Towards Motivic Homotopy Theory

Vihaan Dheer

This document contains work for an independent project on motivic homotopy theory; see the project outline here. The goal is to properly learn most of the prerequisites for the subject, to eventually be able to begin $\mathbf{A}^1$-algebraic topology. The fundamental category of spaces in this theory is the category of Nisnevich sheaves on the category of smooth k-schemes valued in Kan complexes. We first study enough category theory to make sense of this.

1 The Nisnevich Topology

1.1 Sites and Sheaves

In this section, we will use roman letters A, B, C for small categories (e.g. in some universe), lowercase $c \in C$ for elements of small categories, script $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ for possibly large categories. We use P, Q for presheaves, and $\mathcal{F}, \mathcal{G}$ for sheaves, and write $\mathrm{PSh}(C, \mathcal{D})$ for the category of functors $C^{\mathrm{op}} \to \mathcal{D}$; $\mathrm{PSh}(C)$ without qualification will mean $\mathrm{PSh}(C, \mathbf{Set})$, or $\mathrm{PSh}(C, \mathcal{D})$ if $\mathcal{D}$ is clear from context.

In this section, we will generalize the classical notion of a sheaf on a topological space to a sheaf on any category. This is done by defining a 'topology' on a category, from which we can formulate a sheaf condition. We will use this when defining the étale and Nisnevich topologies on the categories of schemes; in these cases the sheaf condition is called 'descent', and the point is that we should be able to construct some object associated to a scheme using 'covering' morphisms by ètale or Nisnevich maps.

Remark 1.1.1 Eventually, we would like to consider sheaves which associate to each scheme a category. This presents two challenges: (1) usually functoriality is only given up to natural isomorphism, and (2) we will want to express the sheaf condition not as a limit/equalizer as we will do now, but as a homotopy limit. In effect, this means that we can glue together categories associated to an open cover using isomorphisms, not equalities. The first problem can be solved by replacing the appropriate category of categories with its homotopy category, but the second cannot. Thus, after Sec. 2, we will consider sheaves in which the sheaf condition is expressed as homotopy limit. But, at the end of this section, we can see an example looking at 2-categories.

The main reference for this section is MacLane-Moerdijk [1, Ch. 2]. Let C be a category with $c \in C$. A *sieve* on c is a subfunctor $S \subset \text{ɬ}(c)$ of the functor represented by c. We identify S with the collection $\bigcup_{d \in C} S(d)$, so that we can write S as a collection of maps to c such that

$$g \in S \implies g \circ f \in S$$

whenever $\operatorname{dom} g = \operatorname{cod} f$. There is always the 'maximal sieve' $\text{ɬ}(c)$, which we write $t_c = \{f \colon d \to c : \forall f\}$. If $f \colon d \to c$ is a map and S is a sieve on c, then the *pullback sieve* f^*S is a sieve on d given by

$$f^*S = \{g \colon e \to d : fg \in S\},$$

which is named as such because it is the pullback

$$\begin{array}{ccc} f^*S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow \\ \text{ɬ}(d) & \xrightarrow{f_*} & \text{ɬ}(c) \end{array}$$

in $\mathrm{PSh}(C)$. We can now define a topology on a category.

Definition 1.1.2 Let C be a small category. A *Grothendieck topology* on C is, for each $c \in C$, a collection $J(c)$ of sieves on c satisfying the following properties:

(G1) $t_c \in J(c)$ for all $c \in C$.

(G2) For any $S \in J(c)$ and any $f \colon d \to c$, we have $f^*S \in J(d)$.

(G3) Suppose $S \in J(c)$, and T is a sieve on c such that for all d and $f \colon d \to c \in S$, $f^*T \in J(d)$. Then $T \in J(c)$.

A small category C equipped with a Grothendieck topology J is called a *site*, which we write (C, J).

These axioms have some basic properties, all of which are relatively apparent from the definitions. See my notes for more details.

Proposition 1.1.3 *Let (C, J) be a site.*

1. *If $S \subset R$ are sieves on c and $S \in J(c)$, then $R \in J(c)$.*
2. *If $R, S \in J(c)$, then $R \cap S \in J(c)$.*
3. *Let $S \in J(c)$, and suppose for each $f \colon d_f \to c \in S$ we have a sieve $R_f \in J(d_f)$. Then*
$$\{f \circ g : f \in S, g \in R_f\}$$
is a sieve in $J(c)$.

If (C, J) is a site and $S \in J(c)$, we say S *covers* c or is a *covering sieve* for c. Just as in normal topology, we usually omit J and just call C a site equipped

with covering sieves for each object. Our motivating example is $C = \mathrm{Open}(X)$, the poset of open subsets of a topological space. A sieve on $U \subset X$ is a downwards-closed collection of open subsets, and we declare the covering sieves to be the ones whose union is all of U. This is called the *standard topology* on $\mathrm{Open}(X)$, and its sheaf theory (as we will see) is the normal sheaf theory on a space. In practice, we want to define a topology using a different set of data:

Definition 1.1.4 Let C be a small category with pullbacks. A *basis* (or *pretopology* or *coverage*) on C is, for each $c \in C$, a collection $K(c)$ of families of morphisms to c with the following properties:

(B1) If $f\colon d \to c$ is an isomorphism, then $\{f\} \in K(c)$.

(B2) If $\{f_i \in c_i \to c : i \in I\} \in K(c)$ and $g\colon d \to c$ is any map, then the family of pullbacks along g

$$\{\bar{f}_i\colon c_i \times_c d \to d : i \in I\}$$

is in $K(d)$.

(B3) Let $\{f_i\colon c_i \to d : i \in I\} \in K(c)$, and suppose for each $i \in I$ we are given a family $\{g_{ij} : d_{ij} \to c_i : j \in J_i\} \in K(c_i)$. Then the composite family

$$\{f_i \circ g_{ij} : i \in I, j \in J_i\}$$

is in $K(c)$.

A family $\{f_i\} \in K(c)$ is called a *covering family* for c. The data for a pretopology is related but not identical to that of a topology; the pretopology is easier to satisfy and thus less unique. But, every pretopology generates a topology in a canonical way: Given K, set

$$J(c) = \{S \text{ sieve on } c : \exists R \in K(c) \text{ s.t. } R \subset S\}.$$

Then $J =: (K)$ is a Grothendieck topology. If $R = \{f_i\colon c_i \to c\}$, the *sieve generated by* R is

$$S = (R) = \{g \circ f_i : i \in I, g\colon d \to c_i\},$$

which by construction is a covering sieve if R is a covering family. Given a topology J, there is a canonical pretopology K associated to it, in which we declare the covering families to be all families of morphisms which generate a sieve.

Two pretopologies K, K' on a category C are *equivalent* if $(K) = (K')$, i.e. they generate the same topology. Explicitly, this means that for each $c \in C$, a sieve on c contains a K-cover if and only if it contains a K'-cover; or equivalently: for any K-cover R, the sieve (R) contains a J-cover, and for any K'-cover R', the sieve (R') contains a J-cover.

Example 1.1.5 Let **Top** be a small (full sub-) category of topological spaces closed under finite limits and taking open subspaces. The *open cover* topology

on **Top** is the one where $\{f_i \colon Y_i \to X\}$ is a covering family of X if and only if each f_i is an open embedding and $X = \bigcup_i \operatorname{im} f_i$. The *étale cover topology* on **Top** is the one where $\{f_i \colon Y_i \to X\}$ is a covering family if and only if each f_i is étale, and $X = \bigcup_i \operatorname{im} f_i$. In fact, these pretopologies are equivalent because if $\{f_i\}$ is an étale cover, take $\{U_{ij} \hookrightarrow Y_i\}$ an open cover of Y_i by opens of X, then

$$\{U_{ij} \hookrightarrow Y_i \xrightarrow{f_i} X : i, j\}$$

is an open cover of X contained in the sieve generated by $\{f_i\}$.

An example we consider in more detail is the *Zariski topology* on **Aff**, the category of affine schemes. (To make this more precise, we should work in a universe so that **Aff** is small, but let's ignore this). We construct this using a pretopology where the covering families are (isomorphism classes of) open immersions

$$\{j_i \colon D_A(f_i) \hookrightarrow \operatorname{Spec}(A) : i \in I\},$$

where $(f_i : i \in I) = A$. It is again standard to check the axioms, inducing the (big) Zariski site structure on **Aff**.

