $\operatorname{char} k = p$ and let p^a be the largest power of p dividing $|G|$. If P is projective over kG, then $p^a \mid \dim_k P$.*

Proof Let H be a Sylow p-subgroup, then $P{\downarrow}^G_H$ is finite projective and hence free over the p-group algebra kH. So $|H|$, being the dimension of kH over k, divides the dimension. □

Proposition 2.5.4 *Let V be an RG-module which is free over R, and P a projective RG-module. Then $V \otimes_R P$ is projective over RG.*

Proposition 2.5.5 *Let X be a kG module and $0 \to U \to V \to W \to 0$ an exact sequence of kG-modules. Then the sequence*

$$0 \to U \otimes_k X \to V \otimes_k X \to W \otimes_k X \to 0$$

is exact. In particular, if $U \subset V$ then

$$(V/U) \otimes_k X \cong (V \otimes_k X)/(U \otimes_k X).$$

Proof Exactness is just a proeprty of underlying pointed sets after the existence of the maps, but as vector spaces is obvious because X is free over k and hence flat. □

Now we describe the structure of products and then semidirect products. Let $\operatorname{char} k = p > 0$ and write $G = H \times K$ where H is a p-group and $p \nmid |K|$. Then we have a trivial isomorphism

$$kH \otimes_k kK \xrightarrow{\sim} k[H \times K]$$

as k-algebras, induced by the bilinear map extending $h \otimes k \mapsto (h, k)$. Then $H = O_p(G)$, and we can write $kK = S_1^{n_1} \oplus \cdots \oplus S_r^{n_r}$ by semisimplicity, so these are all of the simple kG-modules, with (h, k) acting as k. Therefore

$$kG = (kH \otimes S_1)^{n_1} \oplus \cdots \oplus (kH \otimes S_r)^{n_r},$$

where the $kH \otimes S_i$ are projective over kG becaues they are summands of a free module. We proved that $k(O_p(G))$ was the common kernel of all the simple representations, i.e. any $x \in kH$ acts as the identity on S. But an element $x = \sum_{h,k} a_{(h,k)}(h,k)$ acts on all simple modules as

$$\sum_{h,k}(a_{h,k})k = \sum_{k}\Big(\sum_{h} a_{h,k}\Big)k,$$

which is zero only if that is in the radical of kK, which is zero by semisimplicity. Therefore $\sum_h a_{h,k} = 0$ for all fixed k, i.e. $\sum_h a_{h,k}h \in IH = \operatorname{Rad}(kH)$, so we have shown that $\operatorname{Rad}(kG) = \operatorname{Rad}(kH) \otimes kK$. The quotient becomes

$$kG/\operatorname{Rad}(kG) = (kH/\operatorname{Rad}(kH)) \otimes kK \cong k \otimes_k kK = kK$$

because $kH/\operatorname{Rad}(kK) = k$. Since the summands are the same as in the simple decomposition of kK, this means that we have found all of the indecomposable projective modules, and they are $P_{S_i} = kH \otimes S_i$ equipped with the map $P_{S_i} \to S_i$ sending $h \otimes s \mapsto s$.

Suppose that we have a composition series $0 \subset P_1 \subset \cdots \subset P_n = kH$. There is only one simple kH-module k, so all of the comopsition factors are k. The functor $(-) \otimes_k S_i$ preserves exact sequences, giving

$$0 \subset P_1 \otimes_k S_i \subset \cdots \subset P_n \otimes S_i = P_{S_i},$$

which has composition factors $k \otimes_k S_i = S_i$, so there is only one composition factor in P_{S_i}. Hence, for simple modules S_i, S_j, the multiplicity of S_i in P_{S_j} is 0, unless $i = j$ in which case it is the length n of the module kG. So the Cartan matrix is a diagonal $r \times r$ with n as each diagonal entry.

Alternatively, assume k is algebraically closed so that $\operatorname{End}_{kG}(S_i) = k$ for all i, and $kK = \prod_{i=1}^{r} M_{n_i}(k)$ as rings with $n_i = \dim(S_i)$. Then, as rings, we have $kG = \prod_i M_{n_i}(kH)$. This implies that

$$\operatorname{Rad}(kG) = \operatorname{Rad}(kH) \otimes kK = \prod_i M_{n_i}(\operatorname{Rad}(kH)).$$

We can also compute the powers, and we get $\operatorname{Rad}^n(kG) = \prod_i M_{n_i}((IH)^n)$. Now let's do this more generally.

Lemma 2.5.6 *Let R be a ring and $G = H \rtimes K$ a semidirect product. Then RG acts on RH naturally so that H acts by left multiplication and K by conjugation. This makes RH an RG module. Under this action,*

$$RH \cong R\uparrow_K^G \cong RG \cdot \overline{K} \qquad \overline{K} = \sum_{x \in K} x,$$

and we have a decomposition $RG \cong (R\uparrow_K^G) \otimes_R (R\uparrow_H^G)$ where $R\uparrow_H^G = RK$ with H acting trivially.

Assume $\operatorname{char} k = p > 0$ and $G = H \rtimes K$ with H a p-group and K having order coprime to p. This is the same as saying that H is a normal Sylow p-subgroup of G. (This always occurs when, e.g., $|G| = qp^n$ where q is a prime less than p).

Proposition 2.5.7 *In the scenario just described:*

1. *Let* $e_K = \sum_{|K|} x$. *Then the indecomposable projective* P_k *has*

$$P_k = kGe_K = kH = k\uparrow_H^G,$$

   where kH *means the semidirect action of* G *as earlier described.*

2. $\operatorname{Rad}(kG)$ *is the kernel of* $kG \to kK$ *induced by the quotient map, so that* $kG/\operatorname{Rad}(kG)$ *both as rings and* kG*-modules. Furthermore,* $\operatorname{Rad}(kG)$ *is generated as a left ideal and right ideal by* $\operatorname{Rad}(kH) = IH$.

3. *The simple* kG*-modules are precisely the simple* kK*-modules, regarded as* kG*-modules via the quotient map. If* S *is any simple* kG*-module with projective cover* P_S, *then* $P_S = P_k \otimes_k S = S\uparrow_K^G$ *over* kG.

4. *For each simple* kG*-module* S *with projective cover* P_S, *the radical series of* P_S, *has terms* $\operatorname{Rad}^n(P_S) = (IH)^n \otimes S$ *with radical layers*

$$(IH)^{n-1}/(IH)^n \otimes S,$$

   where $IH \subset kH$ *has the semidirect action. Hence, all the indecomposable projective modules have the same Loewy length, namely that of* kH *under the semidirect action.*

5. *We have* $P_k \otimes kK \cong G$ *over* kG.

Example 2.5.8 Example 8.3.4 from the book uses Prop. 8.3.3 to compute the modular representation theory of the nonabelian group $G = C_7 \rtimes C_3$, where $C_3 = \langle y \rangle$ acts on $C_7 = \langle x \rangle$ by $y \cdot x = x^2$. We take $k = C_7$. There are three simple $\mathbb{F}_7 G$-modules, labeled k_1, k_2, and k_4 by their actions. We get uniserial projective covers, and we can compute directly the Cartan matrix.

The multiplicities are as follows: k_i appears as a composition factor in P_{k_j} exactly 2 times whem $i \neq j$, and 3 times when $i = j$. This holds for all i, j, and so the martix is

$$C = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

for the group $C_7 \rtimes C_3$ over $\mathbb{F}_7$.

Definition 2.5.9 Let $p > 0$ be a prime. A group G has a *normal p-complement* if it has a normal subgroup $K \trianglelefteq G$ of order coprime to p, and $[G : K]$ is a pth power.

