Topics in Algebraic Geometry

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1 Smoothness

The goal of this section is to briefly discuss 'smoothness' in algbraic geometry, because I want to unify the notions for a variety over an algebraically closed field, a scheme over an arbitrary field, and the sheaf of Kahler differentials. Ultimately we should have enough to discuss unramified and étale morphisms in some detail. The sources for this section are both [GW23, Ch. 17] and [STK].

1.1 Smoothness in commutative algebra

1.1.1 Ala stacks project

Here we discuss the module of Kahler differentials in commutative algebra. Related but very different material on regularity (i.e. regular local rings) will be elsewhere.

2 Divisors

2.1 Cartier divisors over integral schemes

Here we present the basic foundational material on divisors, and restrict to the case of an integral scheme. Let us recall some standard definitions from ring theory, which will be useful momentarily.

2.1.1 Recap: Invertible fractional ideals

Definition 2.1.1 Let A be a domain, and $K \coloneqq \text{Frac}(A)$. Then K is an A-module, and any A-submodule of K is called a *fractional ideal*. Given I, J fractional ideals, we define their product

$$IJ \coloneqq \{\sum_i x_i y_i : x_i \in I, y_i \in J\} \subset K,$$

so clearly IJ is a fractional ideal. This is a commutative associative product. $(1) = A \subset K$ is a fractional ideal which is the 'unit ideal', and $I(1) = (1)I = I$ for all fractional ideals I. I is *invertible* if there is a fractional ideal I^{-1} such that $II^{-1} = (1)$.

Lemma 2.1.2 *Let A be a domain and I a fractional ideal of A. The following are equivalent:*

1. *I is invertible.*
2. *I is a finite projective module.*
3. *I is a projective module.*

Proof (1) $\Rightarrow$ (2) Choose $a_i \in I$ and $b_i \in I^{-1}$ such that $\sum_i a_i b_i = 1$. If $M \xrightarrow{\phi} I \to 0$ is a surjection, then $\phi(x_i) = a_i$ for some $x_i \in M$. Note that for all $a \in I$, $xb_i \in R$, thus

$$\psi(a) = (ab_1)x_1 + \cdots + (ab_n)x_n$$

is a well-defined A-module map $I \to M$ and $\phi(\psi(a)) = a$. Note that the image of ψ is contained in $\langle x_1, \ldots, x_n \rangle$, so in the special case $M = I$ with $\phi = \mathrm{Id}$, this shows that $I \subset \langle a_1, \ldots, a_n \rangle \subset I$.

(3) $\Rightarrow$ (1): If I is projective and generated by $\{a_\alpha\}_\alpha$, then the surjection $\bigoplus_\alpha A a_\alpha \twoheadrightarrow I$ splits, and we get a right inverse g given by components $g = (g_\alpha \colon I \to A)_\alpha$. Because I is fractional, we can choose $a \in A \cap I$ nonzero, and define $b_\alpha = a^{-1} g_\alpha(a) \in \mathrm{Frac}(A)$, only finitely many of which are nonzero. Let $J = \langle b_\alpha \rangle_\alpha$. Then for $(x/y) \in I$

$$g_\alpha(x/y) = \frac{ay}{ay} g_\alpha(x/y) = \frac{1}{ay} g_\alpha(ax) = \frac{x}{ay} g_\alpha(a) = (x/y) b_\alpha.$$

In other words, $z b_\alpha \in A$ for $z \in I$, so $IJ \subset A$. On the other hand, the splitting condition says that

$$z = \sum_\alpha g_\alpha(z) a_\alpha = \sum_\alpha a_\alpha b_\alpha z$$

for any $x \in I$, so choosing z nonzero we conclude $1 \in IJ$. □

Note that if I is an invertible fractional ideal, then since I^{-1} is flat, $I \otimes I^{-1} \xrightarrow{\sim} IJ = A$, so I is locally free of rank 1. This identifies invertible fractional ideals with the line bundles contained in the fraction field. The study of divisors of an integral scheme is a globalization of this notion.

2.1.2 Cartier divisors and associated line bundles

In this section let X be an integral scheme and write $\mathcal{K}_X$ for the sheaf $U \mapsto \mathrm{Frac}(\Gamma(U, \mathcal{O}_X))$. We will identify $\mathcal{K}_X$ with the constant sheaf at the function field $K(X)$, which is literally constant as X is connected. We always have the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{K}_X$, which means fixing an inclusion $\Gamma(U, \mathcal{O}_X) \subset K(X)$ for all opens U, so we can ask whether an element of $K(X)$ is a section over U.

Definition 2.1.3

1. A *Cartier divisor* D on X is an equivalence class of a tuple $(U_i, f_i)_i$ where $\{U_i\}$ is an open cover of X and $f_i \in \mathcal{K}_X(U)^\times = K(X)^\times$ satisfying the condition that $f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)^\times$ for all i, j. The equivalence relation is defined by $(U_i, f_i) \sim (V_j, g_j)$ if and only if $f_i g_j^{-1} \in \Gamma(U_i \cap V_j, \mathcal{O}_X)^\times$ for all i, j.

2. The set of divisors is denoted $\mathrm{Div}(X)$. It becomes an abelian group as follows: if D is given by (U_i, f_i) and E is given by (V_j, g_j), then $D+E$ is given by $(U_i \cap V_j, f_i g_j)$.
3. A divisor is *principal* if it is given by (X, f) for some $f \in K(X)^\times$. Thus there is a group homomorphism $\mathrm{div}\colon K(X)^\times \to \mathrm{Div}(X)$ sending $f \mapsto [(X, f)]$, whose image is the subgroup of principal divisors. The quotient by the principal divisors is $\mathrm{DivCl}(X)$, the *divisor class group* of X.

Many comments are in order regarding the definition. First of all, it is easy to check that the addition is well-defined, associative, and respects the relation. A useful fact is that if a divisor D is given by $(U_i, f_i)_{i\in I}$ and and $J \subset I$ indexes a subcover of X, then $(U_j, f_j)_{j\in J}$ gives the same divisor. Indeed, it clearly induces a divisor, and $f_i f_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)^\times$.

Therefore, if (U_i, f_i) and (U_i, g_i) induce divisors D and E via the same open cover, then $D+E$ is given by $(U_i, f_i g_i)$. So, e.g., if D is given by (U_i, f_i), then $-D$ is given by (U_i, f_i^{-1}), and then $D + (-D) = (U_i, 1)$, which is equivalent to the principal divisor $(X, 1)$, forming the identity element of the group.

We have the defining exact sequence

$$1 \to \Gamma(X, \mathcal{O}_X)^\times \to K(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{DivCl}(X) \to 0.$$

Eventually, we will want to relate the divisor class group of X to the Picard group. Let D be a divisor given by (U_i, f_i); we define a sheaf $\mathcal{O}_X(D)$ by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in K(X) : f f_i \in \Gamma(U \cap U_i, \mathcal{O}_X)\}.$$

This is a very useful construction and an easy way of writing down line bundles, as we will see.

Lemma 2.1.4 *We have the following properties:*

1. *$\mathcal{O}_X(D)$ is a well-defined sub-O_X-module of $\mathcal{K}_X$ independent of the choice of representation of D.*
2. *$\mathcal{O}_X(D)$ is a line bundle on X. If D is given by (U_i, f_i), then $\mathcal{O}_X(D)$ is trivial on U_i with generating section f_i^{-1}.*
3. *$\mathcal{O}_X(D)$ is trivial if and only if D is a principal divisor.*
4. *$\mathcal{O}_X(D+E) \cong \mathcal{O}_X(D) \otimes \mathcal{O}_X(E)$.*

Proof (1) Suppose $(U_i, f_i) \sim (V_j, g_j)$. If $f f_i \in \mathcal{O}_X(U_i \cap V)$ for all i, then

$$f g_j = f f_i (f_i^{-1} g_j) \in \mathcal{O}_X(U_i \cap V_j \cap V)$$

for all i, hence $f g_j \in \mathcal{O}_X(V_j \cap V)$ since $\mathcal{O}_X$ is a subsheaf. By symmetry, the inclusion in the other direction holds, showing $\Gamma(U, \mathcal{O}_X(D))$ is independent of the choice of representation.

It is clearly a subpresheaf of $\mathcal{K}_X$; to check it's a sheaf, let $f \in K(X)$ and suppose $f \in \Gamma(V_j, \mathcal{O}_X(D))$ for all j where $\bigcup_j V_j = V$. Then $ff_i \in \mathcal{O}_X(U_i \cap V_j)$ for each j, and thus $ff_i \in \mathcal{O}_X(U_i \cap V)$, i.e. $f \in \Gamma(V, \mathcal{O}_X(D))$. Finally, it is an $\mathcal{O}_X$-submodule of $\mathcal{K}_X$, because if $s \in \mathcal{O}_X(U)$ and $f \in \Gamma(U, \mathcal{O}_X(D))$, then $sf \in \Gamma(U, \mathcal{O}_X(D))$, since $sff_i \in \Gamma(U \cap U_i, \mathcal{O}_X)$ for all i.

