Locally finitely presentable categories

Vihaan Dheer

The goal of this document is to rigorously prove enough about locally finitely presentable categories to discuss sheafification with values in such a category. We start 'from the beginning', i.e. by discussing filtered colimits in Sets.

1 Filtered and sifted colimits

1.1 The category of sets

The central definition to this work is as follows: A category J is *filtered* if it is nonempty and satisfies the following conditions:

1. For any $i, j \in J$, there is an object $k \in J$ and morphisms $i \to k$ and $j \to k$; and
2. For parallel morphisms $u, v \colon i \to j$, there is a morphism $w \colon j \to k$ in J such that $wu = wv$.

This definition is good for examples: Our primary example will be poset categories $(J, \leq)$, in which case the second condition is automatically satisfied by taking $k = j$. The first holds iff every pair of objects has an upper bound.

Lemma 1.1.1 *A category J is filtered iff every finite diagram has a cocone.*

Proof ($\Leftarrow$) A cocone for the empty diagram is an object, showing nonemptiness. Considering the diagram $\{0, 1\} \to J$ sending $0 \mapsto i$ and $1 \mapsto j$, a cocone is an object k with maps from i and j. Finally, a cocone from $i \rightrightarrows j$ is an object k with a map $j \to k$ satisfying the desired condition.

($\Rightarrow$) Let us prove the following special cases:

(a) Given objects $a_1, \ldots, a_n \in J$, there is a $b \in J$ and maps $u_i \colon a_i \to b$.

(b) Given maps $u_1, \ldots, u_n \colon a \to b$ in J, there is a $v \colon b \to c$ such that $vu_i = vu_j$ for i, j.

Both are by induction: for (a), take maps $u_i' \colon a_i \to b'$ for $i = 1, \ldots, n-1$, and then take $f \colon a_n \to b$ and $g \colon b' \to b$ so that we have maps $gu_1', \ldots, gu_{n-1}', f$. For (b), choose $g' \colon b \to c'$ such that $g'f_1 = \cdots = g'f_{n-1}$, then choose $g \colon c' \to c$ such that $gg'f_n = gg'f_{n-1}$. Then

$$gg'f_n = gg'f_{n-1} = g(g'f_{n-1}) = \cdots = g(g'f_1)$$

as desired using gg'.

To conclude, let $a_1, \ldots, a_n$ be the objects of the finite diagram and fix $f_i \colon a_i \to b$. For each morphism $u \colon a_i \to a_j$ in the diagram, f_i and $f_j u$ are parallel arrows so there is $g_u \colon b \to c_u$ such that $g_u f_i = g_u f_j u$. Running over all such u, there is $g'_u \colon c_u \to d$, and hence we have a collection of maps $g'_u g_u \colon b \to d$ so there is $g'' \colon d \to e$ such that $g'' g'_u g_u = h$ is some fixed h for varying u.

We claim that $hf_i \colon a_i \to e$ is a cocone. Indeed, for any $u \colon a_i \to a_j$ we have

$$(hf_j)u = g'' g'_u (g_u f_j u) = g'' g'_u g_u f_i = hf_i,$$

completing the proof. □

The next theorem, namely the explicit description of filtered colimits in **Set**, is the foundational result of this section. Suppose $X_\bullet \colon J \to \mathbf{Set}$ is a filtered diagram, i.e. a diagram from a *small* filtered category. Consider the relation $\sim$ on $\bigsqcup_{i \in J} X_i$ defined by $(x, i) \sim (y, j)$ iff there exists $k \in J$ and $i \xrightarrow{u} k \xleftarrow{v} j$ such that $u_*(x) = v_*(y)$ in X_k.

Let us see that $\sim$ is an equivalence relation. Reflexivity and symmetry are obvious. Suppose we have $i \xrightarrow{u} k \xleftarrow{v} j$ with $u_*(x) = v_*(y)$ and $j \xrightarrow{w} n \xleftarrow{z} m$ with $w_*(y) = z_*(y')$. Then take a cocone from the finite diagram

$$k \xleftarrow{v} j \xrightarrow{w} n,$$

say p, equipped with maps $a \colon k \to p$ and $b \colon n \to p$ with $av = bw$. Then we have $i \xrightarrow{au} p \xleftarrow{bz} m$ such that

$$(au)_*(x) = a_* v_*(y) = b_* w_*(y) = b_* z_*(y'),$$

which is what we needed to show $(x, i) \sim (y', m)$.