Proposition 2.5.10 *Let* $\operatorname{char} k = p > 0$. *Then the following are equivalent:*

1. G *has a normal* p*-complement.*

2. *For every simple kG-module S, the composition factors of P_S are all isomorphic to S.*

3. *The composition factors of P_k are all isomorphic to k.*

In particular, when one of these equivalent conditions hold, the Cartan matrix is a diagonal with diagonal entries the composition lengths of the corresponding simple projective P_S.

Proof (1) $\Rightarrow$ (2). Writing $G = K \rtimes H$ with H a Sylow p-subgroup, consider kH as a kG module with the quotient map $G \to H$. We claim it is projective. By semisimplicity, write $kK = k \oplus U$, and then

$$kG = kK \uparrow_G^K = k \uparrow_G^K \ \oplus \ U \uparrow_K^G \ .$$

But then $k \uparrow_K^G = kH$ over kG, so kH is a direct summand of kG and is therefore projective.

Then if S is simple over kG, we know $S \otimes_k kH$ is projective, and all of its composition factors are $S \otimes_k k = S$ because k is the only simple kH-module. But now the surjcetion $S \otimes_k kH \to S$ from a projective module, we know this factors through a projective cover P_S, where $P_S \oplus Q = S \otimes_k KH$ is a summand. Hence P_S cannot have new composition factors, so they all must be k.[1]

It remains to show (3) implies (1). Recall that an element $g \in G$ is *p-regular* if its order is coprime to p. If g is a p-regular element, then $k\,|g|$ is a semisimple ring then we know that $P_k \downarrow_{|g|}^G$ must be of the form k^t, because all of the composition factors of P_k are just k (but why does this force that?[2]). □

Now we discuss duality. If U is a kG-module, then $U^* \coloneqq \mathrm{Hom}_k(U, k)$ is naturally a kG-module, where $(g \cdot \varphi)(u) = \varphi(g^{-1}u)$ is the action. It is standard to check that $U^{**} \cong U$ over kG, U is semisimple iff U^* is semisimple, U is indecomposable iff U^* is indecomposable, and $f \colon U \to V$ is mono (resp., epi) iff $f^* \colon V^* \to U^*$ is epi (resp.) mono.

Proposition 2.5.11 *Let k be a field.*

1. *Then $(kG)^* \cong kG$ over kG.*

2. *A finitely generated kG-module P is projective if and only if P^* is projective.*

Proof (2) trivially folows from the first, since $(-)^*$ commutes with direct sums. For (1) just observe that kG is equipped with a canonical basis over k, so we can just send the basis to the dual basis. This is an isomorphism over k, and all that needs to be checked is that is respects the kG-action. □

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[1] Note that the surjection $S \otimes_\star kH \to S$ is induced by the bilinear map $s \otimes h \mapsto s$.

[2] Here's why; since restriction is exact, a commposition series of a module restricts to a filtration whose factors are the restrictions of the composition factors, namely k in this case, and so the restriction remains a composition series. But for a semisimple module, the composition factors are the same as the simple summands.

Corollary 2.5.12 *Let k be a field.*

1. *Finite projective modules are the same finite injective modules over kG.*
2. *Each indecomposable projcetive kG-module has a simple socle.*

Corollary 2.5.13 *Let U be a kG-module for which there are $U_0 \subset U_1 \subset U$ with $U_1/U_0 = P$ projective. Then $U = P \oplus U'$ is a direct summand.*

Proof The sequence $0 \to U_0 \to U_1 \to P \to 0$ splits so $U_1 = P \oplus U_1$, but since $P \hookrightarrow U$ this mono must be split. □

Definition 2.5.14 A finite dimensional algebra A over a field k is a *symmetric algebra* if there is a non-degenerate symmetric bilinear pairing $(-,-)\colon A \times A \to k$ such that

$$(ab, c) = (a, bc) \qquad a, b, c \in A.$$

For example, $A = M_n(k)$ is a symmetric algebra with the trace form. By the result we just proved, namely self-duality, the pairing on kG is given by sending (g, h) to be 0, unless $h = g^{-1}$ in which case it is 1. In particular, for $a, b \in kG$, the pairing (a, b) is the coefficient of 1 in ab.

Theorem 2.5.15 *Let P be an indecomposable projective module over kG. Then $P/\operatorname{Rad}(P) = \operatorname{Soc}(P)$. In particular, for any simple module S, $\operatorname{Soc}(P_S) = S$ so (1) S is a submodule of P_S, and (1) the only simple submodule of P_S is S.*

The proof using the pairing on kG, and in fact is true in any symmetric algebra. Now, recall that if U is an RG-module then U^G is the fixed point subspace. The *fixed quotient* of G on U is $U_G = U/\{(g-1)u : u, 1 \neq g \in G\}$.

Corollary 2.5.16 *Let k be a field.*

1. *If P is projective and S is simple over kG, the multiplicity of S in $P/\operatorname{Rad}(P)$ equals the multiplicity of S in $\operatorname{Soc}(P)$. In particular,*

$$\dim(P^G) = \dim(P_G) = \dim(P^*)^G = \dim(P^*)_G.$$

2. *For every simple kG-module S, $(P_S)^* = P_{S^*}$.*

Theorem 2.5.17 *Let S and T be simple. Then*

$$c_{ST} \cdot \dim_k \operatorname{End}_{kG}(T) = c_{TS} \cdot \dim_k \operatorname{End}_{kG}(S),$$

so that c_{ST} is symmetric if $\dim \operatorname{End}_{kG}(S) = 1$ for all S, e.g. if k is algebraically closed.

2.6 Chapter 9

Now we come to the hardest and most annoying chapter, which is so expansive that it contains a not insignificant body of results pertaining to Grothendieck groups. There's a lot of stuff about discrete valuation rings in this chapter, so let's quickly review the basic results regarding those.

Definition 2.6.1 Let F be a field. A *multiplicative valuation* on F is a map $\phi\colon F \to \mathbb{R}_{\geq 0}$ such that:

1. $\phi(a) = 0$ iff $a = 0$.
2. $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in F$.
3. $\phi(a + b) \leq \phi(a) + \phi(b)$ for all $a, b \in F$.

There is a 'trivial valuation' on any field F given by $0 \mapsto 0$ and $a \mapsto 1$ otherwise. We will generally not consider it. If $F \subset \mathbb{C}$, $F \to \mathbb{R}_{\geq 0}$ can be defined by $|-|$. A valuation ϕ is called *discrete* if $\{\phi(a) : a \in K, a \neq 0\}$ is an infinite cyclic subgroup of $(\mathbb{R}_{>0}, \cdot)$. We define $R_\phi = \{a \in F : \phi(a) \leq 1\} \subset F$ to be the associated valuation ring.

Proposition 2.6.2 *Let F be a field with a discrete valuation ϕ, valuation ring R_ϕ and valuation ideal P_ϕ.*

1. *$a \in R_\phi$ is a unit iff $\phi(a) = 1$.*
2. *(R_ϕ, P_ϕ) is a local ring.*
3. *$P_\phi = (\pi)$, where π is any element such that $\phi(\pi)$ generates the value group.*
4. *Every element of R_ϕ is uniquely expressible as $a = \pi^\nu a'$ for $a' \in R_\phi^\times$. In this case $\phi(a) = \phi(\pi)^\nu$.*
5. *The ideals of R_ϕ are precisely the powers $P_\phi^n = (\pi^n)$ (i.e. R is PID).*
6. *F is the fraction field of R.*

Definition 2.6.3

1. A *discrete valuation ring* (DVR) R is a ring for which there exists a field (F, ϕ) equipped with a discrete valuation, such that $R = R_\phi$.
2. An *algebraic number field* is a finite extension of $\mathbb{Q}$, and it has a ring of integes.