(2) Note that $f_j^{-1} \in \Gamma(U_j, \mathcal{O}_X(D))$, because $f_i f_j^{-1} \in \mathcal{O}_X(U_i \cap U_j)$ for all i. So there is a corresponding map

$$\mathcal{O}_X|_{U_j} \to \mathcal{O}_X(D)|_{U_j}$$

given by multiplication by f_j^{-1}, which is clearly injective as there are no zero-divisors. If $g \in \Gamma(V, \mathcal{O}_X(D))$ for $V \subset U_j$, then $gf_j \in \mathcal{O}_X(V)$, hence is the image of gf_j under multiplication by f_j^{-1}.

(3) If D is principal, given by (X, f), then $\mathcal{O}_X(D)$ is trivial on all of X by the generating section f^{-1}, by (2). Conversely, if $\mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X(D)$ is an isomorphism, the image of 1 is an element $f \in \Gamma(X, \mathcal{O}_X(D))$. On U_i, we have a composite isomorphism

$$\mathcal{O}_X|_{U_i} \xrightarrow{-\cdot f} \mathcal{O}_X(D)|_{U_i} \xrightarrow{-\cdot f_i} \mathcal{O}_X|_{U_i},$$

where the second isomorphism comes from (2). It follows that $ff_i \in \mathcal{O}_X(U_i)^\times$ for all i, which shows that $D = \operatorname{div}(f^{-1})$.

(4) There is a morphism from the tensor product of the presheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(E)$ to $\mathcal{O}_X(D+E)$ sending $f \otimes g \mapsto fg$, well-defined because $(fg)(f_i g_j) = (ff_i)(gg_j) \in \mathcal{O}_X(U_i \cap V_j)$. This induces an associated multiplication map

$$\mathcal{O}_X(D) \otimes \mathcal{O}_X(E) \to \mathcal{O}_X(D+E)$$

which we claim is an isomorphism. Indeed, consider the restriction to the sets of the form $W_{ij} \coloneqq U_i \cap V_j$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{W_{ij}} \otimes \mathcal{O}_{W_{ij}} & \xrightarrow{\quad\cong\quad} & \mathcal{O}_{W_{ij}} \\ \cong\downarrow & & \downarrow\cong \\ \mathcal{O}_X(D)|_{W_{ij}} \otimes \mathcal{O}_X(E)|_{W_{ij}} & \longrightarrow & \mathcal{O}_X(D+E)|_{W_{ij}} \end{array}$$

Here, the top map is the canonical isomorphism, and the vertical maps are induced by the local trivializations of $\mathcal{O}_X(D)$, $\mathcal{O}_X(E)$, and $\mathcal{O}_X(D+E)$. The diagram commutes because the top is sectionwise multplication, the right is $h \mapsto hf_i^{-1}g_j^{-1}$, and the left is $(f \otimes g) \mapsto ff_i^{-1} \otimes gg_j^{-1}$. □

This lemma already gives us a lot of information. For example, it says that the map

$$\operatorname{Div}(X) \to \operatorname{Pic}(X) \qquad D \mapsto \mathcal{O}_X(D)$$

is a well-defined group homomorphism, and factors through a monomorphism

$$0 \to \operatorname{DivCl}(X) \to \operatorname{Pic}(X).$$

Let us briefly see how to use this formalism to discuss the line bundles on $\mathbb{P}^n$, before continuing to develop the theory of Cartier divisors.

2.1.3 Twisting sheaves on projective space

Fix a domain A. For the purposes of this section, we will consider $\mathbb{P}^n = \mathbb{P}^n_A$ to be a scheme equipped with an affine open cover $U_0, \dots, U_n$ where

$$U_i = \operatorname{Spec} A\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right] \cong \mathbb{A}^n_A,$$

where we set all of the embeddings $U_i \cap U_j \to U_i$ by specifying that everything factors through inclusions into the field

$$K(\mathbb{P}^n) = \operatorname{Frac}(A)\left(\frac{x_i}{x_j} \mid i, j\right).$$

It is clear that $\mathbb{P}^n$ is irreducible and that this is the fraction field. The above specification means that $U_i \cap U_j$ should have global sections given by the subring of $K(\mathbb{P}^n)$ generated by $x_k/x_i, x_k/x_j$ for all $k = 0, \dots, n$, and the restriction maps correspond to the canonical inclusions. Let us also write $K = \operatorname{Frac}(A)$.

The most important line bundles in algebraic geometry are the so-called *Serre twisting sheaves on* $\mathbb{P}^n$. Let us use our definitions from the previous section to construct them.

Let H be the divisor on $\mathbb{P}^n$ given by $(U_i, x_0/x_i)_{i=0}^n$, which indeed defines $H \in \operatorname{Div}(X)$ because $(x_0/x_i)(x_j/x_0) = x_j/x_i$, which is a unit in

$$\Gamma(U_i \cap U_j, \mathbb{P}^n) = A[x_k/x_i, x_k/x_j : k = 0, \dots, n].$$

We write $\mathcal{O}_{\mathbb{P}^n}(1) \coloneqq \mathcal{O}_{\mathbb{P}^n}(H)$. By the remarks of the previous section, for each $d \in \mathbb{Z}$, we have the divisor $dH = H + \cdots + H$, given by $(U_i, (x_0/x_i)^d)$. We write $\mathcal{O}_{\mathbb{P}^n}(d) \coloneqq \mathcal{O}_{\mathbb{P}^n}(dH)$. Since $\operatorname{Div}(X) \to \operatorname{Pic}(X)$ is a homomorphism,

$$\mathcal{O}_{\mathbb{P}^n}(d) \otimes \mathcal{O}_{\mathbb{P}^n}(e) \cong \mathcal{O}_{\mathbb{P}^n}(d+e) \qquad d, e \in \mathbb{Z}.$$

Moreover, we can say that these are all nontrivial and distinct:

Lemma 2.1.5 *The divisor dH is not principal for each $d \neq 0$.*

Proof We may assume without loss that $d \geq 1$. Suppose dH is given by (X, f) for some $f \in K(\mathbb{P}^n)$. By definition, $f x_i^d / x_0^d \in \mathcal{O}_{\mathbb{P}^n}(U_i)^\times = A^\times$, so in particular taking $i = 0$ shows $f = a \in A^\times$. But then taking $i = 1$, this shows $a x_1^d / x_0^d = b \in A^\times$, which is obviously a contradiction for $d > 0$. □

Corollary 2.1.6 *The line bundles $\{\mathcal{O}_{\mathbb{P}^n}(d) : d \in \mathbb{Z}\}$ are all distinct (and nontrivial for $d \neq 0$), and form a free abelian subgroup of $\operatorname{Pic}(\mathbb{P}^n)$ of rank 1, generated by $\mathcal{O}_{\mathbb{P}^n}(\pm 1)$.*

We will say more about these line bundles shortly.

2.1.4 Invertible fractional ideals and the Picard group

Recall that we call an $\mathcal{O}_X$-module $\mathcal{F}$ *invertible* if there is another $\mathcal{O}_X$-module $\mathcal{G}$ such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$. When X is a scheme, then this condition implies that $\mathcal{F}$ is in fact locally free of rank one, i.e. a line bundle, and vice versa. Let us keep the same conventions regarding an integral scheme X as in the last two sections.

Definition 2.1.7 A *(quasicoherent) fractional ideal sheaf* of $\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_X$-submodule of $\mathcal{K}_X$. If $\mathcal{I}, \mathcal{J}$ are fractional ideal sheaves, then $\mathcal{I}(U) \subset K(X)$ and $\mathcal{J}(U) \subset K(X)$ are fractional ideals of $\mathcal{O}_X(U)$, so we can take their product $\mathcal{I}(U)\mathcal{J}(U)$ as fractional ideals. Then $U \mapsto \mathcal{I}(U)\mathcal{J}(U) \subset K(X)$ defines a subpresheaf of $\mathcal{K}_X$, and sheafifying we obtain a new fractional ideal sheaf $\mathcal{I}\mathcal{J} \subset \mathcal{K}_X$.

The case we care about is that of *invertible fractional ideals* in $\mathcal{O}_X$.

Lemma 2.1.8 *The following are equivalent for a fractional ideal $\mathcal{I} \subset \mathcal{K}_X$:*

1. *$\mathcal{I}$ is invertible as an $\mathcal{O}_X$-module.*
2. *$\mathcal{I}$ is a line bundle over X.*
3. *There exists a fractional ideal $\mathcal{J}$ such that $\mathcal{I}\mathcal{J} = \mathcal{O}_X$.*

Proof We know the first two are equivalent. For (3) $\Rightarrow$ (2), we know that $\mathcal{I}$ is locally equal to $\widetilde{I}$ where I is a fractional ideal, hence it is locally a line bundle by our previous results. On the other hand, suppose $\mathcal{I}$ is a line bundle. Define $\mathcal{J}$ by

$$\Gamma(U, \mathcal{J}) = \{f \in K(X) : f \cdot \mathcal{I}(U) \subset \mathcal{O}_X(U)\}.$$

Then this is an $\mathcal{O}_X$-submodule of $\mathcal{K}_X$, and both $\mathcal{I}\mathcal{J} \subset \mathcal{O}_X$ and $\mathcal{O}_X \subset \mathcal{I}\mathcal{J}$ can be checked after restricting to affine opens, in which case it's true by our previous results. □

Viewing our line bundles in this way helps us understand slightly better the operations addition of divisors. So, for $D \in \mathrm{Div}(X)$, the associated line bundle $\mathcal{O}_X(D)$ is canonically an invertible $\mathcal{O}_X$-submodule of $\mathcal{K}_X$. We know how to compute the inverse either via the formula above, or by taking $\mathcal{O}_X(-D)$.