Theorem 1.1.2 *Let $X_\bullet \colon J \to \mathbf{Set}$ be a filtered diagram and consider the quotient*

$$X \coloneqq \left(\bigsqcup_{i \in J} X_i \right) / \sim$$

by the equivalence relation defined above. Then the canonical maps $c = c_i \colon X_i \to X$ form a cocone to $X_\bullet$ and exhibit $X = \varinjlim_i X_i$.

Proof To see that (X, c) is a cocone, note that if $u \colon i \to j$ in J, then for any $x \in X_i$, $(x, i) \sim (u_*(x), j)$ so $c_i = c_j u_*$.

To verify the universal property; suppose we have a collection of maps $f_i \colon X_i \to Y$ such that for all $u \colon i \to j$ in J, $f_j u_* = f_i$. These maps induce $f \colon \bigsqcup_i X_i \to Y$. If $(x, i) \sim (y, j)$, i.e. there is $i \xrightarrow{u} k \xleftarrow{v} j$ with $u_*(x) = v_*(y)$. Hence

$$f(x, i) = f_i(x) = f_k u_*(x) = f_k v_*(y) = f_j(y) = f(y, j),$$

so f descends to the quotient $\widetilde{f}\colon X \to Y$. Note that

$$\widetilde{f}c_i(x) = \widetilde{f}([x,i]) = f_i(x)$$

by definition.

To see that $\widetilde{f}$ is unique, let $q\colon \bigsqcup_i X_i \to X$ be the quotient map and $\mathrm{inj}_i\colon X_i \to \bigsqcup_i X_i$ be the canonical injections. If g is a map with $gc_i = f_i$ for all i, then since $c_i = q\mathrm{inj}_i$ we have

$$f_i = gc_i = (gq)\mathrm{inj}_i \qquad i \in J,$$

hence $f = gq$. But $f = \widetilde{f}q$ as well so $\widetilde{f} = g$ because q is epi. □

We have just specified a filtered colimit for each filtered diagram, so for each small filtered category J this defines a colimit functor

$$\varinjlim_J\colon \mathbf{Set}^J \to \mathbf{Set}$$

with its right adjoint the constant functor $\Delta\colon \mathbf{Set} \to \mathbf{Set}^J$. This of course preserves colimits, but we will soon see it preserves some limits as well. We will first need some properties.

Example 1.1.3 If J is a filtered category and $\mathbf{C}$ is a category, then the filtered colimit of constant diagram $\Delta_X\colon J \to \mathbf{C}$ for some $X \in \mathbf{C}$ always exists and equals X with the identity cocone. It is clear that the latter is a cocone because Δ_X takes all maps to 1_X. Given maps $f_i\colon X \to Y$ in $\mathbf{C}$, whenever there is a morphism $u\colon i \to j$, the cone condition implies $f_i = f_j$. By filtered-ness, for any i, j there are always maps to a common k, so $f_i = f_k = f_j$. The unique map $X \to Y$ commuting with the identity is just $f = f_i$ for any i.

Lemma 1.1.4 *Let $f\colon X_\bullet \to Y_\bullet$ be a natural transformation between filtered diagrams in* **Set**. *If each f is a (componentwise) monomorphism, then the induced map $\varinjlim_i f_i = f\colon \varinjlim_i X_i \to \varinjlim_i Y_i$ is a monomorphism.*

Proof We use the explicit construction of the filtered colimit in **Set**. Suppose equivalence classes $[x,i]$ and $[x',i']$ have a common image under the induced map f, that is $\big(f_i(x), i\big) \sim \big(f_{i'}(x'), i'\big)$. There exists $i \xrightarrow{u} j \xleftarrow{v} i'$ so that

$$u_* f_i(x) = v_* f_{i'}(x'),$$

and thus by naturality of f this implies that

$$f_j u_*(x) = f_j v_*(x'),$$

so $u_*(x) = v_*(x')$ by assumption and $(x,i) \sim (x',i')$. □

Example 1.1.5 Most often our examples of filtered colimits will be the following one: X is a set and $\mathcal{C} \subset \mathcal{P}(X)$ is a collection of subsets with the property that if $X_1, X_2 \in \mathcal{C}$, then there is an $X_3 \in \mathcal{C}$ such that $X_1 \cup X_2 \subset X_3$. Recall that this means $(\mathcal{C}, \leq)$ is a filtered catgeory.