Proposition 2.6.4 *Let F be an algebraic number field, R its ring of integers, and let ϕ be the discrete valuation on F associated to a maxial ideal $\mathfrak{p} \subset R$. Let $(R_\mathfrak{p}, P_\mathfrak{p})$ be the corresponding DVR. Then $R_\mathfrak{p}$ is the localization of R away from $\mathfrak{p}$, with maximal ideal $\mathfrak{p}R_\mathfrak{p} = P_\mathfrak{p}$, and the quotient field is isomoric to the quotient $R/\mathfrak{p}$.*

Given a valuation ϕ on F, we can make (F, d) a metric space by defining $d(x, y) = \phi(x - y)$. Its ring completion extends the ring structure on F, and is a field; ϕ uniquely extends, and if it was discrete then the extension is too. If R is a DVR, we therefore can take its completion to get a complete DVR. A sequence (a_i) of elements of R_ϕ is a Cauchy sequence iff for all n there is N such that when $i, j > N$, we have $a_i \equiv a_j \pmod{P_\phi^n}$.

Now let's return to representation theory. Let A be an R-algebra, and U an A-module. Then a homomorphism $R \to R'$ lets us form the base changed R'-algebra $R' \otimes_R A$, and then $R' \otimes_R U$ is an $R' \otimes_R A$ module. If, here, an $R' \otimes_R A$-module V can be wrriten in the form $R' \otimes_R U$ for an A-module U, then we say V can be written in R.

Suppose U is free over R (e.g. $R = k$ is a field). Then U is identified with the subset $1 \otimes U \subset R' \otimes_R U$, and any R-basis of U becomes an R'-basis of $R' \otimes_U$. If $R' = R/I$ and the map is the quotient, then the tensor product is the same as reducing U modulo I, which amounts to taking U/IU as an R/I-module.

Proposition 2.6.5 *Let E/F be an algebraic extension and A a finite-dimensional F-algebra. Let V be a finite-dimensional $E \otimes_F A$-module. Then there is an intermediate extension $F \subset K \subset E$ finite over F such that V can be written in F (i.e. $V = E \otimes_K U$ for some U).*

Definition 2.6.6 Let A be a finite-dimensional F-algebra.

1. A simple A-module U is *absolutely simple* if for any extension E/F the base change $E \otimes_F U$ is simple over $E \otimes_F A$.
2. An extension E/F is as splitting field for A if every simple $E \otimes_F A$-module is absolutely simple. If $A = FG$ is a group algebra, then in this case we say E is a splitting field for G.

If the base field F is note specified but the group is, then we mean that its a splitting field over the prime field. This is always a non-issue because if E/F and $A = FG$, then

$$E \otimes_F A = E \otimes_F FG = EG.$$

To see this, we observe that we have an E-basis consisting of the elements of G, i.e. $1 \otimes g$, and $x \in E$ acts on this by $x(1 \otimes g) = x \otimes g$; if $x \in F$, then $x(1 \otimes g) = x \otimes g = 1 \otimes xg$, but we can just ignore that.The point of splitting fields is to address the fact that sometimes representations have weird properties because the field is not big enough; usually it's because we're lacking roots of unity.

For example, if $G = \langle g \rangle = C_n$, then g acts on $\mathbb{C}G$ as a direct sum of 1-dim eigenspaces with eigenvalues $e^{2\pi i/n}$. For $n \geq 3$, these are outside of $\mathbb{Q}$, so $\mathbb{Q}G$ is semisimple but it might have higher dimensional semisimple factors. However, if we take $F = \mathbb{Q}(\zeta_n)$ or any extension of it, then it is a splitting field of $\mathbb{Q}G$, because all simple modules are 1-dimensional in F, and because of the fact that

$$\dim_E(E \otimes_F U) = \dim_F(U),$$

they remain 1-dimensional, hence simple, upon base extension.

Lemma 2.6.7 *For F any field, F is a splitting field for $M_n(F)$.*

Proof Let e_{ij} be the standard $n \times n$ basis of $M_n(F)$. Then clearly $E \otimes_F M_n(F) = M_n(E)$. The maximal ideals of $M_n(F)$ are all of matrices which have one column fixed, so all the simple modules are just F^n with the canonical action. This goes to E^n under base change. □

Remark 2.6.8 $\mathbb{F}_p$ (so any field of characteristic p) is a splitting field for every p-group, since there's only one $\mathbb{F}_p G$-module, i.e. $\mathbb{F}_p$ the trivial representation.

Proposition 2.6.9 *Let U be a simple A-module, where A is a finite dimensional k-algebra. Then the following are equivalent:*

1. *U is absolutely simple.*
2. *$\mathrm{End}_A(U) = F$.*
3. *The matrix algebra summand of $A/\operatorname{Rad} A$ corresponding to U has the form $M_n(F)$, where $n = \dim_F(U)$.*

Proof (2) implies (3) trivially because the summand is exactly $M_n(\mathrm{End}_A(U))$ where n is the dimension of U over the division algebra $\mathrm{End}_A(U)$. (3) $\Rightarrow$ (1): If E/F is an extension, then $E \otimes_F A$ has a matrix algebra summand which is $E \otimes_F M_n(F) = M_n(E)$ corresonding to $E \otimes_F U$, which is therefore simple. The other one is harder. □

Theorem 2.6.10 *Let A be an f.d. F-algebra. Then A has a splitting field of finite degree over F. If $A = FG$ for a finite group G, then it has splitting fields that are finite extensions of $\mathbb{F}_p$ or $\mathbb{Q}$ based on* $\operatorname{char} F$.

In the proof of the theorem, the crucial remark here is made: Every simple $E \otimes_F A$-module arises in the algebraic closure $\overline{F}$. If V is such a module, then let $e^2 = e \in E \otimes_F A$ be an idempotent such that $eV \neq 0$ but $eW = 0$ for all $W \ncong V$ simple. But over $\overline{F} \otimes_F A$, we can always find a simple W-module not annihilated by e. Then we must have $V \cong \overline{F} \otimes_F W$ over $\overline{F} \otimes_F A$, so we're done.

Recall that the exponent m of a finite group G is the l.c.m. of the orders of all elements, or equivalently the smallest number m such that $g^m = 1$ for all m.

Theorem 2.6.11 *Let G be a finite group, F a field, and suppose F contains a primitive mth root of unity for m the exponent of G. Then F is splitting field of G.*

Suppose that $p \nmid |G|$. Then $\mathbb{Q}(e^{2\pi i/m})$ or $\mathbb{F}_p(\zeta_m)$ will be splitting fields; if we want them to be splitting fields for all the subgroups of G, then it is the best estimate because a splitting field has to contain the mth roots of unity to have the standard simple representations for $|g|$ where $g \in G$.

On the other hand, if $p \mid |G|$ it is more complicated. For instance we have already seen that $\mathbb{F}_p$ is a splitting field for every p-group. If we are in a semidirect product like situation where all of the simple reps from from some other semisimple group algebra, then we don't need to take the mth root of unity, but only the lcm of the orders of elements which p doesn't divide.