Given an invertible fractional ideal $\mathcal{I} \subset \mathcal{K}_X$, there is an open cover $\{U_i\}$ and sections $f_i \in K(X)^\times$ such that $\mathcal{I}|_{U_i} = f_i \mathcal{O}_{U_i}$, so this corresponds to a divisor $D = (U_i, f_i^{-1})$ as before. This gives an isomorphism between $\mathrm{Div}(X)$ and invertible fractional ideals of $\mathcal{O}_X$.

Definition 2.1.9 Let D be a divisor on X given by (U_i, f_i). D is *effective* if $f_i \in \Gamma(U_i, \mathcal{O}_X)$ for each i. We write $D \leq E$ if $E - D$ is effective.

By definition, D is effective if and only if $1 \in \Gamma(U, \mathcal{O}_X(D))$ for all U, i.e. iff $\mathcal{O}_X \subset \mathcal{O}_X(D)$. Therefore, being effective is independent both of the choice of

representation and of the divisor class. Observe that

$$D \leq E \iff \mathcal{O}_X \subset \mathcal{O}_X(E-D) = \mathcal{O}_X(E)\mathcal{O}_X(-D),$$

and we can multiply both sides by $\mathcal{O}_X(D)$, which preserves the inclusion, to obtain that $D \leq E$ iff $\mathcal{O}_X(D) \subset \mathcal{O}_X(E)$. It follows that $\mathrm{Div}(X)$ and $\mathrm{DivCl}(X)$ are both partially ordered, and the bijection with invertible fractional ideals preserves the ordering.

Given a divisor D given by (U_i, f_i), we write

$$\operatorname{Supp} D = \{x \in X : (f_i)_x \notin \mathcal{O}_{X,x}^\times \text{ for some } i \text{ with } x \in U_i\}.$$

If $x \in X \setminus \operatorname{Supp} D$, then $(f_i)_x \in \mathcal{O}_{X,x}^\times$ for all i with $x \in U_i$, hence there exists a nbhd V of x such that $f_i \in \Gamma(V, \mathcal{O}_X)^\times$ for $x \in U_i$. Thus $\operatorname{Supp} D$ is closed. It is easy to see that this is independent of the chosen representation of D.

Now, suppose D is an effective divisor, i.e. $D \geq 0$. Then $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ is (in particular) a quasicoherent ideal sheaf, and hence we denote the closed subscheme $V(\mathcal{O}_X(-D))$ defined by this ideal by D as well. If $i\colon D \to X$ is the corresponding closed immersion, then we have an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to i_*\mathcal{O}_D \to 0.$$

Note that all closed subschemes whose ideals are invertible arise in this sequence. If D is represented by (U_i, f_i), then D is defined by $D \cap U_i = V_{U_i}(f_i)$.

Theorem 2.1.10 *The map* $\mathrm{Div}(X) \to \mathrm{Pic}(X)$ *is surjective, and hence* $\mathrm{DivCl}(X) \xrightarrow{\sim} \mathrm{Pic}(X)$.

Proof Let $\mathcal{L}$ be a line bundle on X; we want to show that $\mathcal{L}$ embeds into $\mathcal{K}_X$. Suppose that we already have an embedding on a (dense) open set U, i.e. $\mathcal{L}|_U \hookrightarrow \mathcal{K}|_U$. Then we claim it can be extended to all of X:

1. If V is an open set on which $\mathcal{L}|_V \cong \mathcal{O}_X|_V$, then there is a unique extension of the embedding from $U \cap V$ to V. Indeed, embeddings $\mathcal{O}_X|_{U\cap V} \to \mathcal{K}_X$ are just sections of $K(X)^\times$, which obviously extend uniquely to a section $\mathcal{L}_V \cong \mathcal{O}_X|_V \to \mathcal{K}_X|_V$.
2. So, let $\phi\colon \mathcal{L}|_U \hookrightarrow \mathcal{K}_X|_U$ be the embedding. For each V on which $\mathcal{L}$ is trivial, there is a unique extension $\widetilde{\phi}_V\colon \mathcal{L}|_V \to \mathcal{K}|_V$ such that $\widetilde{\phi}_V|_{U\cap V} = \phi|_{U\cap V}$. For V, V' different such opens, clearly $\widetilde{\phi}_V|_{V\cap V'} = \widetilde{\phi}_{V'}|_{V\cap V'}$, so the fact that $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{K}_X)$ is a sheaf implies that we have an extension, which is injective since it is locally injective.

Therefore, by choosing a nonempty affine open we may assume $X = \operatorname{Spec} A$ is affine for A a domain, and $\mathcal{L} = \widetilde{M}$ for M a rank-1 projective A-module. We wish to find an A-linear injection $M \to K(X) = \mathrm{Frac}(A)$. Indeed, $M \to M \otimes_A K(X)$ is injective since M is flat, and $M \otimes_A K(X) \cong K(X)$ as $K(X)$-modules since both are 1-dimensional vector spaces, so in particular they are

isomorphic as A-modules and the composite

$$M \to M \otimes_A K(X) \xrightarrow{\sim} K(X)$$

solves the problem. □

3 Blow-ups and Normalization

3.1 In the classical setting

3.1.1 Blow-ups

3.1.2 Normalization

4 Examples

This section should contain some interesting examples and counterexamples in algebraic geometry. These are divided into two categories: First are scheme-theoretic pathologies, i.e. things which go wrong because a scheme is not a variety over an algebraically closed field. The other category consists of minimal nontrivial examples of phenomena in the nicest cases, and full details are given on how to prove them.

4.1 Scheme-theoretic examples

4.1.1 Closed points

Let X be a space and $x, y \in X$. Recall that we say x is a *specialization* of y - and write $y \rightsquigarrow x$ - if $x \in \overline{\{y\}}$. A lot of scheme-theoreic properties are described as pointwise conditions on the local rings (i.e. reduced, regular, normal, etc.), and we often want to check these on the closed points of the schemes. This is because the geometry of varieties are essentially fully determined by their closed points, so for a property to be determined by the closed points is a niceness condition.

Usually, the approach of reducing to closed points works as follows: we would like to be able to specialize arbitrary points to closed points. In the affine case, this works well: take $X = \operatorname{Spec}(A)$, and recall that $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Spec}(A)$. Hence,

$$\mathfrak{p} \rightsquigarrow \mathfrak{q} \iff \mathfrak{q} \in V(\mathfrak{p}) \iff \mathfrak{p} \subset \mathfrak{q},$$

which means that every nbhd of $\mathfrak{q}$ is a nbhd of $\mathfrak{p}$ as well. Moreover, $A \backslash \mathfrak{q} \subset A \backslash \mathfrak{p}$, so $\mathfrak{p}$ corresponds to a prime $\mathfrak{p}A_{\mathfrak{q}}$ in $A_{\mathfrak{q}}$, and $A_{\mathfrak{p}} = (A_{\mathfrak{q}})_{\mathfrak{p}A_{\mathfrak{q}}}$ is the localization of $A_{\mathfrak{q}}$ at this prime. So, in the affine case, for all $\mathfrak{p}$ there's an $\mathfrak{m}$ maximal such that $\mathfrak{p} \rightsquigarrow \mathfrak{m}$, and $A_{\mathfrak{p}}$ is obtained as a localization of $A_{\mathfrak{m}}$. Thus if the property is reasonably stable under localization, then it can be checked on closed points.

The main obstruction to this technique working is the existence of closed points in the first place. We now construct an example of a scheme which has no closed points. Let $K = k(x_1, x_2, \dots)$. For each i, we have a discrete valuation

$$\nu_i \colon k(x_i, x_{i+1}, \dots,)^\times \to \mathbb{Z}$$

given by the divisibility by x_i, where we note that the residue field of v_i is $k(x_{i+1}, x_{i+2}, \dots)$. There is a composite lexicographic valuation $\nu \colon K^\times \to \mathbb{Z}^{(\mathbb{N})}$, where for each f, define the ith component $\nu(f)_i$ as follows: set $g_1 = f$, and

inductively let

$$g_{i+1} = \overline{x_i^{-v_i(g_i)} g_i} \in k(x_{i+1}, x_{i+2}, \dots),$$

where taking the residue is possible because

$$v_i(x_i^{-v_i(g_i)} g_i) = -v_i(g_i) + v_i(g_i) = 0.$$

Then we set $\nu(f) = \nu_i(g_i)$.