The main reason to consider sites is to learn about what sheaves on the site are. We begin by working over **Set**; let (C, J) be a site, $P \in \operatorname{PSh}(C)$ a presheaf of sets, and $c \in C$ some object.

Definition 1.1.6 A *matching family* of P for $S \in J(c)$ is a family of elements $(x_f)_{f \in S}$ where $x_f \in P(d_f)$ for each $f \colon d_f \to c$, and for every $g \colon e \to d_f$ we have

$$g^*(x_f) \coloneqq P(g)(x_f) = x_{fg}.$$

An *amalgamation* of the family $(x_f)_f$ is an element $x \in P(c)$ such that $f^*(x) = x_f$ for all $f \in S$.

If S is a sieve on $c \in C$, then we may view $S \subset C/c$ as a full subcategory of the slice over c. By the explicit description of limits in **Set**, we see that a matching family is the same as an element

$$(x_f)_{f \in S} \in \varprojlim_{f \in S} P(\operatorname{dom} f),$$

where the limit is being taken over the composite functor

$$S^{\mathrm{op}} \hookrightarrow (C/c)^{\mathrm{op}} \to C^{\mathrm{op}} \xrightarrow{P} \mathbf{Set}.$$

If every matching family of P with respect to S has a *unique* amalgamation, we say that P satisfies the *sheaf condition* for the sieve S. Then it is clear that P satisfies the sheaf condition for S if and only if the family $\{P(c) \to P(c_f) : f \colon c_f \to c \in S\}$ exhibits $P(c)$ as the limit

$$P(c) \cong \varprojlim_{f \in S} P(c_f) \quad \text{in } \mathbf{Set}.$$

Definition 1.1.7 Let (C, J) be a site and $P \in \operatorname{PSh}(C, \mathbf{Set})$.

1. P is a *sheaf* on C if for each $c \in C$ and covering sieve $S \in J(c)$, P satisfies the sheaf condition for S.
2. We write $\mathrm{Sh}((C,J),\mathbf{Set}) \subset \mathrm{PSh}(C,\mathbf{Set})$ for the full subcategory of presheaves which are sheaves, or just $\mathrm{Sh}(C,\mathbf{Set})$ when the Grothendieck topology on C is clear.
3. A *Grothendieck topos* is a category that is equivalent to $\mathrm{Sh}(C,\mathbf{Set})$ for some site (C,J).

We usually abbreviate to $\mathrm{Sh}(C) \subset \mathrm{PSh}(C)$ to talk about **Set**. We may naturally extend this to a more general setting using our description by limits. Namely, suppose $P \in \mathrm{PSh}(C,\mathcal{D})$ for some locally small, complete category $\mathcal{D}$. Once again, we say that P satisfies the sheaf condition with respect to a covering sieve S on $c \in C$ if the natural map

$$P(c) \xrightarrow{\sim} \varprojlim_{f \in S} P(\mathrm{dom}\, f) \qquad \text{in } \mathcal{D}$$

is an isomorphism. Then P is a sheaf if it satisfies the sheaf condition for all covering sieves on all objects, and the corresponding full subcategory of $\mathrm{PSh}(C,\mathcal{D})$ is denoted $\mathrm{Sh}(C,\mathcal{D})$. From now on, when we refer to $\mathrm{Sh}(C,\mathcal{D})$, we mean (C,J) is any site and $\mathcal{D}$ is any locally small complete category.

Proposition 1.1.8 $\mathrm{Sh}(C,\mathcal{D})$ *is complete for any* $C,\mathcal{D}$*, and its limits are computed in* $\mathrm{PSh}(C,\mathcal{D})$*. In other words,*

$$i\colon \mathrm{Sh}(C,\mathcal{D}) \hookrightarrow \mathrm{PSh}(C,\mathcal{D})$$

creates limits.

Proof It suffices to show that $\mathrm{Sh}(C,\mathcal{D})$ is closed under limits of $\mathrm{PSh}(C,\mathcal{D})$ as the inclusion is fully faithful. Suppose we have a diagram $\mathcal{F}_\bullet\colon I \to \mathrm{PSh}(C,\mathcal{D})$ such that each $\mathcal{F}_i$ is a sheaf. To see that the limit is a sheaf, we use the fact that limits in presheaf categories are computed objectwise. For $c \in C$ and $S = \{f\colon c_f \to c\}$ covering c:

$$\begin{aligned}\Big(\varprojlim_{i\in I} \mathcal{F}_i\Big)(c) &\cong \varprojlim_{i\in I} \mathcal{F}_i(c) \cong \varprojlim_{i\in I}\varprojlim_{f\in S} \mathcal{F}_i(c_f)\\ &\cong \varprojlim_{f\in S}\varprojlim_{i\in I} \mathcal{F}_i(c_f) \cong \varprojlim_{f\in S}\Big(\varprojlim_{i\in I} \mathcal{F}_i\Big)(c_f).\end{aligned}$$

Hence the presheaf $\varprojlim_i \mathcal{F}_i$ is a sheaf. □

Aside from $\mathrm{Open}(X)$ for a space X, on which we directly defined its topology, it is hard to talk explicitly about about sheaves on a site since most of our sites have been constructed using pretopologies. Fortunately, sheaves can be characterized directly using those.

Definition 1.1.9 Let C be a small category equipped with a pretopology K, $\mathcal{D}$ be a locally small complete category, and $P \in \mathrm{PSh}(C, \mathcal{D})$ be a presheaf. If $\{f_i \colon c_i \to c : i \in I\}$ is a covering family for c, we say P satisfies the *sheaf condition* with respect to $\{f_i\}$ if the natural diagram

$$P(c) \to \prod_{i \in I} P(c_i) \rightrightarrows \prod_{i,j \in I} P(c_i \times_c c_j)$$

is an equalizer in $\mathcal{D}$.

From now on, we will say C is a site to mean either that C has a pretopology or a topology, and the case will be clear from context.

Lemma 1.1.10 *Let (C, K) be a pre-site, $P \in \mathrm{PSh}(C, \mathcal{D})$ a presheaf, and $R \coloneqq \{f_i \colon c_i \to c\}$ a covering family of some object $c \in C$. P has the sheaf condition with respect to R if and only if it has the sheaf condition with respect to the sieve (R).*

Proof For one direction, assume that P satisfies the sheaf condition with respect to $(R) = \{f_i \circ g : \forall i, g\}$. Fix an object $D \in \mathcal{D}$, and let $(x_i)_{i \in I}$ be a cone from D the diagram, namely $x_i \colon D \to P(c_i)$ for each i and the diagrams

$$\begin{array}{ccc} D & \xrightarrow{x_j} & P(c_j) \\ {\scriptstyle x_i}\downarrow & & \downarrow{\scriptstyle \pi_{ij}^{2,*}} \\ P(c_i) & \xrightarrow{\pi_{ij}^{1,*}} & P(c_i \times_c c_j) \end{array}$$

commute for each $i, j \in I$. We will construct a cone to the diagram defining the sheaf condition for (R). For any $g \colon d \to c_i$ and i so $h = f_i g \in S$, define y_h by

$$\begin{array}{ccc} D & & \\ {\scriptstyle x_i}\downarrow & \searrow^{y_h} & \\ P(c_i) & \xrightarrow[g^*]{} & P(d) \end{array}$$

This is independent of i and g: If $h = f_i g = f_j g_2$, then the universal property of the pullback gives a unique map $u \colon d \to c_i \times_c c_j$ making the diagram commute:

$$\begin{array}{ccccc} & & d & & \\ & {\scriptstyle g}\swarrow & {\scriptstyle u}\downarrow & \searrow{\scriptstyle g_2} & \\ c_i & \xleftarrow[\pi_{ij}^1]{} & c_i \times_c c_j & \xrightarrow[\pi_{ij}^2]{} & c_j \end{array}$$

Therefore,

$$g^* x_i = u^* \pi_{ij}^{1,*} x_i = u^* \pi_{ij}^{2,*} x_j = g_2^* x_j$$

showing independence, i.e. $\{y_h : h \in (R)\}$ is well-defined. In particular, $y_{f_i} = x_i$. We claim $\{y_h\}$ is actually a cone (e.g. matching family) for (R), namely for $g \colon e \to d$, $g^* y_h = y_{hg_2}$. Indeed,

$$g_2^* y_h = g_2^* g^* x_i = (g g_2)^* y_{f_i} = y_{f_i g g_2} = y_{h g_2}.$$

So, by our assumption, there is a unique map $y\colon D \to P(c)$ such that $h^*y = y_h$ for all h, so in particular $f_i^*y = y_{f_i} = x_i$. Indeed y is unique so that $f_i^*y = x_i$ too; if x satisfies that condition too, then for any $h = f_i g$ we have

$$h^*x = g^*f_i^*y = g^*x_i = y_h = h^*y,$$

so $x = y$ by uniqueness. This proves one direction.