Theorem 2.6.12 (Noether-Deuring) *Let A be an f.d. k-algebra and E/F an extension. Then $E \otimes_F U \cong E \otimes_F V$ implies $U \cong F$ over A.*

Problem 1 (Exercise 8) *Let A be a finite dimensional F-algebra.*

1. *Suppose F is a splitting field for A and let E/F be an extension. Then every simple $E \otimes_F A$-module can be written in F. In particular, A and $E \otimes_F A$ have the same number of isomorphism classes of simple modules.*

Proof Recall by Exercise 4 that $\operatorname{Rad}(E \otimes_F A) \cong E \otimes_F \operatorname{Rad}(A)$. It follows that as algebras

$$(E \otimes_F A)/\operatorname{Rad}(E \otimes_F A) = (E \otimes_F A)/(E \otimes_F \operatorname{Rad}(A)) = E \otimes_F (A/\operatorname{Rad}(A))$$

so if $A/\operatorname{Rad}(F) = \bigoplus_S S^{n_S}$ are the simple modules, then

$$(E \otimes_F A)/\operatorname{Rad}(E \otimes_F A) = E \otimes_F \Big(\bigoplus_S S^{n_S}\Big) = \bigoplus_S (E \otimes_F S)^{n_S}.$$

Since F is a splitting field, all the modules $E \otimes_F S$ are simple and so it follows that $E \otimes_F S$ running over all S is a list of simple modules of $E \otimes_F A$. They are also all non-isomorphic; we could just wave our hands and appeal to the Noether-Deuring theorem, but there's a better way. Namely, take a primitive idempotent $f \in A$ such that $fS \neq 0$ yet $fT = 0$ for all other simple modules T. Then $1 \otimes f \in E \otimes_F A$ is an idempotent which annihilates all $E \otimes_F T$ for $T \ncong S$. Finally, since F is a field, we can identify S with the subset

$$1 \otimes_F S \subset E \otimes_F S,$$

and $1 \otimes f$ acts nontrivially on this subset, so it can't annihilate the whole module. Hence the decomposition above is exactly that of $E \otimes_F A$ into (projective covers of) simple modules, and thus the number of isomorphism types, and the numbers n_S, are all constant once we reach a splitting field. □

Definition 2.6.13 An element of G is *p-regular* if its order is coprime to p, and *p-singular* if its order is divisible by p. The only element that is both is the identity.

Lemma 2.6.14 *Let G be a finite group and p a prime. Each $x \in G$ can be uniquely written $x = st$ where s is p-regular, t is p-singular, and $st = ts$. If $x_1 = s_1 t_1$ is such a decomposition of x_1 conjugate to x, then s_1 is conjugate to s and t_1 is conjugate to t.*

Theorem 2.6.15 (Brauer) *Let k be a splitting field of characteristic p for a finite group G. The number of non-isomorphic simple kG-modules equals the number of conjugacy classes of p-regular elements of G.*

Corollary 2.6.16 *Let k be a splitting field of characteristic p for finite groups G_1 and G_2. The simple $k[G_1 \times G_2]$-modules are precisely the external tensor products $S_1 \otimes S_2$ where S_i is a simple kG_i-module and the action is the tensor-wise one. Two such products are isomorphic iff each simple tensor factor is isomorphic.*

Proof The main point is that the number of p-regular conjugacy classes in $G_1 \times G_2$ is the numebr of those in G_1 times the number of those in G_2 so that all the ones we can write down give the right answer. □

Now we begin the difficult portion of this theory, where the goal will be to 'reduce' representations in characteristic 0, which we understand well, to characteristic p.

Definition 2.6.17

1. A *p-modular system* is a triple (F, R, k) where F is a field of characteristic 0 equipped with a discrete valuation $F \to \mathbb{R}_{\geq 0}$, $(R, (\pi))$ is the corresponding discrete valuation ring of elements with valuation ≤ 1, and $k = R/(\pi)$ is the residue field, where we require $\operatorname{char} k = p > 0$.
2. If G is a finite group and (F, R, k) is a p-modular system, we say it is a *splitting* p-modular system for G if both F and k are splitting fields for G.

We will not worry about the question of existence, but in general the point will be to take k to be the splitting field in characteristic p we're interested in, which we now know can always be a finite field $\mathbb{F}_q$. We can always find a DVR with fraction field a finite extension of $\mathbb{Q}$, and moreover we can assume it's complete. Note that if m is the exponent of G, and R and k contain a primitive mth root of unity, then the system (F, R, k) is automatically splitting.

Lemma 2.6.18 *Let $(R, (\pi))$ be a DVR with residue k.*

1. *If S is a simple RG-modulel, then $\pi S = 0$.*
2. *The simple RG-modules are exactly the simple kG-modules made into RG-modules by quotient $RG \to kG$.*
3. *For each RG-module U, we have $\pi U \subset \operatorname{Rad}(U)$. Thus $\pi RG \subset \operatorname{Rad}(RG)$.*
4. *We have $(\operatorname{Rad} U)/\pi U = \operatorname{Rad}(U/\pi U)$.*

Proof Most of the properties are basically that $\operatorname{Rad}(R) = (\pi)$; e.g. πS is either S or 0, but $S \to S/\pi S$ is an essential epimorphism, so $\pi S \neq S$ cannot surject. The next is just because $kG = RG/(\pi)G$ and $\pi(G)$ annihilates the simple modules. Everything else is trivial. □

Corollary 2.6.19 *Let $(R,(\pi))$ be a DVR with residue k. Let P,Q be finite projective RG-modules. Then $P \cong Q$ over RG if and only if $P/\pi P \cong Q/\pi Q$ over kG.*

Proof Suppose $P/\pi P \cong Q/\pi Q$ as kG-modules, then by (4) of the lemma $P/\operatorname{Rad}(P) \cong Q/\operatorname{Rad}(Q)$, so P and Q are projective covers of those quotients and we conclude by uniqueness of projective cover. □

Lemma 2.6.20 *Let $(R,(\pi))$ be a complete DVR with residue k. Each expression*

$$1 = e_1 + \cdots e_n \qquad \text{in } kG$$

as a sum of orthogonal idempotents lifts to an expression

$$1 = \hat{e_1} + \cdots + \hat{e_n} \qquad \text{in } RG$$

via the quotient map. Each e_i is primitive iff $\hat{e_i}$ is primitive.

Proof The proof constructs a Cauchy sequence in RG and letting its limit be the lift. □

Definition 2.6.21 Let R be a PID. An RG-module L is an *RG-lattice* if it is a finite free module over R.

Let $I \subset R$ and L be an RG-lattice, then $V \coloneqq L/(IL)$ is naturally an $(R/I)G$-lattice. V is the reduction modulo I, and we say that L is a lift from R/I to R. But, not every (R/I)-modules are liftable. Observe that if L has basis $\{e_1,\ldots,e_n\}$ over R, then their images form a basis (maybe) over R/I, because

$$(R^n)/IR^n = (R/IR)^n \quad \text{over } R/I.$$

Theorem 2.6.22 *Let $(R,(\pi))$ be a complete DVR with residue k.*

1. *For each simple RG-module S, there is a unique indecomposable projective RG-module $\hat{P}_S$ which is the projective cover of S. It has the form $\hat{P}_S = RG\hat{e_S}$, where $\hat{e_S}$ is a primitive idempotent in RG with $\hat{e_S}S \neq 0$.*
2. *The kG-module $P_S \coloneqq \hat{P}_S/\pi\hat{P}_S$ is the projective cover of S as a kG-module. Viewing P_S as an RG-module, $\hat{P}_S$ is also its projective cover.*
3. *S is the unique simple quotient of $\hat{P}_S$ over RG. So, for simple kG-modules S and T, $\hat{P}_S \cong \hat{P}_T$ iff $S \cong T$ (we're using that simple modules of kG are the same as of RG).*
4. *Every f.g. RG-module has a projective cover. Every f.g. indecomposable projective is of the form $\hat{P}_S$ for some simple S over RG (or kG).*
5. *Every f.g. projective kG-module can be lifted to an RG-lattice. Such a lift is unique up to isomorphism and necessarily projective over RG. In particular, an RG-lattice L is projective if and only if $L/(\pi L)$ is projective over kG.*

6. Projective RG-modules L_1, L_2 are isomorphic iff $L_1/\pi L_1 \cong L_2/\pi L_2$.

Proof The last thing was thing was already proven as a corollary before. The 'in particular' part of (5) is obvious from the first part of (5). There is some work to do here, but the main logic is as follows. Since kG is an f.d. algebra over a field, and S is uniquely a simple module both over RG and over kG, let $e_S \in kG$ be a primitive idempotent with $e_S \cdot S \neq 0$, and let $\hat{e}_S \in RG$ be a lift as before. Set $\hat{P}_S = RG\hat{e}_S$, which is an indecomposable projective. Then P_S as written is just kGe_S, which is the projective cover of S. □

Suppose U is an RG-lattice; the reduction U/IU makes sense because $IU \subset U$ is an RG-submodule. Namely, IU is an R-submodule, but $g \cdot (ru) = rgu$ always commutes with the scalar action of R, so IU is an RG-submodule, and thus U/IU is an (R/I)-module. Note that when R is a PID, IU is an RG-sublattice of U.