It is obvious that the inclusions $\{X_i \hookrightarrow X : X_i \in \mathcal{C}\}$ form a cocone to the union $\bigcup \mathcal{C}$. We claim that this exhibits $\bigcup \mathcal{C}$ as the filtered colimit $\varinjlim_{X_i \in \mathcal{C}} X_i$. One way is to verify the universal property directly; alternatively, that $X_i \to \bigcup \mathcal{C}$ form a cocone means that we have a natural transformation from the diagram to the constant functor which is a componentwise monomorphism. Taking the filtered colimit produces a monomorphism

$$\varinjlim_{X_i \in \mathcal{C}} X_i \to \varinjlim_{X_i \in \mathcal{C}} \bigcup \mathcal{C} = \bigcup \mathcal{C},$$

by the two facts above, and precomposing with the canonical maps into the colimit produces the inclusions $X_i \to \mathcal{C}$ so it is the same as the induced map. It is surjective because each $x \in \bigcup \mathcal{C}$ is contained in X_i for some i, so x is the image of $[x, i]$.

Theorem 1.1.6 *Filtered colimits commute with finite limits in* **Set**.

Proof Fix J a small filtered category. It suffices to show that the colimit functor $\varinjlim_J \colon \mathbf{Set}^J \to \mathbf{Set}$ defined above preserves the terminal object, binary products, and equalizers.

(0) A terminal object is the constant functor $\Delta_{\{*\}}$ at the one point set, and we already showed that $\varinjlim_J \Delta_{\{*\}} = \{*\}$ canonically.

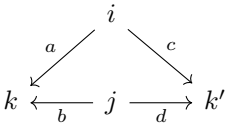
(1) If $X_\bullet, Y_\bullet$ are filtered diagrams, we must show that the natural map

$$\varinjlim_{i \in J}(X_i \times Y_i) \to \left(\varinjlim_{i \in J} X_i\right) \times \left(\varinjlim_{i \in J} Y_i\right)$$

is a bijection. For surjectivity, let $([x, i], [y, j])$ be in the image and choose maps $i \xrightarrow{u} k \xleftarrow{v} j$ in J. Then the element $[(u_*(x), v_*(y)), k]$ in the domain maps to

$$\big([u_*(x), k], [v_*(y), k]\big) = \big([x, i], [y, j]\big).$$

For injectivity, suppose $([x, i], [y, i]) = ([x', j], [y', j])$. This means that we have the maps a, b, c, d as in the following diagram such that $a_*(x) = b_*(x')$ and $c_*(y) = d_*(y')$.



Take a cocone from this diagram to some object ℓ with maps $u \colon k \to \ell$ and $v \colon k' \to \ell$. We claim that

$$(ua)_*(x, y) = (vd)_*(x', y').$$

Indeed, using $ua = vc$ and $ub = vd$,

$$\begin{aligned}(ua)_*(x, y) &= (u_* a_* x, v_* c_* y) \\ &= (u_* b_* x', v_* d_* y')\end{aligned}$$

$$= (vd)_*(x', y'),$$

which shows that $((x, y), i) \sim ((x', y'), j)$.