Now, suppose P has the sheaf condition with respect to $\{f_i\}$ and assume S is a sieve on c containing $\{f_i\}$. Suppose $(x_h)_{h\in S}$ is a cone from D to the appropriate limit, so that for $h\colon d \to c$, $x_h\colon D \to P(d)$, and for any $g\colon e \to d$ we have

$$g^*x_h = x_{hg}.$$

Set $y_i = x_{f_i}$, then we claim (y_i) is a cone for the sheaf condition on $\{f_i\}$. Indeed,

$$\pi_{ij}^{1,*}y_i = \pi_{ij}^{1,*}y_{f_i} = y_{f_i\pi_{ij}^1} = y_{f_j\pi_{ij}^2} = \pi_{ij}^{2,*}y_j,$$

so there's a unique map $y\colon D \to P(c)$, such that $f_i^*y = y_i$. This is an amalgamation for (x_h), because if $h = f_i g$ then

$$h^*y = g^*f_i^*y = g^*x_{f_i} = x_{f_i g} = x_h,$$

and it is obviously unique. □

For $P \in \mathrm{PSh}(C, \mathcal{D})$, we say P is a *separated presheaf* for each c and covering sieve S the natural map

$$P(c) \to \varprojlim_{f\in S} P(\mathrm{dom}\, f)$$

is a monomorphism in $\mathcal{D}$. This means that any matching family out of some object $D \in \mathcal{D}$ has at most one amalgamation.

Theorem 1.1.11 *Let (C, K) be a site and $P \in \mathrm{PSh}(C, \mathcal{D})$. Then P is a sheaf in the induced topology if and only if it has the sheaf condition with respect to every covering family of every object.*

Proof A sieve S is covering if and only if it contains a covering family. In particular, if R is a covering family then (R) is a covering sieve, so if P has the sheaf condition for all covering sieves then it has it for all covering families by Lemma 1.1.10.

On the other hand, suppose it has the sheaf condition for all covering families, and let S be a sieve on c containing a covering family $\{f_i\colon c_i \to d : i \in I\}$. If we have a cone (x_f) for S, then $y_i = x_{f_i}$ is a cone for $\{f_i\}$ because

$$\pi_{ij}^{1,*}y_i = \pi_{ij}^{1,*}y_{f_i} = y_{f_i\pi_{ij}^1} = y_{f_j\pi_{ij}^2} = \pi_{ij}^{2,*}y_j.$$

So there is a unique $y\colon D \to P(c)$ such that $f_i^*y = y_i = x_{f_i}$. Since uniqueness will be obvious, it remains to show that for any $g \in S$, $g^*y = x_g$. Fix this

$g\colon d \to c$, and consider for each i the following pullback square in C:

$$\begin{array}{ccc} d \times_c c_i & \xrightarrow{\rho_{g,i}} & c_i \\ {\scriptstyle \pi_{g,i}}\downarrow & \lrcorner & \downarrow{\scriptstyle f_i} \\ d & \xrightarrow{g} & c \end{array}$$

Then we have

$$\begin{aligned} \pi_{g,i}^* g^* y &= \rho_{g,i}^* f_i^* y = \rho_{g,i}^* x_{f_i} \\ &= x_{f_i \rho_{g,i}} = x_{g \pi_{g,i}} \\ &= \pi_{g,i}^* x_g. \end{aligned}$$

The axioms say that $\{\pi_{g,i} : i \in I\}$ cover d, so the sheaf condition implies that the natural map

$$P(d) \to \prod_{i \in I} P(d \times_c c_i)$$

is a monomorphism. It follows from our computation above that $g^* y = x_g$, which is what we wanted. □

Example 1.1.12 The theorem implies that two equivalent pretopologies on a site admit the same sheaves. For example, in **Top**, a sheaf satisfying the sheaf condition for open covers must also satisfy it for étale covers.

It is clear that the subcategory $\mathrm{Sh}(C, \mathcal{D})$ is closed under isomorphism classes in $\mathrm{PSh}(C, \mathcal{D})$; we say that a site (C, J) is *subcanonical* if $C \subset \mathrm{Sh}(C)$, that is each representable presheaf $\text{ɣ}(c)$ is a sheaf. Most (big) sites we consider will be subcanonical, because a 'regular function' can generally be constructed by gluing together regular functions on overlaps.

Example 1.1.13 The above sentences applies e.g. to **Top**, so that for any space Y, the presheaf of sets $\mathrm{Hom}_{\mathbf{Top}}(-, Y)$ is a sheaf. But if Y has more structure, e.g. $Y = \mathbb{R}$, then $\mathrm{Hom}_{\mathbf{Top}}(-, \mathbb{R})$ is a presheaf and sheaf of rings. Along the same lines we can consider a category like $\mathbf{CGWH} \subset \mathbf{Top}$, and using universes we will assume w.l.o.g. it is small. Omitting some details, **CGWH** is cartesian closed and has the internal hom $\underline{\mathrm{Hom}}(-,-)$, so fixing Y a compactly generated weak Hausdorff space,

$$\underline{\mathrm{Hom}}(-, Y)\colon \mathbf{CGWH}^{\mathrm{op}} \to \mathbf{CGWH}$$

is a sheaf in $\mathrm{Sh}(\mathbf{CGWH}, \mathbf{CGWH})$.

Remark 1.1.14 We might attempt to consider $\mathcal{D} = \mathbf{Cat}$. While the definitions still make sense, in general we will want to have a weaker notion, where our presheaves are pseudopresheaves, and our limits/equalizers defining the sheaf condition are 2-limits/2-equalizers. Such a 'sheaf' is called a *stack*, and the sheaf condition in this case is called *descent*. We will first study ∞-categories and then discuss this idea more generally, so that an ∞-sheaf will be a functor

$N(C)^{\mathrm{op}} \to \mathcal{X}$ for $\mathcal{X}$ an ∞-category, and the sheaf condition will be expressed as a homotopy limit. Taking $\mathcal{X} = \mathbf{Cat}_1$, the ∞-category of 1-categories, this will recover stacks and descent.

For the rest of this section, we will study the *sheafification* construction. Prop. 1.1.8 should make us guess that when $\mathcal{D}$ (and possibly, but not really, C) is nice enough, then the inclusion functor $i\colon \mathrm{Sh}(C,\mathcal{D}) \hookrightarrow \mathrm{PSh}(C,\mathcal{D})$ admits a left adjoint. When this exists, the left adjoint is called the sheafification functor. When $\mathcal{D} = \mathbf{Set}$ it exists, and more generally in most concrete categories (like groups, abelian groups, rings, modules over a ring, etc.) it exists. Even more generally, this works in any locally finitely presentable (LFP) category.

Note: Some reorganization needs to happen here.In what follows, a matching family *from* D for S is a map $h\colon D \to \mathcal{M}(S,P)$, which we may also think of as a collection of maps $D \to P(c_f)$ satisfying the standard property. An amalgamation is a factorization of h through $P(c) \to \mathcal{M}(S,P)$. We have the following easy check, where we write $\mathrm{Hom}_{\mathcal{D}}(D,P)$ for the functor $c \mapsto \mathrm{Hom}_{\mathcal{D}}(D,P(c))$.