Lemma 4.1.1 *This is a well-defined valuation on K.*

Proof Since for all i, $x_i^{-\nu(g_i)} g_i$ has valuation zero, g_{i+1} is a nonzero element of the residue field, hence all of the valuations make sense. Note that if f depends on $x_1, \dots, x_n$, then g_{n+1} doesn't depend on any of these variables and hence must be a constant, so $\nu(f)$ has

$$0 = \nu(f)_{n+1} = \nu(f)_{n+2} = \cdots$$

and gives a well-defined element of $\mathbb{Z}^{(\mathbb{N})}$. Considering ff', we claim the corresponding g is $g_i g_i'$, by induction. Indeed,

$$x_i^{-\nu_i(g_i g_i')}(g_i g_i') = x_i^{-\nu_i(g_i)} g_i \cdot x_i^{-\nu_i(g_i')} g_i',$$

so taking residues the claim follows. Thus ν is a group homomorphism. We skip the proof of the other valuation axiom. Let us also note that the value group is all of $\mathbb{Z}^{(\mathbb{N})}$, because the nth basis element is the valuation of x_n. □

For example, here's how to compute this valuation: consider

$$f = \frac{x_1 x_3^3 + x_2 x_3 + x_3^2}{x_1 x_2^4 + x_1^2 + x_2} \in K.$$

Then $\nu_1(f) = 0$, so we reduce (set $x_1 = 0$) to obtain

$$g_2 = \frac{x_2 x_3 + x_3^2}{x_2},$$

i.e. $\nu(f)_2 = \nu_2(g_2) = -1$, and then

$$g_3 = x_2 g_2 = x_3^2,$$

so $\nu(f)_3 = 2$. Therefore

$$\nu(f) = \big(0, -1, 2, 0, 0, \dots\big).$$

Now, let R_ν be the associated valuation ring. If something has valuation zero, then by induction on n it is independent of x_n, hence $R_\nu^\times = k^\times$, i.e. $\kappa(\nu) = k$. The primes of R_ν, which correspond to convex subgroups of $\mathbb{Z}^{(\mathbb{N})}$, are exactly given by

$$\mathfrak{p}_0 = (0), \qquad \mathfrak{p}_i \coloneqq (x_1, x_2, \dots, x_i), \qquad \mathfrak{m} = (x_1, x_2, \dots),$$

so we get a chain

$$\mathfrak{p}_0 \rightsquigarrow \mathfrak{p}_1 \rightsquigarrow \mathfrak{p}_2 \rightsquigarrow \cdots \rightsquigarrow \mathfrak{m}$$

of specializations. The open subscheme $\mathrm{Spec}(R_\nu) \setminus \{\mathfrak{m}\}$ contains, we claim, no closed points. Indeed, the closure of $\{\mathfrak{p}_i\}$ always contains $\mathfrak{p}_{i+1}$, hence no point in the subspace topology is closed. We've shown:

Proposition 4.1.2 *There exists a scheme X (over any field k) with no closed points.*

Now, let us see the weakest condition to fix this pathology.

Lemma 4.1.3 *Let X be a nonempty T_0 space which is quasicompact. Then X has a closed point.*

Proof Consider the collection $\mathcal{C}$ of closed subsets not including the empty set under reverse inclusion. It is nonempty, and the arbitrary intersection of a decreasing sequence $\{F_i : i \in I\}$ is closed. If it were empty, then $\{X \setminus F_i : i \in I\}$ would have a finite subcover, but a finite decreasing intersection of nonempty subsets must be nonempty.

By Zorn's lemma, let $C \subset X$ be a minimal nonempty closed subset, and suppose there were two points $x, y \in C$ and an open set U such that $x \in U$ and $y \notin U$. Then $C \setminus U$ would be a strictly smaller nonempty closed subset. □

Corollary 4.1.4 *Let X be a quasicompact scheme. Every nonempty closed subset of X contains a closed point of X. In particular, every point specializes to a closed point.*

Proof A closed subset of X is a quasicompact T_0 space and hence has a closed point, which will be a closed point of X. If $x \in X$, then $\overline{\{x\}}$ is closed. □

Corollary 4.1.5 *Let X be a quasicompact scheme. Then X is normal if and only if $\mathcal{O}_{X,x}$ is an integrally closed domain for each closed point $x \in X$.*

Proof The condition is obviously necessary, and if y is an arbitrary point, pick x closed with $y \rightsquigarrow x$. Since $\mathcal{O}_{X,y}$ is a localization of $\mathcal{O}_{X,x}$, which is an integrally closed domain, $\mathcal{O}_{X,y}$ is also an integrally closed domain. □

Corollary 4.1.6 *Let X be a quasicompact scheme. Then X is reduced if and only if $\mathcal{O}_{X,x}$ is a reduced ring for each closed point $x \in X$.*

Proof For the same reason as above, we have to show that if A is reduced then $A[S^{-1}]$ is reduced. Indeed, if $(a/s)^n = 0$, then $a^n = 0$ in $A[S^{-1}]$, so there exists s' such that $s'a^n = 0$. In particular $(s'a)^n = 0$, so $s'a = 0$, thus $a = 0$ in the localization. □

The same property does *not* in general work for regularity, but I don't know any example. In any case, the following proof does not work for many schemes. Recall that a scheme X is J_1 if its singular locus $\mathrm{Sing}(X)$ is closed.

Corollary 4.1.7 *Let X be a Noetherian scheme. Then X is regular iff $\mathcal{O}_{X,x}$ is a regular local ring for all closed points $x \in X$.*

Proof For any point $y \in X$, pick a closed point x so that $y \rightsquigarrow x$ and $\mathcal{O}_{X,x}$ is a regular local ring. There is a prime $\mathfrak{p} \subset \mathcal{O}_{X,x}$ such that $\mathcal{O}_{X,y} = (\mathcal{O}_{X,x})_{\mathfrak{p}}$, and it is a standard (but nontrivial) result that localizations of regular local rings at primes are regular local rings [STK, Lemma 0AFS]. □

Corollary 4.1.8 *Let X be a quasicompact scheme and $f\colon X \to S$ a morphism locally of finite presentation. Let $\mathcal{F}$ be a quasicoherent sheaf on X of finite presentation. Then $\mathcal{F}$ is flat over S if and only if it is flat at each closed point of X.*

Proof This follows from the difficult theorem that in this case, the flat locus over S is open [STK, Thm. 0399]. □

Lemma 4.1.9 *Let X be a quasicompact scheme. If $\mathcal{O}_{X,x}$ is a domain for each closed point $x \in X$, then $\mathcal{O}_{X,y}$ is a domain for all $y \in X$.*

Proof Same trick + the fact that a localization of a domain is a domain. □

Lemma 4.1.10 *If X is a scheme and X_1, X_2 are distinct irreducible components, then $\mathcal{O}_{X,x}$ is not a domain for each $x \in X_1 \cap X_2$.*

Proof By taking an affine nbhd of x, we may consider the case where $\mathfrak{p} \subset A$ is a prime and $\mathfrak{p}_1, \mathfrak{p}_2$ are distinct minimal primes of A contained in $\mathfrak{p}$. Then $\mathfrak{p}_1 A_{\mathfrak{p}}$ and $\mathfrak{p}_2 A_{\mathfrak{p}}$ are distinct minimal primes of $A_{\mathfrak{p}}$, hence the latter cannot be a domain. □

Corollary 4.1.11 *Let X be a scheme.*

1. *Closed irreducible subsets intersect at singular (even non-normal) points.*
2. *Suppose X is quasicompact, connected, and has at most finitely many irreducible components. If $\mathcal{O}_{X,x}$ is a domain for each closed point $x \in X$, then X is in fact irreducible and hence integral.*

Proof (1) $\mathcal{O}_{X,x}$ cannot be normal (or regular) without being a domain.

(2) Observe that all of the irreducible components of X are disjoint. Since there are finitely many of them, this implies that they are all open as well and hence there is only one, because the space is connected. Thus X is irreducible. □

Remark 4.1.12 In part (2) of the previous corollary, the assumption that X has only finitely many irreducible components is necessary: There is a connected quasicompact scheme X which has all of its local rings as domains, but it in fact has infinitely many irreducible components and none of them are open.

Note that in the context of connected Noetherian schemes, X is always quasicompact with finitely many components. So in this case (e.g. varieties), we can indeed check irreducibility by checking that the local rings at closed points are all domains.

5 Cohomology

5.1 Singular vs. Sheaf cohomology

In this section we give an exposition of a pretty irrelevant comparison theorem between singular and sheaf cohomology of topological spaces, but I don't know where else to put it. The main inputs we shall use are the standard theory of sheaf cohomology, and very basic results from algebraic topology. The precise statement of the result is as follows:

Theorem 5.1.1 *Let X be a locally contractible, hereditarily paracompact Hausdorff space. For any abelian group G, there is an isomorphism*

$$C^*(X; G) \xrightarrow{\sim} R\Gamma(X, \underline{G}) \qquad \text{in } D^+(\mathbb{Z})$$

which is natural in the two variables:

1. *If Y is a locally contractible, hereditarily paracompact Hausdorff space, and $f\colon X \to Y$ is a continuous map, then the diagram*

$$\begin{array}{ccc} C^*(Y; G) & \xrightarrow{\cong} & R\Gamma(Y, \underline{G}) \\ {\scriptstyle f^*}\downarrow & & \downarrow \\ C^*(X; G) & \xrightarrow{\cong} & R\Gamma(X, f^{-1}\underline{G}) \end{array}$$

   commutes in $D^+(\mathbb{Z})$, under the canonical identification $f^{-1}\underline{G} = \underline{G}$.

2. *If $\phi\colon G \to H$ is a group homomorphism, the diagram*

$$\begin{array}{ccc} C^*(X; G) & \xrightarrow{\cong} & R\Gamma(X, \underline{G}) \\ {\scriptstyle \phi_*}\downarrow & & \downarrow{\scriptstyle R\Gamma(X, \underline{\phi})} \\ C^*(X; H) & \xrightarrow{\cong} & R\Gamma(X, \underline{H}) \end{array}$$

   commutes in $D^+(\mathbb{Z})$.