Definition 2.6.23 Let R be a PID and F its fraction field. For L an RG-lattice, we can identify $L = 1 \otimes_R L \subset F \otimes_R L$ using freeness.

1. If U is an FG-module, a *full RG-lattice* $U_0 \subset U$ is an RG-lattice $U_0 \subset U$ that has an R-basis which is also an F-basis of U.
2. In this case, $U \cong F \otimes_R U_0$ can be written in R, and U_0 is an *R-form* of U.

Lemma 2.6.24 *Let R be a PID with fraction field F, and let U be finite dimensional over F. Then any R-submodule $U_0 \subset U$ which contains an F-basis is a full lattice.*

Proof We use R being a PID be observing that the module U_0 is torsion-free and finitely generated, hence free over R. The R-basis of U_0 accesses the F-basis, so it spans U over F, and it can be shown that it is linaerly independent over F. □

Corollary 2.6.25 *Let R be a PID with fraction field F, and U an f.d. FG-module. Then there is an RG-lattice U_0 which is an R-form for U. I.e. any such U can be written in R.*

Proof Take a basis for U and let U_0 be the R-submodule of U spanned by all group elements applied to all basis vectors. This contains the F-basis and is closed under the action of RG, so is a full RG-lattice inside U. □

Example 2.6.26 Let $G = \{1, g\}$ and let (F, R, k) be a 2-modular system. FG is canonically free over F with basis $\{1, g\}$, and so there is a full sublattice $R1 + Rg$, i.e. an RG-module free over R (it is RG), which contains an F-basis. Its reduction is kG which is a p-group algebra, so it's indecomposable. On the other hand, $R(1+g) + R(1-g)$ is a full sublattice which splits as $R(1+g) \oplus R(1-g)$ as RG-modules, so the two couldn't be isomorphic.

Here we are using the fact that if L is an RG-module and $L/\pi L$ is indecomposable then L is indecomposable. This is because if $L = U \oplus V$ over RG, then (say) $U/\pi U = 0$, so $\pi U = U$, hence $U = \mathrm{Rad}(U)$, a contradiction unless $U = 0$.

Theorem 2.6.27 (Brauer-Nesbitt) *Let (F, R, k) be a p-modular system, and U an f.g. FG-module. Suppose L_1, L_2 are full RG-lattices in U. Then $L_1/\pi L_1$ and $L_2/\pi L_2$ have the same composition factors with the same multiplicities as kG-modules.*

Proof We can easily reduce to the case that $L_1 \subset L_2$ and L_1 is a maximal RG-submodule of L_2, so in particular the quotient is simple, and thus $\pi L_2 \subset L_1$, whence we are done by an easy argument. □

Definition 2.6.28 Let (F, R, k) be a splitting p-modular system for G. The *decomposition matrix* D is the matrix (d_{TS}) with rows indexed by simple FG-modules and columns by simple kG-modules with

$$d_{TS} = \text{composition multiplicity of } S \text{ in } T_0/\pi T_0,$$

where S is a simple kG-module, T is a simple FG-module, and T_0 is any full RG-sublattice in T, but whose composition factors/multiplicities are independent of the choice of T_0 by the Brauer-Nesbitt theorem.

Example 2.6.29 The 'direct method' to computing a decomopsition matrix is: compute all simple representations in characteristic 0 (simple FG-modules), then reduce them to k by finding lattice forms for them, and then compute composition factors. For example:

$$D_{S_3,p=2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad D_{S_3,p=3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The rows represent the simple modules in char 0, of which there are 3, and the columns are the multiplicities for the two ones in $\mathbb{F}_2$ and $\mathbb{F}_3$. (Note that, assuming we are using splitting fields, there might be simple kG-module which don't appear as a composition factor...). Let us briefly state how to do this: The three char 0 reps are: trivial, sign, and the $\mathbb{Z}S_3$-lattice of dim 2. In char 2, they are the trivial and the 2-dim one interpreted as lying in $\mathbb{F}_2$. These are all of them by Brauer's theorem. Then we can compute the R-forms.

Note that if G is a p-group and k has $\mathrm{char}\, k = p$, then the only simple kG-module is k, so there is only one column in D. If T a simple FG-module let T_0 be a full RG-sublattice in T Of course all of the composition factors of $T_0/\pi T_0$ must be k; since T_0 has the same rank over R as the dimension of R over F, and so $T_0/\pi T_0$ has the same dimension over k, and this dimension is the number of k must appear in the decomposition.

Theorem 2.6.30 *Let (F, R, k) be a splitting p-modular system for G where p does not divide the group order. Then each simple FG-module reduces to a simple kG-module of the same dimension. This establishes a bijective corresopndence between the simple FG-modules and the simple kG-modules so that taken in the correct order, $D = Id$. The bijection preserves tensor products, homs, and induction and restriction.*

If $|G|$ is coprime to p an RG-module L is projective iff it is projective over R, and for each FG module U, all R-forms of U are isomorphic as RG-modules.

2.6.1 CDE and blocks of defect zero

Let (F, R, k) be a splitting p-modular system for G and assume F and R are complete. We define $G_0(FG)$ and $G_0(kG)$ to be the free abelian groups with basis the isomorphism classes of simple FG-modules and simple kG-modules, respectively. We let $K_0(kG)$ be the free abelian group with basis the indecomposable projectives in kG.

So, the rank of $G_0(FG)$ is the number of conjugacy classes of G, and the ranks of $G_0(kG)$ and $K_0(kG)$ are the number of p-regular conjugacy classes of G. If T is a siple FG-module, we write $[T]$ for the corresponding basis element and likewise for the other groups. Here's the point: If U is any kG-module then we can break it up into its composition factors $\{S_i\}$ and the corresponding multiplicities n_i in a composition series. Then we set

$$[U] = n_1[S_1] + \cdots + n_r[S_r] \in G_0(kG).$$

By Jordan-Holder, the numbers n_i and isomorphism classes $[S_i]$ are independent of the composition series, and only on the isomorphism type of U.

We do the same thing for an FG-module $[U] = n_1[S_1] + \cdots$. But by semisimplicity of FG, the compsition factors and multiplicities uniquely determine the module back by taking direct sums, so the map from iso. types FG-modules into $G_0(FG)$ is injective. Finally, if P is projective then we can define $[P]$ by breaing it up into its indecomposable projective summands giving a map into $K_0(kG)$.

Simple FG-modules biject with their characters, so we can identify $G_0(FG)$ with the subspace of class functions $\mathbb{C} \to cc(G)$ consisting of the $\mathbb{Z}$-linear combos of simple characters. These are called *virtual characters*; they give rise to a representation exactly when all the coefficients are nonnegative.

The CDE triangle is the following diagram of Grothendieck groups:

$$\begin{array}{ccc} & G_0(FG) & \\ {\scriptstyle e}\nearrow & & \searrow{\scriptstyle d} \\ K_0(kG) & \xrightarrow[c]{} & G_0(kG) \end{array} \qquad (*)$$

The maps are defined as follows, and by freeness: for e we need to say where to send an indecomposable projective to a virtual character, for c we need to say where to send it to a sum of kG-module types, and for d where to send a simple character.