Lemma 1.1.15 *Let $P \in \mathrm{PSh}(C,\mathcal{D})$ and $D \in \mathcal{D}$; then the following are equivalent:*

1. *For every $c \in C$, matching families from D have a unique amalgamation.*
2. *$\mathrm{Hom}_{\mathcal{D}}(D,P) \in \mathrm{PSh}(C)$ is a sheaf of sets.*

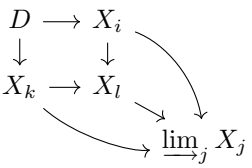
In particular, $P \in \mathrm{Sh}(C,\mathcal{D})$ if and only if $\mathrm{Hom}_{\mathcal{D}}(D,P) \in \mathrm{Sh}(C)$ for all $D \in \mathcal{D}$.

Definition 1.1.16 Let $\mathcal{D}$ be a category with filtered colimits.

1. An object $D \in \mathcal{D}$ is *finitely presentable* (or *fp*) if the functor
$$\mathrm{Hom}_{\mathcal{D}}(D,-)\colon \mathcal{D} \to \mathbf{Set}$$
preserves filtered colimits. We denote by $\mathcal{D}_{\mathrm{fp}}$ the full subcategory of all finitely presentable objects of $\mathcal{D}$.
2. Assume further that $\mathcal{D}$ is cocomplete and locally small. We say $\mathcal{D}$ is *locally finitely presentable* if there is a (small) set $S \subset \mathcal{D}_{\mathrm{fp}}$ of objects so that every object of $\mathcal{D}$ is filtered colimit over objects of S.

Concretely, suppose $D \in \mathcal{D}$ is finitely presentable and $X_\bullet\colon J \to \mathcal{D}$ is a filtered diagram. Then all maps $D \to \varinjlim_j X_j$ factor through $X_k \to \varinjlim_j X_j$ for some $j \in J$, and whenever we have a commutative diagram like
$$\begin{array}{ccc} D & \longrightarrow & X_i \\ \downarrow & & \downarrow \\ X_k & \longrightarrow & \varinjlim_j X_j \end{array}$$

there is always some $l \in J$ with maps $i \to l \leftarrow k$ in J such that



commutes. We will begin to use this condition soon, but a quick proposition first:

Proposition 1.1.17 *Let $\mathcal{D}$ be an lfp category with finite limits. Then finite limits commute with filtered colimits in $\mathcal{D}$.*

Proof Let A and I be finite and filtered categories respectively, and suppose we have a functor $F\colon A \times I \to \mathcal{D}$. Then $\varprojlim_a \varinjlim_i F(a,i) = \varinjlim_i \varprojlim_a F(a,i)$ if and only if

$$\mathrm{Hom}_{\mathcal{D}}(D, \varprojlim_{a\in A} \varinjlim_{i\in I} F(a,i)) = \mathrm{Hom}_{\mathcal{D}}(D, \varinjlim_{i\in I} \varprojlim_{a\in A} F(a,i))$$

for all $D \in \mathcal{D}$. Since any such D is the colimit of finitely presentable objects, we may assume D is finitely presentable. Then the result is obvious because we can move limit/colimits out of Hom and use the same result for **Set**. □

Let (C, J) be a site, and consider $(J(c), \supset)$ as a poset category. Then this is filtered since if S, T cover c, then $S \cap T$ covers c. Observe that a map $f\colon d \to c$ in C induces a functor $f^*\colon J(c) \to J(d)$, because $S \supset T$ implies $f^*S \supset f^*T$. To define sheafification, we will need to take filtered colimits over $J(c)$, so from now on we will write $\mathcal{D}$ for an arbitrary locally small category which is complete and admits filtered colimits. For convenience, if $P \in \mathrm{PSh}(C, \mathcal{D})$ and S is a sieve on c, we set

$$\mathcal{M}(S, P) \coloneqq \varprojlim_{f\in S} P(\mathrm{dom}\, f) \qquad \text{in } \mathcal{D},$$

where as before the diagram category is $S \subset C/c$ as a full subcategory. Then $\mathcal{M}$ is equipped with the projection maps $\mathcal{M}(S,P) \to P(d)$ for each $f\colon d \to c \in S$. Also as before, for each such f the maps $P(c) \to P(c_f)$ induce a canonical morphism

$$P(c) \longrightarrow \mathcal{M}(S,P) \quad \text{in } \mathcal{D},$$

which is an isomorphism when P is a sheaf and a monomorphism when P is a separated presheaf. If $S \supset T$ are covering sieves, then for each $f \in T$ the projection $\mathcal{M}(S,P) \to P(\mathrm{dom}\, f)$ induces restriction

$$\mathcal{M}(S,P) \to \mathcal{M}(T,P) \qquad \text{when } S \supset T.$$

This is transitive, so we get a functor $J(c) \to \mathcal{D}$ given by $S \mapsto \mathcal{M}(S, P)$. We define

$$P^+(c) \coloneqq \varinjlim J(c) \to \mathcal{D} = \varinjlim_{S \in J(c)} \mathcal{M}(S, P)$$

to be the (filtered) colimit of this functor in $\mathcal{D}$. Hence this is equipped with natural maps $\mathcal{M}(S, P) \to P^+(c)$ for all covering sieves S, and in particular a composite

$$\eta_c = P(c) \to \mathcal{M}(t_c, P) \to P^+(c).$$

The maximal sieve t_c is just used here for definiteness, but in fact we can take any covering sieve S because the following diagram has to commute:

$$\begin{array}{ccc} P(c) & \longrightarrow & \mathcal{M}(t_c, P) \\ \downarrow & \swarrow & \downarrow \\ \mathcal{M}(S, P) & \longrightarrow & P^+(c) \end{array}$$

The commutativity of this upper triangle shows that η can be realized as a colimit of the *morphisms*

$$\eta_c = \varinjlim_{S \in J(c)} \Big(P(c) \to \mathcal{M}(S, P) \Big) \tag{1}$$

where for $P(c)$ we just consider the constant diagram. Next, suppose we have a map $f \colon d \to c$ in C. For each S, there is a morphism in $\mathcal{D}$

$$\mathcal{M}(S, P) \to \mathcal{M}(f^*S, P)$$

induced by, for each $g \colon e \to d \in f^*S$, the projection map $\mathcal{M}(S, P) \to P(e)$ associated to $gf \in S$. Composing with $\mathcal{M}(f^*S, P) \to P^+(d)$ for all S induces a morphism of the colimits

$$P^+(c) \to P^+(d)$$

which is functorial in f; thus P^+ is a presheaf in $\mathrm{PSh}(C, \mathcal{D})$. One can use the naturality of all of the involved constructions here that $\eta \colon P \to P^+$ is in fact a natural transformation.

Proposition 1.1.18 *Let $P \in \mathrm{PSh}(C, \mathcal{D})$ and consider the canonical morphism $\eta \colon P \to P^+$.*

1. *If η is a monomorphism, then P is separated.*
2. *If P is a sheaf, then η is an isomorphism.*
3. *Suppose that $\mathcal{D}$ is an lfp category. Then P is a separated presheaf (resp. sheaf) if and only if η is a monomorphism (isomorphism, resp.).*

Proof Since C is small and $\mathcal{D}$ is complete, being a monomorphism in $\mathrm{PSh}(C, \mathcal{D})$ is equivalent to being a pointwise monomorphism in $\mathcal{D}$. (1) If η is a monomorphism, then for any $c \in C$ and any covering sieve S we have a commutative

diagram:

$$\begin{array}{ccc} P(c) & \longrightarrow & \mathcal{M}(S,P) \\ & \searrow^{\eta} & \downarrow \\ & & P^+(c) \end{array}$$

This immediately implies that $P(c) \to \mathcal{M}(S,P)$ is a monomorphism, so P is a separated presheaf. (2) If P is a sheaf, then for each c and S the map $P(c) \to \mathcal{M}(S,P)$ is an isomorphism. But then Eq. (1) shows that η_c is an isomorphism for all c, i.e. η is an isomorphism. (3) By Prop. 1.1.17, in this case filtered colimits commute with pullbacks, and hence they preserve monomorphisms. So Eq. (1) means that η_c is an monomorphism for all c.