In particular, this theorem says that for sufficiently nice spaces X, the singular cohomology $H^i(X; G)$ agrees with $H^i(X, \underline{G})$. More precisely, if we restrict to a nice subcategory $\mathcal{C} \subset \mathbf{Top}$, then the two ith cohomology functors $\mathcal{C} \to \mathbf{Ab}$ are isomorphic. Naturality here is nice: e.g., we can extract the degree of a map $f\colon S^n \to S^n$ from the corresponding morphism of sheaves $\underline{\mathbb{Z}} \to f_* f^{-1}\underline{\mathbb{Z}}$, i.e. $\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}$.

Finally, let us mention that the space restriction still retains most non-pathological spaces, in particular all topological manifolds (when they are

assumed to be Hausdorff and second-countable). Slightly less trivially, it has locally finite CW complexes, because such spaces are both locally contractible and metrizable, and metrizable spaces are hereditarily paracompact.

5.1.1 Topology

Let $C_*(X)$ denote the singular complex of a space; if $\mathcal{U}$ is an open cover, then let $C_*^{\mathcal{U}}(X)$ denote the free abelian group on those simplices $\sigma\colon \Delta^n \to X$ whose image is contained in U for some $U \in \mathcal{U}$. Then $C_*^{\mathcal{U}}(X) \subset C_*(X)$ naturally.

Proposition 5.1.2 (barycentric subdivision) *There is a barycentric subdivision map $b\colon C_*(X) \to C_*^{\mathcal{U}}(X)$ exhibiting $C_*^{\mathcal{U}}(X)$ as a deformation retract; i.e. $bi = \mathrm{Id}$ and $ib \sim \mathrm{Id}$.*

We prove the other topological facts we need now. The only standard fact we have to use is the following: if X is paracompact Hausdorff, then it is normal, and if X is normal, then every locally finite open cover admits a shrinking. Here, a shrinking of an open cover $\{U_i\}_i$ is an open cover $\{V_i\}_i$ with the same index set such that $\overline{V_i} \subset U_i$.

Lemma 5.1.3 *Let $\{U_i\}$ be a locally finite open cover and $\{V_i\}$ a shrinking. For each $x \in X$, we can choose nbhds $\{M_x : x \in X\}$ with the following properties:*

1. *For each x, M_x intersects only finitely many of the $\{V_i\}$.*
2. *If M_x intersects V_i, then $x \in \overline{V_i}$ and $M_x \subset U_i$.*
3. *If M_x intersects M_y, then $M_x \cup M_y \subset U_i$ for some i.*

Proof Let N_x be a nbhd intersecting finitely many of the U_i (hence V_i); then, for each j such that $V_j \cap N_x \neq \varnothing$ but $x \in V_j$, we can subtract $\overline{V_j}$ from N_x to obtain a nbhd N_x'. Now $M_x = N_x' \cap \bigcap_i U_i$ can be taken as a finite intersection, and then this M_x satisfies the first two conditions.

If $z \in M_x \cap M_y$, then choose $V_j \ni z$, then $M_x \subset U_j$ and $M_y \subset U_J$ as desired, by the properties. □

This lemma is used to proof the following result. Here, given a presheaf $\mathcal{F}$, we denote by $L\mathcal{F}$ its sheafification, and $\eta\colon \mathcal{F} \to L\mathcal{F}$ the canonical map. A presheaf is satisfies the gluing condition if it satisfies the existence part of the sheaf condition.

Lemma 5.1.4 *Let X be a paracompact Hausdorff space, and let $\mathcal{F}$ be a presheaf (of sets) satisfying the gluing condition. Then the canonical map*

$$\eta_X\colon \mathcal{F}(X) \to L\mathcal{F}(X)$$

is surjective.

Proof If $s \in L\mathcal{F}(X)$, there's a locally finite open cover $\{U_i\}$ of X with $t_i \in \mathcal{F}(U_i)$ such that $\eta(t_i) = s|_{U_i}$, and each x has a nbhd N_x such that $t_i|_{N_x} = t_j|_{N_x}$ if $x \in U_i \cap U_j$. Let $\{V_i\}$ be a shrinking and apply the above lemma to obtain nbhds M_x, and consider $P_x \coloneqq M_x \cap N_x$. Let t^x be the common value of any of the restrictions $t_i|_{P_x}$ for which $x \in U_i$.

Now, if P_x intersects P_y, then $P_x \cup P_y \subset U_j$ for some j, so

$$t^x|_{P_x \cap P_y} = t_j|_{P_x \cap P_y} = t^y|_{P_x \cap P_y},$$

i.e. the $\{t^x\}$ glue to give a global t, and $\eta(t)$ agrees with s on small enough opens, demonstrating surjectivity. □

5.1.2 Constant sheaf cohomology

For a fixed space X, $C_n(-)$ is naturally a functor $\mathrm{Open}(X) \to \mathbf{Ab}$, hence $C^n(-\,; G)$ is naturally a presheaf for any abelian group G. We write

$$\widetilde{C}^n(-\,; G) \coloneqq LC^n(-\,; G)$$

for its sheafification. The differentials of $C^*(-\,; G)$ are obviously morphisms of presheaves, so $C^*(-\,; G)$ is a complex of presheaves, and by functoriality $\widetilde{C}^*(-\,; G)$ is a complex of sheaves.

Corollary 5.1.5 *Let X be a hereditarily paracompact Hausdorff space. Then for each open $U \subset X$, the natural map*

$$C^n(U; G) \to \widetilde{C}^n(U; G)$$

is surjective for each $n \geq 0$.

Proof By the lemma, we just need to check that $C^n(-\,; G)$ satisfies the gluing condition. If $\mathcal{U} = \{U_i\}$ is an open cover of U and $\alpha_i \in C^n(U_i; G)$ are cochains which agree on overlaps, then as maps $C_n(U_i) \to G$ these induce a map $\alpha\colon C_n^{\mathcal{U}}(X) \to G$ (by agreement on overlaps). Letting b be the barycentric subdivision map from Prop. 5.1.2, the cochain αb does the trick because for each i, we have a factorization

$$(\alpha b)|_{U_i} = C_n(U_i) \to C_n^{\mathcal{U}}(X) \to C_n(X) \xrightarrow{b} C_n^{\mathcal{U}}(X) \xrightarrow{\alpha} G,$$

and the retract property implies that this is just α_i. □

Now, until we finish the proof of Thm. 5.1.1, we will fix the following notation: The space X and coefficient group G are fixed; so, C^n and $\widetilde{C}^n$ mean the presheaf $U \mapsto C^n(U; G)$ and its sheafification respectively. Assume that X is hereditarily paracompact Hausdorff, so that the conclusion of the previous lemma applies.

Let $\pi_U\colon C^n(U) \to \widetilde{C}^n(U)$ be the canonical surjection, and form its kernel as in

$$0 \to C_0^n(U) \hookrightarrow C^n(U) \xrightarrow{\pi_U} \widetilde{C}^n(U) \to 0,$$

then $C_0^n(U)$ consists of the chains which restrict to zero on an open cover of U.

On the other hand, let $\mathrm{Cov}(U)$ be the poset of covers of U, where $\mathcal{V} \to \mathcal{U}$ if $\mathcal{U}$ is a refinement of $\mathcal{V}$. In this case, then there's a canonical inclusion map $C_*^{\mathcal{U}}(U) \to C_*^{\mathcal{V}}(U)$, and hence a restriction map

$$C_{\mathcal{V}}^*(U) \to C_{\mathcal{U}}^*(U)$$

turning $C_-^*(U)$ into a functor $\mathrm{Cov}(U) \to \mathrm{Ch}(\mathbf{Ab})$. Now define $C_{0,\mathcal{U}}^*$ to be the kernel in

$$0 \to C_{0,\mathcal{U}}^n(U) \to C^n(U) \xrightarrow{\pi_U^{\mathcal{U}}} C_{\mathcal{U}}^n(U) \to 0,$$

where surjectivity comes from the same argument we applied earlier, namely that we can lift a lift a cochain in $C_{\mathcal{U}}^n(U)$ using the barycentric subdivision map b of Prop. 5.1.2. Then $C_{0,\mathcal{U}}^n(U)$ consists of cochains over U whose restriction to the cover $\mathcal{U}$ is trivial; by definition, if we take the filtered colimit of this we obtain $C_0^n(U)$, and thus

$$0 \to C_0^n(U) \to C^n(U) \to \varinjlim_{\mathcal{U}} C_{\mathcal{U}}^n(U) \to 0$$

is exact, so we can identify the cokernel here with $\widetilde{C}^n(U)$. We have basically shown the following:

Lemma 5.1.6 *In the case that X is hereditarily paracompact Hausdorff and $U \subset X$ is open, the surjective homotopy equivalences*

$$\pi_U^{\mathcal{U}} \colon C^n(U) \to C_{\mathcal{U}}^n(U)$$

induce a surjective quasi-isomorphism

$$\pi_U \colon C^n(U) \to \varinjlim_{\mathcal{U}} C_{\mathcal{U}}^n(U) = \widetilde{C}^n(U)$$

upon taking the filtered colimit over $\mathcal{U} \in \mathrm{Cov}(U)$.