(2) Let $E_i \to X_i \rightrightarrows Y_i$ be a natural family equalizer diagrams for $i \in J$ by maps $f_i, g_i \colon X_i \to Y_i$. We use the description $E_i = \{x \in X_i : f_i(x) = g_i(x)\}$. Taking the filtered colimit we obtain a commutative diagram

$$\varinjlim_{i \in J} E_i \to \varinjlim_{i \in J} X_i \underset{g}{\overset{f}{\rightrightarrows}} \varinjlim_{i \in J} Y_i$$

where the first map is injective, hence identified with a subset of $\varinjlim_{i \in J} X_i$. We need to show it is the subset on which f and g agree. Let $\alpha \in \varinjlim_i X_i$. Then

$$\begin{aligned} \alpha \in \varinjlim_{i \in J} E_i \quad &\leftrightarrow \quad \exists i \text{ s.t. } x \in E_i, \alpha = [x, i] \\ &\leftrightarrow \quad \exists i \text{ s.t. } \alpha = [x, i], f_i(x) = g_i(x). \end{aligned}$$

On one hand, this implies that $f(\alpha) = [f_i(\alpha), i] = [g_i(\alpha), i] = g(\alpha)$. On the other hand, assume $f(\alpha) = g(\alpha)$. Then writing $\alpha = [x, j]$, we have $[f_j(x), j] = [g_j(x), j]$, i.e. there are parallel maps $u, v \colon j \to k$ such that $u_* f_j(x) = v_* g_j(x)$. Further choose $w \colon k \to \ell$ with $wu = vu$, then

$$(wu)_* f_j(x) = w_* v_* g_j(x) = (wu)_* g_j(x),$$

so by naturality $f_\ell (wu)_*(x) = g_\ell (wu)_*(x)$, so $(wu)_* x \in E_\ell$, hence $[x, j] = [(wu)_* x, \ell] \in \varinjlim_i E_i$.

Recall that this suffices to show commutativity with finite limits because we can always write the limit of a finite diagram $X_\bullet \colon I \to \mathbf{Set}$ as the equalizer of

$$\prod_{i \in I} X_i \rightrightarrows \prod_{u \colon i \to j \in I} X_j,$$

where the u component of each map is given by

$$\prod_{i \in I} X_i \underset{\mathrm{pr}_j}{\overset{u_* \mathrm{pr}_i}{\rightrightarrows}} X_j,$$

so we are done. □

There is also a sort of (mostly useless) converse to this statement. Suppose J is a nonempty small category for which J-colimits in **Set** commute with finite limits. Then we claim J is filtered. Indeed, let $x_\bullet \colon I \to J$ be a finite diagram. For each $j \in J$, we have by definition

$$\mathrm{Cocone}(x_\bullet, j) = \varprojlim_{i \in I} \mathrm{Hom}_J(x_i, j).$$

If J is not filtered, then the LHS is empty for all $j \in J$, so taking the J-colimit we obtain

$$\varinjlim_{j\in J} \varprojlim_{i\in I} \mathrm{Hom}_J(x_i, j) = \varnothing,$$

(because e.g. the colimit admits a surjection from $\bigsqcup_j \mathrm{Cocone}(x_\bullet, j)$).

On the other hand, for any i, observe that we have isomorphisms

$$\begin{aligned}\mathrm{Cocone}(\mathrm{Hom}_J(x_i, -), Y) &= \mathrm{Hom}_{\mathbf{Set}^J}\big(\mathrm{Hom}_J(x_i, -), \Delta_Y\big) \\ &\cong \Delta_Y(x_i) = Y \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(*, Y)\end{aligned}$$

(natural in a set Y) by the Yoneda lemma, so $\varinjlim_j \mathrm{Hom}_J(x_i, j) = *$. Then

$$\begin{aligned}\varprojlim_{i\in I} \varinjlim_{j\in J} \mathrm{Hom}_J(x_i, j) &= \varprojlim_{i\in I} * = \mathrm{Cone}(*, \Delta_*) \\ &= \mathrm{Hom}_{\mathbf{Set}^I}(\Delta_*, \Delta_*) \ni \mathrm{Id}_{\Delta_*}\end{aligned}$$

is nonempty, which contradicts the commutativity assumption. Note, finally, that in the case $J = \varnothing$, colimits don't even commute with the empty diagram: doing $\varinjlim_\varnothing \varprojlim_\varnothing$ must produce $\varnothing$, and with $\lim_\varnothing \varinjlim_\varnothing$ must give $*$. So:

Proposition 1.1.7 *A small category J is filtered if and only if J-shaped colimits commute with finite limits in* **Set**.