Finally, now assume that η is an isomorphism, and let $D \in \mathcal{D}$ be a finitely presentable object. Suppose $(y_h)_{h\in S}$ is matching family out of D for a covering sieve S on c; we claim it has an amalgamation. Indeed, consider the composite

$$y = D \xrightarrow{(y_h)} \mathcal{M}(S,P) \to P^+(c) \xrightarrow{\eta^{-1}} P(c).$$

By definition, the diagram

$$\begin{array}{ccccc} D & \xrightarrow{y} & P(c) & \longrightarrow & \mathcal{M}(t_c,P) \\ {\scriptstyle (y_h)}\downarrow & & & \searrow^{\eta} & \downarrow \\ \mathcal{M}(S,P) & & \longrightarrow & & P_+(c) \end{array}$$

commutes. Since D is finitely presentable, the maps to $P_+(c)$ must jointly factor through some sieve $T \subset S \cap t_c = S$, so that the diagram below commutes,

$$\begin{array}{ccc} D & \xrightarrow{y} & P(c) \\ {\scriptstyle (y_h)}\downarrow & & \downarrow \\ \mathcal{M}(S,P) & \longrightarrow & \mathcal{M}(T,P) \end{array}$$

which means that for all $f \in T$, we have $f^*y = y_f$. We want to extend this to all of S: fix $g\colon d \to c \in S$, then by axiom g^*T covers d, and for each $h\colon e \to d \in g^*T$ we have the computation below.

$$h^*g^*y = (gh)^*y = y_{gh} = h^*y_g$$

(as $gh \in T$). Since h was arbitrary and P is separated by (1), $g^*y = y_g$; since $g \in S$ was arbitrary, y is indeed an amalgamation for (y_h), and it is unique because P is separated. By Lemma 1.1.15, this means that $\mathrm{Hom}_{\mathcal{D}}(D,P)$ is a sheaf of sets.

Now, any arbitrary object $E \in \mathcal{D}$ is a filtered colimit $E = \varinjlim_j D_j$ of finitely presentable objects D_j. Therefore

$$\Big(\varprojlim_j \mathrm{Hom}_{\mathcal{D}}(D_j,P)\Big)(c) = \varprojlim_j \mathrm{Hom}_{\mathcal{D}}(D_j,P(c)) = \Big(\mathrm{Hom}_{\mathcal{D}}(E,P)\Big)(c),$$

implies that in $\mathrm{PSh}(C)$, $\varprojlim_j \mathrm{Hom}_{\mathcal{D}}(D_j, P) = \mathrm{Hom}_{\mathcal{D}}(E, J)$. Since $\mathrm{Sh}(C)$ is closed under limits of $\mathrm{PSh}(C)$, this shows $\mathrm{Hom}_{\mathcal{D}}(E, P)$ is a sheaf of sets for any E, which means $P \in \mathrm{Sh}(C, \mathcal{D})$. □

We show the $(-)^+$ construction is also functorial. Indeed, suppose $\phi\colon P \to Q$ is a morphism of presheaves. Then if $S \in J(c)$, this gives a natural transformation between the diagrams

$$\phi\colon \left(S^{\mathrm{op}} \to C^{\mathrm{op}} \xrightarrow{P} \mathcal{D}\right) \to \left(S^{\mathrm{op}} \to C^{\mathrm{op}} \xrightarrow{Q} \mathcal{D}\right),$$

and therefore a map $\phi_S\colon \mathcal{M}(S, P) \to \mathcal{M}(S, Q)$. The family $\{\phi_S : S \in J(c)\}$ is natural with respect to sieve restriction, so it induces a morphism of colimits $\phi_c^+\colon P^+(c) \to Q^+(c)$. This is natural for maps $d \to c$ in C, so $\phi^+\colon P^+ \to Q^+$ is a morphism of presheaves, and finally this is and so

$$(-)^+\colon \mathrm{PSh}(C, \mathcal{D}) \to \mathrm{PSh}(C, \mathcal{D})$$

is a functor on the presheaf category.

We still have some more work to do regarding sheafification. In particular, I need to show that P^+ is always separated, and that P^+ is a sheaf if P is separated.

Lemma 1.1.19 *A presheaf $P \in \mathrm{PSh}(C, \mathcal{D})$ is separated (a sheaf) if and only if $\mathrm{Hom}_{\mathcal{D}}(D, P) \in \mathrm{PSh}(C)$ is separated (a sheaf) for all $D \in \mathcal{D}$. If $\mathcal{D}$ is lfp, it suffices to check this for D finitely presentable.*

Proof We already know both statements for sheaves. Observe that, because $\mathrm{Hom}_{\mathcal{D}}(D, -)$ preserves limits, we have

$$\mathrm{Hom}_{\mathcal{D}}(D, \mathcal{M}(S, P)) = \mathcal{M}(S, \mathrm{Hom}_{\mathcal{D}}(D, P)).$$

Thus the first result follows from the fact that $\mathrm{Hom}_{\mathcal{D}}(D, -)$ preserves and jointly reflects monomorphisms in D. If $\mathcal{D}$ is lfp and we have the statement for D finitely presentable, then note that limits preserve monomorphisms and the map

$$\mathrm{Hom}_{\mathcal{D}}(E, P(c)) \to \mathcal{M}(S, \mathrm{Hom}_{\mathcal{D}}(D, P))$$

is a (cofiltered) limit of monomorphisms. □

1.2 Smoothness and Étale Morphisms

Let us now move on to geometric considerations. Here is the main motivation for our study of étale morphisms of schemes. In the previous section we discussed the open cover site structure on **Top**, on which a sheaf attaches to each space an object of a complete (and usually locally finitely presentable (lfp)) category, and when a space is covered by open subsets the resulting objects are glued together on a common intersection.

We use this global perspective often even in geometry. For instance, on the category **Aff** of affine schemes we have the the Zariski coverage. We can define a sheaf of rings

$$\mathcal{O}\colon \mathbf{Aff}^{\mathrm{op}} \to \mathbf{Ring}$$

sending $\mathrm{Spec}(A) \mapsto A$, which acts as a global structure sheaf. Once we introduce the Nisnevich topology on the category $\mathbf{Sm}_k$ of smooth k-schemes, we will actually consider Nisnevich schemes

$$\mathcal{S}\colon \mathbf{Sm}_k^{\mathrm{op}} \to \mathcal{D},$$

where $\mathcal{D}$ might be various categories: for instance the category of abelian groups, Kan complexes, the ∞-category of 1-categories, and many others. But, in general, our main considerations will be on *relative* site structures. That is, we want to consider sheaves on a specific space, scheme, or object. Let us consider smooth manifolds **Man**, so fix a manifold M; our underlying category C will be the full subcategory of the slice $\mathbf{Man}/M$ consisting of smooth embeddings. C is essentially small because it is equivalent to $\mathrm{Open}(M)$, and so $\mathrm{Sh}(C)$ will just recover the standard sheaves on the manifold. C is strictly contained in the full subcategory $\acute{\mathrm{E}}\mathrm{t}(M)$ of local diffeomorphisms. As we saw in Sec. 1.1, this will have the sheaf theory as the open cover topology on M, because any étale open cover can be refined to give an open cover.

In algebraic geometry, one comes to a fundamental topological issue: namely, the Zariski topology on $\mathrm{Spec}(A)$ is completely unlike the 'Euclidean' topology on more standard spaces one might consider. This makes normal topological techniques unusable; instead, we will want to consider *étale open subsets* of a scheme, i.e. (isomorphism classes of) étale morphisms into the scheme. In schemes, though, our étale morphisms are not *literally* local homeomorphisms, but rather maps which pointwise induce isomorphisms on the tangent spaces of a scheme. In this section, we develop this notion in some detail.

In geometry, there are two ways to approach smoothness. One is pointwise, and roughly corresponds to the approach of tangent spaces in differential settings. The other is section-wise, and is analogous to considering the differential forms, that is smooth sections of the tangent bundle. This hints at how the approaches are related. While we will briefly discuss the first method for interpretational value, we are more interested in the second: the module (or sheaf) of Kahler differentials.

Let A be a ring, $I \subset A$ an ideal, M an A-module. If $I \subset \mathrm{Ann}_A(M)$, then M is canonically an A/I-module, where the obvious action $(a + I)m = am$ is well-defined because $Im = 0$. This agrees with the base change along $A \twoheadrightarrow A/I$, that is M viewed as an A/I-module is $M \otimes_A A/I$.