Proof Note that each $\pi_U^{\mathcal{U}}$ is homotopy equivalence because $C_n^{\mathcal{U}}(U) \to C_n(U)$ is, by Prop. 5.1.2. In particular, they are quasi-isomorphisms. But cohomology commutes with filtered colimits, showing π_U is a quasi-isomorphism. □

The rest of the proof is easy: first, each sheaf $\widetilde{C}^n$ is flasque, because we have a commutative diagram:

$$\begin{array}{ccc} C^n(X) & \twoheadrightarrow & \widetilde{C}^n(X) \\ \downarrow & & \vdots \\ C^n(U) & \twoheadrightarrow & \widetilde{C}^n(U) \end{array}$$

Here the left hand map is surjective because a cochain on U can clearly be extended to one on X, e.g. by declaring simplices not contained in U to map to zero.

Now assume X is locally contractible. We claim $\widetilde{C}^*$ is acyclic, i.e. $H^i(\widetilde{C}^*) = 0$ for $i > 0$. Indeed, let $\{U_i\}$ be an open cover of U with $U_i \to U$ nullhomotopic, then we have a commutative diagram

$$\begin{array}{ccc} H^n(C^*(U)) & \xrightarrow{0} & H^n(C^*(U_i)) \\ \downarrow\cong & & \cong\downarrow \\ H^n(\widetilde{C}^*(U)) & \longrightarrow & H^n(\widetilde{C}^*(U_i)) \end{array}$$

with vertical isomorphisms. Hence the bottom map is zero, which means that for each cocycle $z \in \widetilde{C}^n(U)$, $z|_{U_i} \in (\operatorname{im} d)(U_i)$, thus $z \in \operatorname{im} d$ since $\operatorname{im} d$ is a subsheaf. This shows that $\widetilde{C}^*(U)$ is acyclic for every $U \subset X$ open, which certainly implies that $\widetilde{C}^*$ is acyclic.

Proof (of Thm. 5.1.1) To conclude, note that since X is locally contractible it is in particular locally path connected, so the path components are in bijection with connected components, and thus $\underline{G} \cong H^0(C^*) \cong H^0(\widetilde{C}^*)$. We have a diagram

$$\begin{array}{ccc} & & \underline{G}[0] \\ & & \downarrow \\ C^* & \longrightarrow & \widetilde{C}^* \end{array}$$

and we've shown that both maps are quasi-isomorphisms. Considering this in $D^+(\mathrm{Sh}(X))$, we can take derived global sections and use the fact that $\widetilde{C}^*$ is a resolution by flasque sheaves to obtain the diagram

$$\begin{array}{ccc} C^*(X; G) & \dashrightarrow & R\Gamma(X, \underline{G}) \\ \downarrow\cong & & \downarrow\cong \\ (\widetilde{C}^*)(X) & \xrightarrow{\cong} & R\Gamma(\widetilde{C}^*) \end{array}$$

The upper map is the desired isomorphism. That this is natural with respect to coefficient maps is obvious from this diagram by naturality of all constructions with respect to morphisms of sheaves. We prove naturality with respect to continuous maps of spaces carefully below. □

In general, suppose $F, G \colon \mathcal{A} \to \mathcal{B}$ are (say) left exact additive functors between abelian categories, where $\mathcal{A}$ is abelian and with enough injectives. If $u \colon F \to G$ is a natural transformation, there is an induced natural transformation

$$u \colon \mathbf{R}F \to \mathbf{R}G$$

between right derived functors $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Explicitly, we can compute it as follows: given a defining injective resolution $A^* \to I^*$ of a complex A^*, naturality ensures that the induced map

$$u \colon F(I^*) = \mathbf{R}F(A^*) \to G(I^*) = \mathbf{R}G(A^*)$$

is a chain map. Naturality is then easy to check. This has a second naturality property which is that it commutes with the canonical maps $F \to \mathbf{R}F$. More precisely, for each complex A^*, the square on the left

$$\begin{array}{ccccc} F(A^*) & \longrightarrow & \mathbf{R}F(A^*) & = & F(I^*) \\ {\scriptstyle u}\downarrow & & {\scriptstyle \mathbf{R}u}\downarrow & & {\scriptstyle u}\downarrow \\ G(A^*) & \longrightarrow & \mathbf{R}G(A^*) & = & G(I^*) \end{array}$$

commutes. This is obvious from the defining square on the right because u extends to a natural transformation on the extensions of F and G to the whole categories of chain complexes.

In our case, suppose $f\colon X \to Y$ be continuous, and consider the unit map $\mathrm{Id} \to f_* f^{-1}$, which induces $\Gamma(Y, \mathcal{F}) \to \Gamma(X, f^{-1}\mathcal{F})$, i.e. a natural transformation

$$\eta\colon \Gamma_Y \to \Gamma_X \circ f^{-1}.$$

Since $\mathbf{R}(\Gamma_X \circ f^{-1}) = \mathbf{R}\Gamma_X \circ f^{-1}$ by exactness of f^{-1}, we obtain a natural transformation $\mathbf{R}\Gamma(Y, \mathcal{F}) \to \mathbf{R}\Gamma(X, f^{-1}\mathcal{F})$ in the derived category. We can compute it using the following easy lemma:

Lemma 5.1.7 *Let $\mathcal{F} \to \mathcal{I}^*$ and $f^{-1}\mathcal{F} \to \mathcal{J}^*$ be resolutions such that $\mathcal{I}^*$ and $\mathcal{J}^*$ are (bounded below) complexes of Γ_Y- and Γ_X-acyclic objects respectively. Suppose we have a map $v\colon f^{-1}\mathcal{I}^* \to \mathcal{J}^*$ making the diagram*

$$\begin{array}{ccc} & & f^{-1}\mathcal{F} \\ & \swarrow & \downarrow \\ f^{-1}\mathcal{I}^* & \xrightarrow[v]{} & \mathcal{J}^* \end{array}$$

commute. Then the diagram

$$\begin{array}{ccccc} \mathbf{R}\Gamma(Y, \mathcal{F}) & \xrightarrow{\cong} & \Gamma(Y, \mathcal{I}^*) & \longrightarrow & \Gamma(X, f^{-1}\mathcal{I}^*) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle \Gamma(X, v^\#)}\downarrow & \swarrow_{\Gamma(X,v)} & \\ \mathbf{R}\Gamma(X, f^{-1}\mathcal{F}) & \xrightarrow{\cong} & \Gamma(X, \mathcal{J}^*) & & \end{array}$$

commutes in $D^+(\mathbb{Z})$.

Proof Consider the diagram:

$$\begin{array}{ccccc} \mathbf{R}\Gamma(Y, \mathcal{F}) & \longrightarrow & \mathbf{R}\Gamma(Y, \mathcal{I}^*) & \longleftarrow & \Gamma(Y, I^*) \\ {\scriptstyle f^*}\downarrow & & {\scriptstyle f^*}\downarrow & & \downarrow \\ \mathbf{R}\Gamma(X, f^{-1}\mathcal{F}) & \longrightarrow & \mathbf{R}\Gamma(X, f^{-1}\mathcal{I}^*) & \longleftarrow & \Gamma(X, f^{-1}\mathcal{I}^*) \\ & \searrow & {\scriptstyle \mathbf{R}\Gamma(X,v)}\downarrow & & {\scriptstyle v}\downarrow \\ & & \mathbf{R}\Gamma(X, \mathcal{J}^*) & \longleftarrow & \Gamma(X, \mathcal{J}) \end{array}$$

The top-left square commutes by naturality of the induced map f^*; the top-right square commutes by the property we discussed of commutativity with

canonical maps $F \to \mathbf{R}F$. The bottom-right square commutes by naturality of these canonical maps $F \to \mathbf{R}F$. The bottom-left triangle commutes by our assumption. The outer perimeter is essentially what we want. The final dotted line in the statement comes from the fact that the corresponding triangle composite is global sections applied to $\mathcal{I}^* \to f_* f^{-1}\mathcal{I}^* \xrightarrow{f_* v} \mathcal{J}^*$, which is just the right adjunct to v. □

The lemma tells us how to compute the maps f^* in the derived category if we've resolved by complexes which aren't a priori injective. Let's apply it to our singular/constant sheaf cohomology isomorphism. To this end, fix X and Y locally contractible and hereditarily paracompact, and $f\colon X \to Y$ a continuous map.