Definition 2.6.31 The *CDE triangle* is the diagram $(*)$ with maps defined as follows:

(e) If P_S is an indecomposable projective, then send $e([P_S]) = [F \otimes_R \hat{P}_S]$. By an earlier theorem, the type of $\hat{P}_S$ depends only on the type of S, whose projective cover is P_S.

(d) d is the *decomposition map*: If V is a simple FG-module, then choose any full RG-lattice V_0 inside V and set $d([V]) = [V_0/\pi V_0]$. This is is well-defined and works on all FG-modules. By the Brauer-Nesbitt theorem, this is independent of the choice of V_0, and viewing d as a matrix we see d is the transpose of the decomposition matrix D.

(c) c is the *Cartan map* defined by $c([P_S]) = [P_S]$, where on the left we mean a basis element and on the right we mean to break P_S up into composition factors and multiplicities. Hence by definition the matrix of c is teh Cartan matrix.

Proposition 2.6.32 *$c = de$.*

Proof This is trivial: $\hat{P}_S \subset F \otimes_R \hat{P}_S$ is a full sublattice. □

Proposition 2.6.33 *Let (F, R, k) be a p-modular system and U, V be FG-modules, and choose U_0, V_0 full RG-sublattices. Then:*

1. $\mathrm{Hom}_{RG}(U_0, V_0)$ *is a full R-lattice in* $\mathrm{Hom}_{FG}(U, V)$.
2. $\pi\mathrm{Hom}_{RG}(U_0, V_0) = \mathrm{Hom}_{RG}(U_0, \pi V_0)$ *as a subset.*
3. *If U_0 is a projective RG-lattice, then*

$$\begin{aligned}\mathrm{Hom}_{RG}(U_0, V_0)/\pi\mathrm{Hom}_{RG}(U_0, V_0) &\cong \mathrm{Hom}_{RG}(U_0, V_0/\pi V_0) \\ &\cong \mathrm{Hom}_{kG}(U_0/\pi U_0, V_0/\pi V_0).\end{aligned}$$

Proof Let us remark that to see $\mathrm{Hom}_{RG}(U_0, V_0)$ as a subset of $\mathrm{Hom}_{FG}(U, V)$, choose R-bases for each module and then view it as a matrix embedding $M_{m\times n}(R) \hookrightarrow M_{m\times n}(F)$. □

Corollary 2.6.34 *Let U_0 and V_0 be full RG-lattices in U and V with U_0 projective. Then:*

$$\dim_F \mathrm{Hom}_{FG}(U, V) = \dim_k \mathrm{Hom}_{kG}(U_0/\pi U_0, V_0/\pi V_0).$$

Theorem 2.6.35 *Let (F, R, k) be a splitting p-modular system for G and suppose that R is complete.*

1. *Let S be a simple kG-module and T a simple FG-module with $T_0 \subset T$ a full RG-lattice. The multiplicity of T in $F \otimes_R \hat{P}_S$ equals the multiplicity of S as a composition factor of $T_0/\pi T_0$.*
2. *With respect to the defining bases of the Grothendieck groups, the matrix of e is D and that of d is D^T. In particular, $C = D^T D$.*

This also shows that the decomposition numbers are independent of the choice of splitting p-modular system. This formula also lets us compute the Cartan matrix from the decomposition matrix, though in my opinion it's much easier to compute the Cartan matrix.

We have two canonical pairings on the Grothendieck groups. First,

$$\langle -, - \rangle : G_0(FG) \times G_0(FG) \to \mathbb{Z}$$

given by $\langle [U], [V] \rangle = \dim \mathrm{Hom}_{FG}(U, V)$. Likewise, we get

$$\langle -, - \rangle : K_0(kG) \times G_0(kG) \to \mathbb{Z}$$

given by $\langle [P], [V] \rangle = \dim \mathrm{Hom}_{kG}(P, V)$. Each of the pairings are non-degenerate, and $\langle e(x), y \rangle = \langle x, d(y) \rangle$. So, the projectives and simples in the kG case are dual to each other. Now, we move on to studying blocks of defect zero.

Definition 2.6.36 Let (F, R, k) be a splitting p-modular system for G.

1. A *block of defect zero* is a ring summand (factor) of kG having a projective simple module.
2. We will see that the block can only have one simlpe module, and it is a matrix algebra, and there is a unique simple character which reduces to it. These simple representations are are also called *blocks of defect zero.*

Theorem 2.6.37 *Let (F, R, k) be a splitting p-module system in which R is complete, and let G be a group order $p^d q$ with $(p, q) = 1$. Let T be an FG-module of dimension n containing $T_0 \subset RG$ a full sublattice. Then the following are equivalent:*

1. *$p^d \mid n$ and T is simple over FG.*
2. *The map $RG \to \mathrm{End}_R(T_0)$ (sending g to its action $x \mapsto g \cdot x$) identifies $\mathrm{End}_R(T_0) \cong M_n(R)$ with a ring summand of RG.*
3. *T is a simple FG-module and T_0 is a projective RG-module.*
4. *The map $kG \to \mathrm{End}_k(T_0/\pi T_0)$ identifies $\mathrm{End}_k(T_0/\pi T_0) \cong M_n(k)$ with a ring direct summand of kG.*
5. *$T_0/\pi T_0$ is simple and projective over kG.*

Lemma 2.6.38 *Let (F, R, k) be a p-modular system and let $\hat{P}$ be a projective RG-module and χ the character of $F \otimes_R \hat{P}$. THen $\chi(1)$ is divisible by the order of a Sylow p-subgroup, and if $g \in G$ has order divisible by p then $\chi(g) = 0$.*

Corollary 2.6.39 *Let T be a simple FG-module where F is a splitting field for G of characteristic 0, and let χ_T be the character. Let p be a prime and suppose that the p-part of $|G|$ also divides the degree $\chi_T(1)$. Then if g is any element of order divisible by p, $\chi_T(g) = 0$.*

This shows us where zeros in the characteristic 0 character table might appear, and often it accounts for most or all of them.

2.7 Brauer Characters

Let G be a group. We will generally assume that there is a complete p-modular system (F, R, k) such that both F and k are splitting fields for G *and all of its subgroups*. This means in particular that for each p-regular element $g \in G$ of order n, F and k have the 1-dimensional simple representations of $\langle g \rangle$, and so they must contain primitive nth roots of unity. In particular, if a is the lcm of the orders of all p-regular elements, then F and k must contain a primitive ath root of unity. Let us also note that, because we can always take F to be a finite extension of $\mathbb{Q}$, we will make the assumption that $F \subset \mathbb{C}$.

Write $\mu_F \subset F$ for the ath roots in F and likewise μ_k. These are both cyclic subgroups of order a in each of the respective fields, because $p \nmid a$.

Lemma 2.7.1 *$\mu_F \subset R$, and the quotient map $R \to k$ induces a group isomorphism $\mu_F \xrightarrow{\sim} \mu_k$.*

The proof of the lemma is trivial, and the point is that roots of unity have to have valuation 1 so they're included in R. We write $\lambda \in \mu_k$ and $\hat{\lambda} \in \mu_F$ for the unique lift.

Definition 2.7.2 Let $g \in G$ be a p-regular element and $\rho \colon G \to GL(U)$ a representation over k. $k\langle g \rangle$ is semisimple and all of the eigenvalues lie in k as they're ath roots of unity. So, if the eigenvalues of $\rho(g)$ are $\lambda_1, \ldots, \lambda_n$, then set

$$\rho_U(g) = \hat{\lambda}_1 + \cdots + \hat{\lambda}_n \in \mathbb{C}.$$

The function ρ_U defined on the p-regular elements of G is the *Brauer character* of U, and takes values in $F \subset \mathbb{C}$.

Note that the lifting from $\lambda \mapsto \hat{\lambda}$ does not preserves sums, and we must lift first then add eigenvalues, not the other way around. In particular, we cannot just lift the trace of the matrix. See Example 10.1.2.