In particular, M/IM is always annihilated by I by so it is an I module. If $I = \mathfrak{m}$ is maximal, then $M/\mathfrak{m}M$ is a vector space over the field $A/\mathfrak{m}$. Even more specially, $\mathfrak{m}/\mathfrak{m}^2$ is thus a vector space over $A/\mathfrak{m}$.

Definition 1.2.1

1. If $(R, \mathfrak{m})$ is a local ring, then the cotangent space $T^\vee R$ of R is the $\kappa(R) = R/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$. The tangent space TR is the dual space $\underline{\mathrm{Hom}}_{\kappa(R)}(\mathfrak{m}/\mathfrak{m}^2, \kappa(R))$.
2. If X is a scheme and $x \in X$ is a point, then the cotangent space *at* x, $T_x^\vee X$, is the $\kappa(x)$ vector space that is the cotangent space of $\mathcal{O}_{X,x}$. The tangent space $T_x X$ is its $\kappa(x)$-dual.

Remark 1.2.2 Suppose $X = \mathrm{Spec}(A)$ is affine and $\mathfrak{p} \in X$ is a point. It is an easy check that $(\mathfrak{p}A_\mathfrak{p})^2 = \mathfrak{p}^2 A_\mathfrak{p}$, so the cotangent space at $\mathfrak{p}$ is the vector space $\mathfrak{p}A_\mathfrak{p}/\mathfrak{p}^2 A_\mathfrak{p}$ over $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$. If A models the ring of functions on X, and $\mathfrak{p} \subset A$ are the functions which vanish at $\mathfrak{p} \in X$, then $A_\mathfrak{p}$ is the ring of functions on a neighborhood of $\mathfrak{p}$. Then the maximal ideal $\mathfrak{m}_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$ are those functions on a neighborhood of $\mathfrak{p}$ which vanish at $\mathfrak{p}$, and $\mathfrak{m}_\mathfrak{p}^2$ are those functions which vanish to second order.

Also, $\kappa(\mathfrak{p})$ is the field of functions which are nonvanishing at $\mathfrak{p}$. Therefore, $T_\mathfrak{p}^\vee X$ is the space of functions on a neighborhood of $\mathfrak{p}$ which vanish at $\mathfrak{p}$ with functions having the same linear approximation identified. As multiplying by an invertible function from $\kappa(\mathfrak{p})$ doesn't change this property, it has a natural action.

This generally behaves well in very 'regular settings', especially for a scheme X over a field k, and a *rational point* $x \in X$: i.e. one where the field extension $k \subset \kappa(x)$ is an isomorphism (so we view it as an equality). This is the same as a k-point, i.e. a map $\mathrm{Spec}(k) \to X$ which gives the identity when composed with $X \to \mathrm{Spec}(k)$.

Let us analyze this situation for $\mathbf{A}_k^n$. A k-point or k-rational point is a closed point with residue k, and hence $x = (x_1, \ldots, x_n)$ must correspond to the ideal

$$M = (t_1 - x_1, \ldots, t_n - x_n) \subset k[t_1, \ldots, t_n].$$

Our differential geometric-intuition says that the cotangent space at x should be spanned by basis vectors $b_i = t_i - x_i \in T_x^\vee \mathbf{A}_k^n$. This in fact holds; first we show $\{b_i\}$ span the cotangent space. Indeed, consider $g/s \in \mathfrak{m}_x$ so that $s \notin \mathfrak{m}_x$ and $g = \sum_i g_i(t_i - x_i)$. We can always write g_i as a polynomial in the variables $t_i - x_i$, but then

$$g_i \equiv a_0 + \sum_k a_k(t_k - x_k) \pmod{\mathfrak{m}_x^2}$$

since any higher order terms would be in $\mathfrak{m}_x^2$. Therefore, $g_i(t_i - x_i) \equiv a_0$, so we may write

$$g \equiv \sum_i a_i(t_i - x_i) \qquad \text{in } T_x^\vee \mathbf{A}_k^n.$$

Now, consider $s \notin M$, which means that $s(x_1, \ldots, x_n) \neq 0$. Thus, when expressed with respect to the shifted variables $(t_i - x_i)$, the constant term

$s_0 \in k$ is nonzero. So $s \cdot (t_i - x_i) \equiv a_0(t_i - x_i)$, and thus $(t_i - x_i)/a_0 \equiv (t_i - x_i)/s$. In other words,

$$g/s = \sum_i s_0^{-1} a_i (t_i - x_i) \quad \text{in } T_x^\vee \mathbf{A}_k^n,$$

which shows that $\{b_i\}$ span the vector space. Linear independence is an easy exercise as well.

Though this construction helps build intuition about tangent spaces, it is less convenient to work with; we will now consider a more general and advanced formulation using Kahler differentials.

1.3 The Étale and Nisnevich Topologies

First let us explain the motivation behind taking our fundamental spaces to be sheaves in some Grothendieck topology over the category of schemes. A key motivation for schematic algebraic geometry is the ability to represent scheme constructions as new schemes. This works *sometimes* in topology: For instance, for nice enough spaces (like compactly generated Hausdorff spaces), we can naturally topologize the collection of maps between two of them.

This is much better in geometry: We have projective space characterizing line bundles on a space, Grassmanians characterizing bundles more generally. Another classic construction is $\mathrm{Hilb}(n)$, the Hilbert scheme of closed subschemes of $\mathbb{P}^n$, whose S-points for S locally Noetherian are the set of closed subschemes of $\mathbb{P}^n \times S$ which are flat over S.

We want to define such schemes X by specifying what the functor $\mathrm{Hom}_{\mathbf{Sch}}(-X)$ should be. It may or may not be representable, and often there are interesting concepts (like motivic spaces, as we will see) which are *almost* representable but not quite. Therefore, to characterize what properties such spaces should have we give such functors a sheaf condition, done by specifying a Grothendieck topology. We will always work with subcanonical topologies, i.e. those in which $\mathrm{Hom}(-, X)$ are sheaves for X a scheme. By the Yoneda lemma, these are generalized schemes, which are close to representing a scheme with some conditions relaxed.

2 Infinity and Model Categories

2.1 Infinity Categories and Kan Complexes

We follow [2, Ch. 1] for the very basics of infinity category theory. For the purposes of this project, infinity category, ∞-category, $(\infty, 1)$-category, and quasicategory are all the same thing. An ∞-category models the scenario where composition and associativity of composition are only defined up to natural isomorphism. The perspective is that this isomorphism is a 'homotopy'.

Let us recall some standard constructions. For an integer $n \geq 0$, we write $[n]$ for the set ordered set $\{0, 1, \ldots, n\}$ of size $n + 1$, and then $\mathbf{\Delta}$ is the category

of such $[n]$ with monotone maps. We write

$$\mathbf{sSet} \coloneqq \mathrm{PSh}(\Delta) = \mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Set})$$

for the category of simplicial sets. $\mathbf{\Delta}$ has canonical morphisms: for $0 \leq i \leq n$, we have

$$d^i \colon [n-1] \to [n] \qquad s^i \colon [n+1] \to [n].$$

The d^i maps are the unique monotone maps whose image is $[n] \setminus i$. Likewise (for $i \geq 1$), s^i is the unique surjection mapping to i twice. If X is a simplicial set, we write these maps as

$$d_i \colon X_n \to X_{n-1} \qquad s_i \colon X_n \to X_{n+1}.$$

We write the image of $[n]$ under the Yoneda embedding as Δ^n; we will identify $\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) = X_n$, which happens through the following rule:

$$t = \Delta^n \xrightarrow{s} X \xrightarrow{f} Y \quad \leftrightarrow \quad f_n(s) = t.$$

The map $\mathbf{\Delta} \to \mathbf{Top}$ sending $[n]$ to the topological simplex $|\Delta^n|$ extends by colimits to an adjunction

$$|-| : \mathbf{sSet} \longleftrightarrow \mathbf{Top} : \mathrm{Sing},$$

where $\mathrm{Sing}(Y) \coloneqq \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, Y)$. $|-|$ is the *realization* of a simplicial set, built up by gluing standard simplices Δ^n. Although it does not actually, we can (and will) pretend that $|-|$ preserves finite products, because this is true if we replace **Top** by some better category of spaces like compactly generated Hausdorff spaces, for example. In any case, $|X|$ is always a CW complex for any simplicial set X.