For $U \subset Y$, $f^{-1}(U) \to U$ induces a map $f^*\colon C^n(U) \to C^n(f^{-1}(U))$; note that if $\alpha \in C^n(U)$ is a cochain which vanishes upon restriction to some open cover $\{U_i\}$, then $f^*(\alpha)$ vanishes on $\{f^{-1}(U_i)\}$. It follows that f^* preserves C_0^n, and thus we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_0^n(U) & \longrightarrow & C^n(U) & \longrightarrow & \widetilde{C}^n(U) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \vdots{\scriptstyle f^*} & & \\ 0 & \longrightarrow & C_0^n(f^{-1}(U)) & \longrightarrow & C^n(f^{-1}(U) & \longrightarrow & \widetilde{C}^n(f^{-1}(U)) & \longrightarrow & 0 \end{array}$$

of exact sequences, and in particular, we have a canonical commutative diagram:

$$\begin{array}{ccc} C_Y^* & \longrightarrow & \widetilde{C}_Y^* \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ f_* C_X^* & \longrightarrow & f_* \widetilde{C}_X^* \end{array}$$

Note that taking global sections of the left-hand map is the standard f^*. On the other hand, we can consider the left adjunct $f^\flat\colon f^{-1}\widetilde{C}_Y^* \to \widetilde{C}_X^*$ of the second map, and the above diagram induces a standard diagram:

$$\begin{array}{ccc} f^{-1}\underline{G} & \xrightarrow{\cong} & \underline{G} \\ \downarrow & & \downarrow \\ f^{-1}\widetilde{C}_Y^* & \xrightarrow{f^\flat} & \widetilde{C}_X^* \end{array}$$

Finally, we can apply the above lemma. It says that the diagram

$$\begin{array}{ccccc} & & R\Gamma(Y, \underline{G}) & \xrightarrow{\cong} & \widetilde{C}^*(Y) & \longleftarrow & C^*(Y) \\ & \overset{f^*}{\swarrow} & {\scriptstyle f^*}\downarrow & & {\scriptstyle \Gamma(X, f^*)}\downarrow & & \downarrow{\scriptstyle f^*} \\ R\Gamma(X, \underline{G}) & \xrightarrow{\cong} & R\Gamma(X, f^{-1}\underline{G}) & \longrightarrow & \widetilde{C}^*(X) & \longleftarrow & C^*(X) \end{array}$$

commutes, the outer perimeter is the desired statement.

5.2 Cohomology of locally constant sheaves

In the previous section we demonstrated how to compute the cohomology of constant sheaves, at least on nice spaces. The case of locally constant sheaves is also important in algebraic topology, because these represent so-called 'local systems', which is used for cohomology with local coefficients. This type of cohomology comes up when certain actions of π_1 of things are nontrivial: the Serre spectral sequence, basic obstruction theory, non-orientable Poincaré duality, and more.

5.2.1 Locally constant sheaves and bundles of groups

Let X be a space and G an abelian group. A *constant sheaf* at G is a sheaf $\mathcal{F}$ for which there exists a morphism $G \to \mathcal{F}$ of presheaves inducing an isomorphism on stalks, or equivalently an isomorphism on the sheafification $L(G) \xrightarrow{\sim} \mathcal{F}$. Here G denotes the constant presheaf with value G.

Equivalent ways to describe it are as follows: $\mathcal{F}$ is constant (at G) iff its étale space $\mathrm{Et}(\mathcal{F}) \to X$ is isomorphic to $X \times G \to X$ where G is viewed as discrete. This gives an explicit description as $\mathcal{F} \cong \mathbf{Top}(-, G)$, where G is again discrete. It follows that $\mathcal{F}$ is also isomorphic to the sheaf

$$U \mapsto \prod_{\mathrm{Con}(U)} G$$

where $\mathrm{Con}(U)$ denotes the set of connected components of U, and restriction maps are induced by projections in the obvious way. When the map $G \to \mathcal{F}$ is fixed, we denote the sheafication by $G \to \mathcal{F} = \underline{G}$. Hence, mapping out of $\underline{G}$ is equivalent to providing maps $G \to \mathcal{G}(U)$ commuting with restriction maps, and thus a map $G \to \Gamma(X, \mathcal{G})$. So we have an adjunction

$$\underline{(-)} \colon \mathbf{Ab} \rightleftarrows \mathbf{Ab}(X) : \Gamma(X, -).$$

If $c \colon X \to *$ is the terminal map, clearly $\Gamma(X, -) \cong c_*$ under the identification $\mathbf{Ab}(*) \cong \mathbf{Ab}$, thus we have a natural isomorphism $\underline{G} \cong c^{-1}(G)$. It follows e.g. that $f^{-1}\underline{G} = \underline{G}$, and so on. Observe that if X is connected, then the adjunction is a fully faithful adjoint, and thus induces an equivalence of $\mathbf{Ab}$ with the constant sheaves on X.

Definition 5.2.1 A *locally constant sheaf* is a sheaf $\mathcal{F}$ for which there exists an open cover $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf.

A priori, in this definition, the group at which the sheaf is locally constant could vary across connected components, but we will usually just assume $\mathcal{F}|_{U_i}$ is a constant sheaf at the same group G. Thus, all of the stalks $\mathcal{F}_x$ will be isomorphic to G, but non-canonically. Our first goal is to describe the locally constant sheaves on X in a few different ways.

First of all, we can just recast the definition in terms of étale spaces. By our description of etale spaces of constant sheaves, it is clear that $\mathcal{F}$ is locally

constant iff the corresponding étale space $p\colon E \to X$ admits an open cover $\{U_i\}$ such that $p^{-1}(U_i) \cong U_i \times G$ over U_i. Note that since we are working with etale spaces valued in abelian groups, such an isomorphism must satisfy the following additional condition: $E_x \coloneqq p^{-1}(x)$ must be an abelian group, and the isomorphisms above restrict to to isomorphisms $E_x \cong \{x\} \times G$ which should be a group isomorphism. This data is a 'bundle of groups', so we immediately get an equivalence of categories between locally constant sheaves and bundles of groups over X.

The more interesting interpretation is the following description through local systems. Fix a locally constant sheaf $\mathcal{F}$ (which a priori could have different stalks). Let $\gamma\colon I \to X$ be a path in x; we construct an isomorphism

$$\phi_\gamma\colon \mathcal{F}_{\gamma(0)} \to \mathcal{F}_{\gamma(1)}$$

of abelian groups as follows:

1. Choose a trivializing open cover $\{U_i\}$ with stalkwise isomorphisms $\varphi_i\colon G \to \mathcal{F}|_{U_i}$. Then $\gamma^{-1}(U_i)$ is a cover of I, and by compactness (and reindexing) we may choose

   $$0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$$

   such that $\gamma([t_i, t_{i+1}]) \subset U_i$. Then we have isomorphisms

   $$\varphi_{i,\gamma(t_{i+1})} \circ \varphi_{i,\gamma(t_i)}^{-1}\colon \mathcal{F}_{\gamma(t_i)} \xrightarrow{\sim} \mathcal{F}_{\gamma(t_{i+1})},$$

   and we compose them all to obtain the map ϕ_γ as desired.

2. This is independent of all choices: First of all note that if we keep the same open cover but change all the trivializations, meaning we use $\varphi_i g_i$ for some automorphism g_i of G instead of φ_i, then this clearly doesn't change the map ϕ_γ because the intermediate isomorphisms above become

   $$\big(\varphi_{i,\gamma(t_{i+1})} g_i\big)\big(g_i^{-1}\varphi_{i,\gamma(t_i)}^{-1}\big) = \varphi_{i,\gamma(t_{i+1})}\varphi_{i,\gamma(t_i)}^{-1}.$$

3. Next, observe that we can insert $t_i < t' < t_{i+1}$ and use the same trivialization φ_i for $[t_i, t']$ and $[t', t_{i+1}]$ without changing anything. Thus, if we choose a different $\{V_1, \ldots, V_m\}$ and $0 = s_0, \ldots, s_{m+1} = 1$ such that $\gamma([s_i, s_{i+1}]) \subset V_i$ with some trivializations, we can subdivide I into $\{t_0, t_1, \ldots, s_0, s_1, \ldots\}$ without changing the ϕ_γ corresponding to the $\{t_i\}$ and to $\{s_i\}$. But by (2), the map is independent of choice of local trivialization, completing the proof of uniqueness of ϕ_γ.

Remark 5.2.2 Note that we actually should assume X is locally connected here. This is because we, in (2), used the fact that automorphisms of the sheaf $\underline{G}$ are given by automorphisms of G, and we observed above that this is true when the space is connected (but not in general, as is obvious).

The above shows that stalks $\mathcal{F}_{x_0}, \mathcal{F}_{x_1}$ are canonically isomorphic up to a choice of path from x_0 to x_1. Next, let us show this is a homotopy invariant. Suppose

we have a based homotopy of paths

$$F\colon I \times I \to X \qquad F(s,0) = \gamma(s), F(s,1) = \eta(s), F(i,t) = x_i, i \in \{0,1\}.$$