Proposition 2.7.3 *Let (F, R, k) be a p-modular system and U, V, S finite dimensional kG-modules.*

1. $\phi_U(1) = \dim_k(U)$
2. *ϕ_U is a class function on p-regular conjugacy classes.*
3. $\phi_U(g^{-1}) = \overline{\phi_U(g)} = \phi_{U^*}(g)$.
4. $\phi_{U\otimes V} = \phi_U \cdot \phi_V$.
5. *If $0 \to U \to V \to W \to 0$ is an exact sequence over kG, then $\phi_V = \phi_U + \phi_W$. In particular ϕ_U depends only on the isomorphism type of U; if U has composition factors S, each occurring with multiplicity n_S, then $\phi_U = \sum_S n_S \phi_S$.*
6. *If U lifts to an RG-lattice $\hat{U}$ and the ordinary character of $\hat{U}$ is $\chi_{\hat{U}}$, then $\phi_U(g) = \chi_{\hat{U}}(g)$ on p-regular elements.*

Proof Most of this is easy; for (5), observe that if g is p-regular then $k\langle g\rangle$ is semisimple and $V \cong U \oplus W$ over $k\langle g\rangle$ because all exact sequences split over semisimple rings. It follows that the eigenvalues of g on V are the union of the eigenvalues on V and on W with multiplicities.

For (6), if g acts on $\hat{U}$ with eigenvalues $\mu_1, \ldots, \mu_n$, then $U = \hat{U}/\pi\hat{U}$ and g acts on U with eigenvalues the images of the $\mu_1, \ldots, \mu_n$. But we proved that these are mapped isomorphically from R to k. □

We need to consider Brauer characters not just of the simple modules but of the indecomposable projectives as well. By Brauer's theorem, the number of simple module isomorphism types (and hence the indecomposable projective types) are the number of p-regular conjugacy classes, which is the same as the Brauer character table, so both of these are square tables.

In computing the tables, representations may or may not lift to characteristic zero. If they do, then by (6) of the proposition, the Brauer tables will be embedded from the ordinary character tables in characteristic 0. We proved that every f.g. projective kG-module lifts to a projective RG-lattice which is unique up to isomorphism, but these are not part of the simple character tables so we would need to figure out what they are.

The Brauer characters of indecomposable projectives are always obtained by taking the simple Brauer characters and taking linear combinations according to the columns of the Cartan matrix (see 10.1.4). Let us say how to do in the case of $G = S_4$, for which we already know the ordinary character table. Let $p = 2$; conjugacy classes of G are cycle types, which can be (abc), $(ab)(cd)$, or $(abcd)$. Only the three cycles have orders not divisible by 2, so there are 2 p-regular conjugacy classes and hence 2 simple reps. The nontrivial one lifts from the 2-dimensional module via the quotient map $S_4 \to S_3$. These lift to char 0, so we get $\phi_2((123)) = -1$ as in S_3.

Next, we can reduce all of the simple representations modulo 2, for the classes () and (123). This shows that the sign representation collapses to the trivial module. Next, observe that χ_3 collapses into one-three dimensional representation,

and χ_3 becomes a sum of $\chi_1 + \chi_2$. Now, we know that

$$\phi_3 = n_1\phi_1 + n_2\phi_2$$

and since the characters are visibly independent we must have $n_1 = 1$ and $n_2 = 1$. This let's us easily write down the decomposition matrix (we're using that we already have R-forms given by the lifts to characteristic 0), and so we may compute the Cartan matrix as $D^T D$, which gives

$$C = \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix}.$$

'Taking linear combinations' to obtain the indecomposable projectives table means exactly what we would expect, namely just use matrix multiplication to multiply the Cartan matrix by Brauer simple table. Note of course that we have to arrange the table and the matrix in the right order. For example, doing this to S_4 gives the following table:

g	()	(123)
ϕ_1	8	2
ϕ_2	8	-1

In particular, the projective cover P_k is an eight-dimensional representation in this case, and likewise with the the projective cover of the 2-dimensional rep.

Proposition 2.7.4 *Let (F, R, k) be a p-modular system. Suppose P and U are f.d. kG-modules with P projective. Then*

$$\dim \mathrm{Hom}_{kG}(P, U) = \frac{1}{|G|} \sum_{p\text{-}reg\ g\ \in\ G} \phi_P(g^{-1})\phi_U(g),$$

is satisfied.

Let p-reg$(G) \subset cc(G)$ be the set of p-regular conjugacy classes of G; since the characters are constant on the conjugacy classes, we can regard Brauer characters as elements of the $\mathbb{C}$-vector space $\mathbb{C}^{p\text{-reg}(G)}$ of functions from p-reg(G) to $\mathbb{C}$. Then define

$$\langle \phi, \psi \rangle := \frac{1}{|G|} \sum_{p\text{-reg}\, g \in G} \overline{\phi(g)}\psi(g)$$

Then we have $\langle \phi\theta, \psi \rangle = \langle \phi, \theta^* \psi \rangle$, where $\theta^*(g) = \overline{\theta(g)}$ is the complex conjugate. If ϕ and ψ are Brauer characters of representations, then everything is symmetric and dual. If P is projective, the result says that

$$\langle \phi_P, \phi_U \rangle = \dim \mathrm{Hom}_{kG}(P, U),$$

which is useful. We can now get orthogonality relations.

Theorem 2.7.5 *Let $S_1, \ldots, S_n$ and $P_{S_1}, \ldots, P_{S_n}$ be a complete list of non-isomorphic simple modules and their projective covers over kG. Then the*

simple Brauer characters and indecomposable projective Brauer characters

$$\{\phi_{S_1}, \ldots, \phi_{S_n}\} \qquad \{\phi_{P_{S_1}}, \ldots, \phi_{P_{S_n}}\}$$

both form bases for $\mathbb{C}^{p\text{-}reg(G)}$*. These are dual with respect to* $\langle -, - \rangle$*, so that*

$$\langle \phi_{P_{S_i}}, \phi_{S_j} \rangle = \delta_{i,j} \qquad \forall i, j,$$

and in particular $\langle -, - \rangle$ *is a non-degenerate form.*

In practice, we will compute and use this information for the Brauer character tables, so the better form is to assume that $\{g_1, \ldots, g_k\}$ are p-regular conjugacy class representatives. Then the definition is

$$\langle \phi, \psi \rangle = \sum_{i=1}^{k} \frac{1}{|C_G(g)|} \overline{\phi(g)} \psi(g)$$

because the number of conjugates of g is the index of $C_G(g)$ in G. The linear independence of the simple characters is very useful in practice, because it means that, after we reduce the char 0 representations modulo p, we can try to find linear relations $\phi_U = a\phi_{S_1} + b\phi_{S_2} + \cdots$ in the rows as related to the simple modules, which will tell us the multiplicity appearing in the simple decomposition.

Corollary 2.7.6 *Let* E/k *be an extension and* U *and* V *f.d.* kG*-modules.*

1. *If* S *and* T *are non-isomorphic simple* kG*-modules, then* $E \otimes_k S$ *and* $E \otimes_k T$ *have no composition factors in common.*
2. U *and* V *have teh same composition factors as* kG*-modules iff* $E \otimes_k U$ *have the same composition factors as* EG*-modules.*
3. U *and* V *have the same composition factors iff their Brauer characters are equal.*

Corollary 2.7.7 *Let* P *and* Q *be finite projective* kG*-modules, where* k *is a splitting field of characteristic* p*. Then* P *and* Q *are isomorphic iff they have the same composition factors (with multiplicities). The Cartan matrix is invertible, and the decomposition matrix has maximum rank. The e map in the cde triangle is injective.*

Let Φ be the table of Brauer character values of simple kG-modules, and Π the table of indecomposable projectives. Let B be the diagonal maatrix which has $1/(C_G(g))$ on the diagonal. We already know that

$$\overline{\Pi} B \Phi^T = I$$

by row orthogonality; but since each thing is ilnearly independent, the matrices are both invertible, and moreover $\Pi = C\Phi$. This gives us the column orthogonality relations:

Proposition 2.7.8 *Let Φ be the simple Brauer character matrix and Π the indecomposable projective matrix. Then $\overline{\Phi}^T \Pi = B$, where B has on the diagonal $|C_G(g_i)|$. In particular,*

$$\sum_{S \text{ simple}} \phi_S(g^{-1}) P_S(h) = \begin{cases} |C_G(g)| & g \sim h \\ 0 & \text{otherwise} \end{cases},$$

where $g \sim h$ means g is conjugate to h.