This also leads to the following definition: a small category J is *sifted* if J-shaped colimits commute with finite products in **Set**. A *sifted colimit* is a colimit of a diagram with domain a sifted category. Of course, by Thm. 1.1.6, filtered colimits are a special case of sifted colimits.

1.2 Algebraic categories

Filtered colimits also have an especially good presentation in so-called algebraic categories, such as **Mon**, **CMon**, **Grp**, and so on. We will now prove these most important cases by hand, and then discuss a general method for handling algebraic categories. Note: for now, the phrase algebraic catgeory is defined only by a list of examples.

The idea is as follows: Let **C** be one of these algebraic categories and let $U\colon \mathbf{C} \to \mathbf{Set}$ be the forgetful functor. We want to show that U *creates* filtered colimits and give an explicit description of them. This means that filtered colimits should exist, the underlying set of the filtered colimit should be the filtered colimit of the underlying sets, and it should reflect the filtered colimits. The last part is immediate because U is always conservative.

Let's take a category like monoids. Given a filtered diagram $M_\bullet$ of monoids, a naive approach would be to try to define a monoid structure on $\varinjlim_i U(M_i) \in \mathbf{Set}$. The multiplication should be $[x, i] \cdot [y, j] = [u_*(x)v_*(y), k]$ for some $i \xrightarrow{u}$

$k \xleftarrow{v} j$, but it is cumbersome to check that everything is well-defined and satisfies the axioms.

Instead, let us take an alternative approach by using the equivalence $\mathbf{Mon} \cong \mathrm{Mon}(\mathbf{Set})$ of monoids with monoid objects in $\mathbf{Set}$. If $M_\bullet \colon J \to \mathbf{Mon}$ is a filtered diagram, then we can interpret this as a filtered diagram of monoid objects in $\mathbf{Set}$. Then $UM_\bullet$ is naturally a monoid object in $\mathbf{Set}^J$, because products and the axioms are computed pointwise.

Now $\varinjlim_J \colon \mathbf{Set}^J \to \mathbf{Set}$ preserves finite limits, hence induces a functor $\varinjlim_J \colon \mathrm{Mon}(\mathbf{Set}^J) \to \mathrm{Mon}(\mathbf{Set}) \cong \mathbf{Mon}$, showing that $\varinjlim_i U(M_i)$ is canonically a monoid object where the multiplication operation is the right hand map in the diagram

$$\begin{array}{ccc} \varinjlim_i U(M_i) \times U(M_i) & \xrightarrow{\cong} & \varinjlim_i U(M_i) \times \varinjlim_i U(M_i) \\ & \searrow^{m} & \downarrow \\ & & \varinjlim_i U(M_i) \end{array}$$

and makes it commute. In particular, $[x, i] \cdot [y, i] = [xy, i]$ which implies the multplication relation we expected. Write M for this monoid. The canonical set maps $U(M_i) \to M$ are thus homomorphisms $M_i \to M$, and that they form a cocone is checked in the underlying sets.

Given a cocone $f_i \colon M_i \to N$, there is already a unique set map $U(M) \to U(N)$ commuting with the cocone maps, so it just remains to see that it is a monoid homomorphism. The map is given by $f([x, i]) = f_i(x)$, and any pair of elements can be written $[x, k], [y, k]$, so

$$f([x, k] \cdot [y, k]) = f([xy, k]) = f_k(x) f_k(y) = f([x, k]) f([y, k])$$

gives us what we wanted. Observe that since U is right adjoint to the free functor, it preserves (and hence creates limits). This shows that finite limits and filtered colimits commute in $\mathbf{Mon}$ as well. To summarize:

Proposition 1.2.1 *Filtered colimits exist in the category* $\mathbf{Mon}$ *of monoids, and the forgetful functor* $\mathbf{Mon} \to \mathbf{Set}$ *creates filtered colimits. Moreover, filtered colimits commute with finite limits in* $\mathbf{Mon}$.