A subcomplex of X is a simplicial set S with a map $S \to X$ such that $S_n \to X_n$ is an inclusion. Recall that (co)limits in **sSet** are computed pointwise, so most constructions from **Set** can be transported to **sSet**. For example, suppose $S, T \subset X$ are subcomplexes. Then

$$\begin{array}{ccc} S \cap T & \hookrightarrow & T \\ \downarrow & \lrcorner & \downarrow \\ S & \hookrightarrow & X \end{array}$$

is a pullback because the claim is true at each n, so that $(S \cap T)_n = S_n \cap T_n$. This means we can form unions of subcomplexes by taking the pushout,

$$\begin{array}{ccc} S \cap T & \hookrightarrow & T \\ \downarrow & \ulcorner & \downarrow \\ S & \hookrightarrow & S \cup T \end{array}$$

and the result is a subcomplex $S \cup T \hookrightarrow X$. In general, if $X = \bigcup_i S_i$ is the union of subcomplexes, then we have a coequalizer

$$\coprod_{i,j \in I} S_i \cap S_j \rightrightarrows \coprod_{i \in I} S_i \longrightarrow X$$

of simplicial sets. A central construction to the theory is the following: for $[n] \in \mathbf{\Delta}$, $[n]$ forms a poset category in **Cat**, the category of small categories. A monotone map is the same thing as a functor $[m] \to [n]$, so extending by colimits along the fully faithful inclusion $\mathbf{\Delta} \hookrightarrow \mathbf{Cat}$ gives an adjunction

$$\tau \colon \mathbf{sSet} \rightleftarrows \mathbf{Cat} \colon N,$$

where the right adjoint N is the *nerve*. For a small category C, $N(C)$ is the simplicial set where $N(C)_k$ consists of strings of composable morphisms of length k:

$$a_0 \xrightarrow{f_0} a_1 \to \cdots \to a_{k-1} \xrightarrow{f_k} a_k \qquad a_i \in C$$

Observe that $N([n]) \cong \Delta^n$: a string of composable morphisms is a sequence

$$a_0 \leq \cdots \leq a_k,$$

and the monotone map $i \mapsto a_i$ is the same as a map $[k] \to [n]$. Let $n \geq 0$ and consider $E \subset [n]$ any nonempty subset. Viewing this as a poset category, we write $N(E) \coloneqq \Delta^E \subseteq \Delta^n$ for the corresponding subcomplex. Explicitly, we have

$$(\Delta^E)_k = \{[k] \xrightarrow{\phi} [n] \in (\Delta^n)_k : \operatorname{im} \phi \subset E\}.$$

Of course, we can also view E as an injection $E \colon [|E|] \to [n]$ in $\mathbf{\Delta}$, and there is a unique isomorphism making

$$\begin{array}{ccc} \Delta^E & \searrow & \\ \downarrow \cong & & \Delta^n \\ \Delta^{|E|} & \nearrow_{E} & \end{array}$$

commute. The most important use case will be $E = [n] \setminus k$ for some $k \in [n]$, so the inclusion $\Delta^E \subset \Delta^n$ is isomorphic to the face map $d^k \colon \Delta^{n-1} \hookrightarrow \Delta^n$. Therefore, the corresponding map

$$X_n \to \operatorname{Hom}(\Delta^E, X) \xrightarrow{\sim} X_{n-1}$$

is just the face map d_k.

Definition 2.1.1 Fix $n \geq 1$ and $0 \leq k \leq n$.

1. The *boundary* $\partial\Delta^n$ of Δ^n is the subcomplex

$$\partial\Delta^n \coloneqq \bigcup_{i=0}^{n} \Delta^{[n]\setminus i} \subset \Delta^n.$$

2. The *kth horn* Λ^n_k of Δ^n is the subcomplex

$$\Lambda^n_k \coloneqq \bigcup_{i \neq k} \Delta^{[n]\setminus i} \subset \Delta^n.$$

For an explicit description, we can see that a k-simplex of $\partial\Delta^n$ is a non-surjective map $\phi \colon [k] \to [n]$. Similarly, a k-simplex of the horn Λ^n_i is a map

$\phi\colon [k] \to [n]$ whose image contains excludes an element of $[n]$ which isn't i. In other words, it is a non-surjective map whose image isn't $[n] \setminus \{i\}$. So, in particular, $\Lambda^n_k \subset \partial\Delta^n$ is as well a subcomplex.

Notationally, consider Δ^1, and the two maps $d^1, d^0\colon \Delta^0 = * \to \Delta^1$. We will write $d^1 = 0$ and $d^0 = 1$. So, given a map $f\colon \Delta^1 \to X$ (an element $f \in X_1$), we get a zero simplex $f(0) \coloneqq f \circ d^1\colon \Delta^0 \to X$ and similarly $f(1)$ (i.e. $f(0) = d_1(f)$ and $f(1) = d_0(f)$). Consistent with our nerve notation, we can also write this as

$$f(0) = d_1(f) = f \circ d^1 = f|_{\Delta^0}$$

Along the same lines, let's consider $\Delta^1 \to \Delta^2$. The maps d^0, d^1, d^2 correspond to the edges $1 \to 2$, $0 \to 2$, and $0 \to 1$. Hence the maps d_i are isomorphic to $\Delta^{\{0,1\}}$, etc., depending on i. Therefore, if $\alpha \in X_2$ is a 2-simplex, we may write

$$\alpha(0 \to 2) = d_1(\alpha) = \alpha \circ d_1 = \alpha|_{\Delta^{\{0,2\}}} \in X_1$$

for the same 1-simplex edge of α.

Since neither of these subcomplexes $\partial\Delta^n$ nor Λ^n_k are representable, they don't immediately correspond to X_{n-1} or something similar. Instead, we use the coequalizer description of the union to write them as simplices. Observe that

$$\Delta^{[n]\setminus i} \cap \Delta^{[n]\setminus j} = \Delta^{[n]\setminus\{i,j\}}$$

because the k-simplices of both sides are maps $\phi\colon [k] \to [n]$ which contain neither i nor j in the image. It follows from the coequalizer that a map $f\colon \partial\Delta^n \to X$ corresponds to a collection

$$f = (f_0, \dots, f_n) \qquad f_i \in X_{n-1}$$

of $(n-1)$-simplices such that

$$d_i(f_j) = d_j(f_i) \quad \forall i, j.$$

Using our earlier convention, we can write this 'gluing' condition as

$$f_j|_{\Delta^{[n-1]\setminus i}} = f_i|_{\Delta^{[n-1]\setminus j}},$$

i.e. the simplices must agree on the shared $n-2$-simplices. We observe that if $f \in X_n$, then

$$f|_{\partial\Delta^n} = (d_0 f, d_1 f, \dots, d_n f) = (f|_{\Delta^{[n]\setminus i}})_{i\in[n]}.$$

Similarly, a map $\Lambda^n_k \to X$ is an n-tuple

$$f = (f_0, \dots, f_{\hat{k}}, \dots, f_n)$$

of $(n-1)$-simplices of X such that $d_i f_j = d_j f_i$ for all $i, j \neq k$.

Definition 2.1.2 Let X be a simplicial set. *Objects* of X are 0-simplices, i.e. elements of X_0. Morphisms/arrows/maps of X are 1-simplices. The source and target of a map $f \in X_1$ is $f(0) = d_1(f)$ and $f(1) = d_0(f)$ respectively.

Two morphisms $f, g \in X_1$ are composable if $f(1) = g(0)$. This is the same as saying that $d_0 f = d_1 g$, or in other words that (g, f) defines a morphism $\Lambda^2_1 \to X$. If we let $f(0) = x$, $f(1) = g(0) = y$, and $g(1) = z$, for $x, y, z \in X_0$, then this represents:

$$\begin{array}{ccc} & & z \\ & \overset{f}{\nearrow} & \uparrow{\scriptstyle g} \\ x & & y \end{array}$$

Any morphism $h\colon x \to z$ will make (g, f, h) into a map $\partial\Delta^2 \to X$; we say h is a *composite* of f and g if there is an $\alpha \in X_2$ such that $\alpha|_{\partial\Delta^2} = (g, h, f)$. In other words, f and g have a composite if $\Lambda^2_1 \to X$ extends to Δ^2 along the inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$. Generalizing this allows us to discuss infinity categories.