Note, as a final remark, that if ϕ_S is a simple charcater and χ_T is an ordinary char 0 character, then the Brauer character of a reduction of T is just the restriction of T, i.e. in the character table just cut off all columns which aren't labeled by p-regular elements. This is because of the relation we established between the lifts of particular characters.

Let us also note that if a simple character lifts to characteristic 0, that means that it is isomorphic to a reduction $S/\pi S$ for simple S. Oh, one more technique is to look for blocks of defect zero; so if the ordinary character has dimension n, an the higher power of p dividing thr gruop order also divides n, then the ordinary character remains simple upon reduction mod p. Note that a 1-dimensional character always remains simple upon reduction mod p, because it remains one-dimensional.

2.8 Chapter 11: Indecomposable Modules

Proposition 2.8.1 *Let U be an A-module. Direct sum decompositions*

$$U = U_1 \oplus \cdots \oplus U_n$$

of U as a sum of submodules are in one-to-one correspondence with $1_U = e_1 + \cdots e_n$ where 1_U is in $\mathrm{End}_A(U)$ as a sum of orthogonal idempotents. e_i is obtained as the composite

$$U \to U_i \to U,$$

and U_i is obtained as $e_i(U)$. U_i is an indecomposable summand iff e_i is primitive.

A corollary is than A-module U is indecomposable iff the ring $\mathrm{End}_A(U)$ has only one non-trivial idempotent, namely 1_U.

Theorem 2.8.2 *The following are equivalent for a ring B.*

1. *B is a local ring (i.e. has a unique maximal left ideal).*
2. *B has a unique maximal right ideal.*
3. *$B/\operatorname{Rad}(B)$ is a division ring.*
4. *The non-units form a left (or right, or two-sided) ideal.*

We saw earlier that if G is p-group and $\operatorname{char} k = p > 0$, then kG is a local ring because its radical is IG, and $kG/IG = k$.

Proposition 2.8.3 *The only idempotents in a local ring are 0 and 1. Let B is an R-algebra that is a finitely generated R-moduel, for R a complete DVR or a field. If the only idempotents in B are 0 and 1, then B is a local ring.*

Corollary 2.8.4 *Let U be an A-module, and assume R is a complete DVR or field.*

1. *If $\operatorname{End}_A(U)$ is a local ring, then U is indecomposable.*
2. *Assume further that A is an R-algebra and U is finitely generated over R. Then U is indecomposable iff $\operatorname{End}_A(U)$ is a local ring.*

In particular, if R is a complete DVR or a field, then an RG-module U is indecomposable if and only if $\operatorname{End}_{RG}(U)$ is a local ring. Now, we turn to the Krull-Schmit theorem, which is the goal of this part.

Theorem 2.8.5 *Let U be an A-module with two decompositions*

$$U = U_1 \oplus \cdots U_r = V_1 \oplus V_s$$

with each $\operatorname{End}_A(U_i)$ a local ring, and each V_i an indecomposable A-module. Then $r = s$ and there exists $\sigma \in S_r$ such that $V_{\sigma(i)} \cong U_i$.

Corollary 2.8.6 (Krull-Schmidt) *Let R be a complete DVR or a field, and let U be a finitely generated RG-module with*

$$U = U_1 \oplus \cdots \oplus U_r = V_1 \oplus \cdots \oplus V_s$$

two decompositions into indecomposable RG-modules. Then $r = s$ and there exists $\sigma \in S_r$ such that $U_i = V_{\sigma(i)}$ for all i.

Let k be char p and let $H \trianglelefteq G$ be normal Sylow p-subgroup. Then $G = H \rtimes K$ for $H = \langle x \rangle$ is cyclic of order p^n and K is group with order coprime to p. We know in this case that all indecomposable projectives are uniserial (and thus also the injectives). Recall that a *Nakayama algebra* is an algebra in which all indecomposable projectives and injectives are uniserial.

Proposition 2.8.7 *Let A be a finite-dimensional Nakayama algebra over k. Then every indecomposable module is uniserial, and is a homomorphic image of an indecomposable projective. Thus, the homomorphic images of indecomposable projectives dform a list of indecomposable modules.*

Corollary 2.8.8 *Let k be char p, G has a normal cyclic Sylow p-subgroup H so that $G = H \rtimes K$. The number of isomorphism classes of indecomposable kG-modules is equals p^n times the number of isomorphism classes of simple kK-modules.*

For example, let us consider $D_{2p} = \mathbb{Z}_2 \rtimes \mathbb{Z}_p$, with $p > 2$ so that p is coprime to the order of $\mathbb{Z}_2$, over a field of characteristic p. When p is odd, D_{2p} has $2p$ indecomposable modules.

Definition 2.8.9 Let $H \leq G$ and R a commutative ring.

1. An RG-module is *H-free* if it is of the form $V\uparrow_H^G$ for some RH-module V.
2. It is *H-projective* or *projective relative to H* if it is a direct summand of an H-free module.

So, $RG = R\uparrow_1^G$ is 1-projective, and thus any projective RG-module is 1-projective. Any module is G-projective because it is its own induction. But, every 1-projective RG-module is not projective (e.g. when R is not a field.).

Let $H \leq G$ and U be an RG-module. We define a *relative trace map*

$$\mathrm{tr}_H^G \colon U^H \to U^G$$

on the fixed-point subspaces by first choosing $[G/H] = \{g_1, \ldots, g_n\}$ and then setting $\mathrm{tr}_H^G(u) \coloneqq \sum_i g_i \cdot u$. There are two other operations which we can define:

$$\mathrm{res}_H^G \colon U^G \hookrightarrow U^H \qquad c_g \colon U^H \to U^{({}^gH)}$$

The first is just viewing a G-fixed point as an H one, and the second is the mapping $u \mapsto g \cdot u$.

Lemma 2.8.10 *Let U be an RG-module and $K \leq H \leq H$, and $L \leq G$.*

1. *$\mathrm{tr}_H^G \colon U^G \to U^H$ is well-defined, and in particular independent of a choice of left-coset representatives.*
2. *We have the formulas:*

$$\mathrm{tr}_H^G \mathrm{tr}_K^H = \mathrm{tr}_K^G \qquad \mathrm{res}_K^H \mathrm{res}_H^G = \mathrm{res}_K^G.$$
$$c_g \mathrm{tr}_K^H = \mathrm{tr}_{({}^gK)}^{({}^gH)} c_g \qquad c_g \mathrm{res}_K^H = \mathrm{res}_{({}^gK)}^{({}^gH)} c_g$$
$$\mathrm{tr}_H^H = 1 = \mathrm{res}_H^H$$

3. *$\mathrm{tr}_K^G \mathrm{res}_K^H$ is multiplication by $[H : K]$.*
4. *We have a Mackey formula*

$$\mathrm{res}_L^G \mathrm{tr}_H^G = \sum_{g \in [LG/H]} \mathrm{tr}_{L \cap {}^gH}^L \mathrm{res}_{L \cap {}^gH}^{{}^gH} c_g$$

as in induction/restriction.