As in (1), we can choose trivializations (U_{ij}, φ_{ij}) and subdivisions

$$I = [s_0, s_1] \cup \cdots \cup [s_n, s_{n+1}], \quad I = [t_0, t_1] \cup \cdots \cup [t_m, t_{m+1}],$$

such that $F([s_i, s_{i+1}] \times [t_j, t_{j+1}]) \subset U_{ij}$. Now ϕ_γ is defined using $[s_i, s_{i+1}] \times 0$ and ϕ_η using $[s_i, s_{i+1}] \times 1$, and we have intermediate paths F_t from x_0 to x_1, defined using $[s_i, s_{i+1}] \times \{t\}$. Hence it suffices to show

$$\phi_{F_{t_i}} = \phi_{F_{t_{i+1}}} \qquad i \geq 0.$$

But by our independence results above, we can compute $\Phi_{F_{t_{i+1}}}$ using the trivializations φ_{ij}. Indeed, let $x_{i,j} = F(s_i, t_j)$, so then we have commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}_{x_{i,j+1}} & \xleftarrow{\varphi_{ij}} & G & \xrightarrow{\varphi_{ij}} & \mathcal{F}_{x_{i+1,j+1}} \\
{\scriptstyle\varphi_{ij}}\uparrow & & & & \uparrow{\scriptstyle\varphi_{ij}} \\
G & = & = & = & G \\
{\scriptstyle\varphi_{ij}}\downarrow & & & & \downarrow{\scriptstyle\varphi_{ij}} \\
\mathcal{F}_{x_{i,j}} & \xleftarrow[\varphi_{ij}]{} & G & \xrightarrow[\varphi_{ij}]{} & \mathcal{F}_{x_{i+1,j}}
\end{array}$$

where all of the stalk points on φ_{ij} are supressed. Taking horizontal compositions, we obtain:

$$\begin{array}{ccc}
\mathcal{F}_{x_0} & \xrightarrow{\phi_{F_{t_{j+1}}}} & \mathcal{F}_{x_1} \\
{\scriptstyle(\varphi_{ij})_{x_0}}\uparrow & & \uparrow{\scriptstyle(\varphi_{ij})_{x_1}} \\
G & = & G \\
{\scriptstyle(\varphi_{ij})_{x_0}}\downarrow & & \downarrow{\scriptstyle(\varphi_{ij})_{x_1}} \\
\mathcal{F}_{x_0} & \xrightarrow{\phi_{F_{t_j}}} & \mathcal{F}_{x_1}
\end{array}$$

This shows that the top and bottom are the same. Given a path γ, or based path class $[\gamma]$, we write $M_F\mathcal{F}(\gamma) = M_\mathcal{F}([\gamma]) = \phi_\gamma$.

Lemma 5.2.3 *Let X be locally connected. The association*

$$M_\mathcal{F}\colon \Pi_1(X) \to \mathbf{Ab} \qquad x \mapsto \mathcal{F}_x, [\gamma] \mapsto \mathcal{F}([\gamma])$$

defines a functor. Moreover, it induces a faithful embedding

$$\mathrm{LocConstSh}_{\mathbf{Ab}}(X) \to \mathrm{Fun}(\Pi_1(X), \mathbf{Ab})$$

given by $\mathcal{F} \mapsto M_\mathcal{F}\colon \Pi_1(X) \to \mathbf{Ab}$. The category on the left is the full subcategory of $\mathbf{Ab}(X)$ spanned by the locally constant sheaves.

Proof (1) If γ is the trivial path at x_0, we can choose a trivialization $\varphi\colon G \to \mathcal{F}|_U$ on any nbhd of x_0, and then $\mathcal{F}_{x_0} \to \mathcal{F}_{x_0}$ is $\varphi_{x_0}\varphi_{x_0}^{-1} = 1$. Now suppose $\gamma\colon x \to y$, $\eta\colon y \to z$. Then we want to show that $\mathcal{F}(\gamma * \eta) = \mathcal{F}(\eta) \circ \mathcal{F}(\gamma)$.

This follows from the following fact: if we have subdivisions

$$0 < t_1 < \cdots < t_n < 1 \qquad 0 < s_1 < \cdots < s_m < 1$$

corresponding to trivializations (U_i, φ_i) and (V_j, ψ_j), then we can use the subdivision

$$0 < \frac{t_1}{2} < \cdots < \frac{t_n}{2} < \frac{s_1}{2} + 1 < \cdots < \frac{s_m}{2} + 1 < 1$$

for the composite path $\gamma * \eta$ with the same trivializations, and the result follows immediately.

Now suppose we have a morphism $u\colon \mathcal{F} \to \mathcal{G}$ of locally constant sheaves. Then we get a natural transformation as follows: the stalk map $u_x\colon \mathcal{F}_x \to \mathcal{G}_x$ are the components, and given a path $\gamma\colon x \to y$, the diagram

$$\begin{array}{ccc} \mathcal{F}_x & \xrightarrow{\mathcal{F}(\gamma)} & \mathcal{F}_y \\ {\scriptstyle u_x}\downarrow & & \downarrow{\scriptstyle u_y} \\ \mathcal{G}_x & \xrightarrow{\mathcal{G}(\gamma)} & \mathcal{G}_y \end{array}$$

commutes: Indeed, we can choose connected nbhds on which both sheaves are trivialized, so that the same subdivision of I works; then we use the fact that a map $u\colon \mathcal{F}|_U \to \mathcal{G}|_U$ is (via the trivilizations) given by a map $\phi\colon G \to G$. It follows that the diagram

$$\begin{array}{ccccc} \mathcal{F}_{x_i} & \xleftarrow{\varphi_{x_i}} & G & \xrightarrow{\varphi_{x_{i+1}}} & \mathcal{F}_{x_{i+1}} \\ {\scriptstyle u_{x_i}}\downarrow & & {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle u_{x_{i+1}}} \\ \mathcal{G}_{x_i} & \xleftarrow{\psi_{x_i}} & G & \xrightarrow{\psi_{x_{i+1}}} & \mathcal{G}_{x_{i+1}} \end{array}$$

commutes, as desired. Finally, it is obvious that the association $\mathcal{F} \mapsto M_{\mathcal{F}}$ is faithful, because sheaf maps are equal if they coincide on stalks. □

We have a much stronger result under the assumption that X is in addition locally path connected and seim-locally simply connected. Let $\{U_\alpha\}$ be a basis of X of open, path connected sets for which $\pi_1(U_\alpha) \to \pi_1(X)$ is trivial. Moreover, choose a basepoint $x_\alpha \in U_\alpha$. Consider the inclusions $\Pi_1(U_\alpha) \to \Pi_1(X)$; the assumptions tell us firstly that the domain is a connected groupoid, and that the functor is trivial on automorphisms. Hence the image is a contractible subgroupoid of $\Pi_1(X)$.

Suppose we start with a functor $F\colon \Pi_1(X) \to \mathbf{Ab}$. We are going to define a sheaf on the basis: Set $\mathcal{F}(U_\alpha) = F(x_\alpha)$; if $U_\alpha \subset U_\beta$, then there is a unique map $[\gamma]\colon x_\beta \to x_\alpha$ in the image of $\Pi_1(U_\beta) \to \Pi_1(X)$, so define $\mathcal{F}(U_\beta) \to \mathcal{F}(U_\alpha)$ by $F([\gamma])$. It is easy to see by uniqueness that this is a presheaf on the basis.

To check the sheaf condition, let $\{U_\alpha\}_{\alpha \in A}$ cover some U_β, and suppose we have sections $s_\alpha \in F(x_\alpha)$ satisfying the following property: for all $U_\delta \subset U_{\alpha_1} \cap U_{\alpha_2}$, $s_{\alpha_1}|_{U_\delta} = s_{\alpha_2}|_{U_\delta}$.

(1) Existence: Easy, just pick s_{α_0}, then note that the restriction maps we defined were isomorphisms, so set s to be the inverse of s_{α_0} under $F(x_\beta) \to F(x_{\alpha_0})$. Uniqueness guarantees that its restriction to $F(x_\alpha)$ agree with s_α.

(2) Uniqueness: If $s, s' \in F(x_\beta)$ both restrict to $s_\alpha \in F(x_\alpha)$ for all $\alpha \in A$, then they both must be the unique inverse of the restriction to e.g. $F(x_{\alpha_0})$ as before.

It follows that $\mathcal{F}$ is a sheaf on the basis, and we can choose an extension $\mathcal{F}$ to the whole space X. But then the identity $F(x_\alpha) \to \mathcal{F}(U_\alpha)$ induces $\underline{F(x_\alpha)} \to \mathcal{F}|_{U_\alpha}$, and we can take stalks via a filtered colimit over basis elements. Since all restriction maps are isomorphisms, the stalks $F(x_\alpha) \to \mathcal{F}_x$ are all isomorphisms, showing that $\mathcal{F}$ is a locally constant sheaf on X.

Now suppose that $\mathcal{G}$ was a locally constant sheaf and $F = M_{\mathcal{G}}$. Then the isomorphisms above are isomorphisms $\underline{\mathcal{G}_{x_\alpha}} \to \mathcal{F}|_{U_\alpha}$, so by possibly restricting to a smaller basis element we obtain an isomorphism $\mathcal{G}|_{U_\beta} \xrightarrow{\sim} \mathcal{F}|_{U_\beta}$, and these agree on overlaps for varying β. Hence $\mathcal{G} \cong \mathcal{F}$ naturally. Conversely, if F was arbitrary and $\mathcal{F}$ was constructed as above, consider the functor $M_{\mathcal{F}}$. Then clearly $M_{\mathcal{F}}(x_\alpha) = \mathcal{F}_{x_\alpha} = F(x_\alpha)$, and this can be extended to natural isomorphism $M_{\mathcal{F}} \cong F$. Thus:

Theorem 5.2.4 *If X is locally path connected and semilocally simply connected, then*

$$\mathrm{LCSh}_{\mathbf{Ab}}(X) \to \mathrm{Fun}(\Pi_1(X), \mathbf{Ab}) \qquad \mathcal{F} \mapsto M_{\mathcal{F}}$$

is an equivalence of categories.

Example 5.2.5 If X is locally path connected and simply connected, then locally constant sheaves on X are all constant.

We remark that in the usual ways, one can now connect this to right actions of the fundamental group via a universal cover.

5.2.2 Cohomology with local coefficients

References

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