We want to try to repeat this proof in all of the familiar algebraic categories simultaneously. To do this, we will proceed with a systematic study of Lawvere theories and essentially algebraic theories. We will eventually come to *Gabriel-Ulmer duality*, which shows that essentially algebraic theories and lfp categories are the same thing.

Definition 1.2.2

1. A *Lawvere theory* is an essentially small category T with finite products equipped with distinguished object x such that every object is (isomorphic to) a finite power $x^n = x \times \cdots \times x$ of x.
2. A morphism of Lawvere theories is a functor which preserves finite products and the distinguished object. The category of Lawvere theories is denoted **Law**.
3. If T is a Lawvere theory, the *category of T-algebras* is the full subcategory of $\mathrm{Fun}(T, \mathbf{Set})$ spanned by finite-product preserving functors, and is denoted $\mathrm{Alg}(T)$.

If T is a Lawvere theory, we have a canonical functor $U\colon \mathrm{Alg}(T) \to \mathbf{Set}$ given by evaluation at the distinguished object x, called the forgetful functor. This has many similar properties to the forgetful functors we are familiar with. Consider the composite

$$U = \mathrm{Alg}(T) \hookrightarrow \mathrm{Fun}(T, \mathbf{Set}) \xrightarrow{\mathrm{ev}_x} \mathbf{Set}.$$

We claim that $\mathrm{Alg}(T)$ is closed under limits and sifted colimits. E.g. if $X_\bullet\colon J \to \mathrm{Alg}(T) \to \mathrm{Fun}(T, \mathbf{Set})$ is a sifted diagram, then because (co)limits are computed fiberwise we have

$$\begin{aligned}\big(\varinjlim_{i\in J} X_i\big)(t \times t') &= \varinjlim_{i\in J} X_i(t \times t') = \varinjlim_{i\in J} \big(X_i(t) \times X_i(t')\big) \\ &= \varinjlim_{i\in J} X_i(t) \times \varinjlim_{i\in J} X_i(t') \\ &= \big(\varinjlim_{i\in J} X_i\big)(t) \times \big(\varinjlim_{i\in J} X_i\big)(t')\end{aligned}$$

since sifted colimits commute with finite limits in **Set**, therefore $\varinjlim_i X_i \in \mathrm{Alg}(T)$. The same thing works of course for arbitrary small limits, because they commute with products too. It follows that $\mathrm{Alg}(T)$ is complete and has sifted colimits, and that the embedding in $\mathrm{Fun}(T, \mathbf{Set})$ creates them. In particular, U preserves limits and sifted colimits.

Suppose $\phi\colon X \to Y$ is a morphism in $\mathrm{Alg}(T)$. By naturality and the fact that X and Y preserve finite products, we have a commutative diagram

$$\begin{array}{ccc} X(x^n) & \xrightarrow{\phi_{x^n}} & Y(x^n) \\ \cong\downarrow & & \downarrow\cong \\ U(X) & \xrightarrow{U(\phi)^n} & U(Y) \end{array}$$

(since $U(X) = x$). Hence if $U(\phi)$ is an isomorphism, then ϕ_{x^n} is an isomorphism for all n, and then for any $t \in T$, $t \cong x^n$ implies by naturality that ϕ_t is an isomorphism. Thus U is conservative.

In addition, by the same logic, if $U(\phi) = U(\phi')$, then $\phi_{x^n} = {\phi'}_{x^n}$ by the square above, and hence $\phi_t = {\phi'}_t$ for all t. Therefore U is faithful. To summarize what we have said so far:

Proposition 1.2.3 *Let T be a Lawvere theory and $U\colon \mathrm{Alg}(T) \to \mathbf{Set}$ the forgetful functor. Then U is faithful and conservative, $\mathrm{Alg}(T)$ has small limits and sifted colimits, and U creates both of them. In particular, sifted colimits commute with finite products and filtered colimits commute with finite limits in $\mathrm{Alg}(T)$.*

Proof The creation is just because U is conservative, and the final two statements follow from U creating all the involved kinds of limits and colimits and the same statement in **Set**. □

Remark 1.2.4 It is not necessary for us to use **Set** in these constructions. In other words, if $\mathcal{D}$ is some category with small limits and sifted colimits - where sifted colimits commute with finite products and filtered colimits commute with finite limits - we could define $\mathrm{Alg}(T, \mathcal{D})$ to be product preserving functors $T \to \mathcal{D}$, and then everything we just did still works the same way using the forgetful functor $\mathrm{Alg}(T, \mathcal{D}) \to \mathcal{D}$. The application of this would be to take $\mathcal{D} = \mathrm{Alg}(T')$ for some other Lawvere theory T'.