Definition 2.1.3 A simplicial set X is an ∞-*category* if for each $n \geq 2$ and $0 < k < n$, every map $\Lambda^n_k \to X$ extends to Δ^n. In other words,

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \twoheadrightarrow \mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X)$$

is a surjection for those n, k. X is a *Kan complex* if the same condition holds for all $n \geq 1$ and $0 \leq k \leq n$.

So, in particular, in an ∞-category any two morphisms have a composite, but it may not be unique. We can state our first nontrivial (but still elementary) result:

Theorem 2.1.4 *The nerve functor* $N\colon \mathbf{Cat} \to \mathbf{sSet}$ *is fully faithful, and its essential image consists of those simplicial sets* X *for which the inclusion* $\Lambda^n_k \hookrightarrow \Delta^n$ *induces a bijection*

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{sSet}}(\Lambda^n_k, X)$$

for each $n \geq 2$ *and* $0 < k < n$.

In particular, $\mathbf{Cat} \subset \mathbf{sSet}$ is a reflective subcategory closed under limits, and each small category is an ∞-category. Let x be an object of an infinity category X. Then there is always a morphism $1_x \in X$ from $x \to x$, defined by $1_x = s_0(x)$. We observe that for any $g\colon x \to y$ and $f\colon w \to x$, the diagrams

$$\begin{array}{ccc} & {\scriptstyle x} & \\ {\scriptstyle 1_x}\nearrow & & \searrow{\scriptstyle g} \\ x & \xrightarrow[g]{} & y \end{array} \qquad \begin{array}{ccc} & {\scriptstyle x} & \\ {\scriptstyle f}\nearrow & & \searrow{\scriptstyle 1_x} \\ w & \xrightarrow[f]{} & x \end{array}$$

always commute. Indeed, the 2-simplex $s_0(g)$ satisfies

$$d_0(s_0 g) = d_1(s_0 g) = g \qquad d_2 s_0 g = s_0 d_1(g) = s_0(g(0)) = 1_x$$

by the simplicial identities. Similarly $s_1(f)$ shows that the right-hand square commutes. A morphism $f\colon x \to y$ is *invertible* if there are maps $g, h\colon y \to x$ such that 1_x is a composite of $x \xrightarrow{f} y \xrightarrow{g} x$ and 1_y is a composite of $y \xrightarrow{h}$

$x \xrightarrow{f} y$. An infinity category in which every morphism is invertible is called an ∞-*groupoid*. It is easy to check that a Kan complex is an ∞-groupoid.

The long-term goal is to develop the category theory of ∞-categories; we do this using the inclusion $\mathbf{Cat} \subset \mathbf{sSet}$. Namely, we can take properties of $\mathbf{Cat}$ and use them to define properties of infinity categories. For example, a *functor* of ∞-categories is just a morphism of $\mathbf{sSet}$, because N is fully faithful.

Let $F, G \colon C \to D$ be functors between small categories. A natural transformation $\alpha \colon F \to G$ is the same data as a functor $\alpha \colon C \times [1] \to D$ such that

$$\alpha|_{C\times\{0\}} = F \qquad \alpha|_{C\times\{1\}} = G.$$

So, we can make this an ∞-categorical definition by taking nerves:

$$N(\alpha) \colon N(C) \times N[1] = N(C) \times \Delta^1 \to N(D).$$

Then, again by full faithfulness, we can make the following definition.

Definition 2.1.5 Let $f, g \colon X \to Y$ be functors of ∞-categories. A *natural transformation* α from f to g is a morphism

$$\alpha \colon X \times \Delta^1 \to Y$$

of simplicial sets such that the composites

$$X \times \Delta^0 \xrightarrow{\mathrm{Id}\times 0} X \times \Delta^1 \xrightarrow{\alpha} Y \qquad X \times \Delta^0 \xrightarrow{\mathrm{Id}\times 1} X \times \Delta^1 \xrightarrow{\alpha} Y$$

are f and g respectively.

Now, we will briefly discuss the right adjoint τ to the nerve functor. This will be the first explicit appearance of homotopy, though many homotopy-reminiscent concepts have already been discussed. Let X be an ∞-category and suppose that we have morphisms:

$$x \xrightarrow{f} y \qquad y \xrightarrow{g} z \qquad x \xrightarrow{h} z$$

For the moment we will write $gf \sim h$ if the three form the boundary of a 2-simplex (i.e. h is a composite of f and g).

Lemma 2.1.6 *Let X be an infinity category, x, y be objects in X, and write $S = \{f \in X_1 : f(0) = x, f(1) = y\}$. Consider the following four relations on S:*

1. $f \sim_1 g$ *if* $f1_x \sim g$
2. $f \sim_2 g$ *if* $1_y f \sim g$
3. $f \sim_3 g$ *if* $g1_x \sim f$
4. $f \sim_4 g$ *if* $1_y g \sim f$

Then in fact

$$\sim_1 = \sim_2 = \sim_3 = \sim_4$$

and they are all equivalence relations.

Using the notation of the lemma, if $f \sim_1 g$ then we say that f is homotopic to g. Set

$$\mathrm{Hom}_{\mathrm{ho}(X)}(x, y) \coloneqq S/\sim_1,$$

and let $\mathrm{ho}(X)$ have objects X_0. The following theorem is the main result.

Theorem 2.1.7 *Let X be an ∞-category. Then $\mathrm{ho}(X)$ is a small category equipped with a canonical map*

$$X \to N(\mathrm{ho}(X))$$

that is the identity on objects and the quotient map on morphisms. Moreover there is an isomorphism $\mathrm{ho}(X) \xrightarrow{\sim} \tau(X)$ making the diagram

$$\begin{array}{ccc} X & \longrightarrow & N(\mathrm{ho}(X)) \\ & \searrow & \downarrow \cong \\ & & N(\tau(X)) \end{array}$$

commute.

We call $\mathrm{ho}(X)$ the homotopy category of X. $\mathrm{ho}(X)$ represents the 1-categorical information contained in X. But, another corollary is that a morphism $f\colon x \to y$ in X is invertible if and only if $[f]$ is an isomorphism in $\mathrm{ho}(X)$. Hence the inverse to f is unique up to homotopy, and the same inverse is both a left and right homotopy inverse. It follows that X is an ∞-groupoid if and only if $\mathrm{ho}(X)$ is a groupoid.

Example 2.1.8 This is a quasiexample, because we need some better set-theoretic foundations before formalizing this. But, roughly, there is an infinity category $\mathbf{Cat}_1$ of small categories. The objects and morphisms are small categories and functors between them as usual. The 2-cells are natural transformations $\alpha\colon GF \cong H$ whenever this makes sense. Hence a functor is invertible exactly when it is an equivalence of categories. We 'leave' all the higher cells as strict equalities (i.e. encode no more interesting information in them), but we do not have enough to make this precise yet.

The homotopy category $\mathrm{ho}(\mathbf{Cat}_1)$ in this case is a quotient of **Cat** by the equivalence relation on hom-sets of natural isomorphism. I.e. two categories are isomorphic in the quotient iff they are equivalent.

If Y is a space, another example is $\mathrm{Sing}(Y)$. It is easy to check that $\mathrm{Sing}(Y)$ is a Kan complex, and $\mathrm{ho}(\mathrm{Sing}(Y))$ recovers $\Pi_1(Y)$, the fundamental groupoid of Y. So, $\mathrm{Sing}(Y)$ as an ∞-category retains the fundamental group (as the automorphism group of a basepoint in the homotopy category). An important

question is where the other homotopy groups are; we will be able to answer this eventually.

TODO: Insert material here defining and proving facts about some basic constructions of ∞-categories, especially $\mathbf{Cat}_1$, done more precisely. Consider the case of pseudofunctors between categories, and see what axioms come out of this. Consider the definition of a pseudopresheaf of categories.

2.2 Model Categories

References

[1] Mac Lane, S., Moerdijk, I.: Sheaves in Geometry and Logic. Universitext. Springer, NY (1994)

[2] Cisinski, D.-C.: Higher Categories and Homotopical Algebra. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2019)