Before going further let us discuss some examples of T-algebras, to convince the reader that all of the familiar algebraic categories are in fact algebras of a Lawvere theory. In what follows, Let $\mathbb{N} \subset \mathbf{FinSet}$ be a skeleton $0, 1, 2, \dots$ of finite sets. Since the inclusion $\mathbb{N} \hookrightarrow \mathbf{Set}$ is fully faithful, $\mathbb{N}$ is closed under (and hence has) finite colimits (and limits). In particular, $n = 1 \sqcup \cdots \sqcup 1$.

Let $T = \mathbb{N}^{\mathrm{op}}$, which for the reason stated above is a Lawvere theory with distinguished object 1. We have the forgetful functor $U\colon \mathrm{Alg}(\mathbb{N}^{\mathrm{op}}) \to \mathbf{Set}$, and in the other direction $\mathbf{Set} \to \mathrm{Alg}(\mathbb{N}^{\mathrm{op}})$ we have the Yoneda functor sending $S \mapsto \mathrm{Hom}_{\mathbf{Set}}(-, S)\colon \mathbb{N}^{\mathrm{op}} \to \mathbf{Set}$, which is obviously product preserving.

We claim this is an equivalence. In one direction the composite is just $S \mapsto \mathrm{Hom}_{\mathbf{Set}}(1, S)$, yet $\mathrm{Hom}_{\mathbf{Set}}(1, -) \cong \mathrm{Id}_{\mathbf{Set}}$. The other composite is $X \mapsto \mathrm{Hom}_{\mathbf{Set}}(-, X(1))$, and we have natural identifications

$$\mathrm{Hom}(n, X(1)) = \mathrm{Hom}(1, X(1))^n = X(1)^n = X(n)$$

for each n, giving $\mathrm{Hom}(-, X(1)) \cong X$.

Here is the general recipe for how to convert some algebraic category $\mathbf{C}$ that we care about into a category of algebras for a Lawvere theory. First construct the standard free-forgetful adjunction $F\colon \mathbf{Set} \leftrightarrows \mathbf{C} : U$, and define the category $T_{\mathbf{C}} \coloneqq \{F(0), F(1), F(2), \dots\}^{\mathrm{op}}$ as the opposite of the full subcategory of 'finite free' objects in $\mathbf{C}$. Since $\mathbb{N}$ is closed under finite coproducts and F preserves them, $T_{\mathbf{C}}$ is closed under finite products and the distinguished object $F(1)$ makes it a Lawvere theory.

Note that there's always an obvious functor $\mathbf{C} \to \mathrm{Alg}(T_{\mathbf{C}})$ defined by Yoneda, i.e. $c \mapsto \mathrm{Hom}_{\mathbf{C}}(-, c)$. We will try to define a functor $\mathrm{Alg}(T_{\mathbf{C}}) \to \mathbf{C}$ backwards, but we will always want this to be a functor *over* **Set**, that is we define a $\mathbf{C}$

structure on the underlying set $X(1)$. Note that in this case one of the two composite functors will be

$$X \mapsto \mathrm{Hom}_{\mathbf{C}}(-, X(1)),$$

and we have a natural isomorphism $\mathrm{Hom}(F(n), X(1)) \cong \mathrm{Hom}(F(1), X(1))^n \cong U(X(1))^n = X(n)$, hence $\mathrm{Hom}_{\mathbf{C}}(-, X(1)) \cong X$.

The point is that an n-ary operation involved in defining an object of $\mathbf{C}$ on top of a set should correspond to an element of the set $U(F(n))$. For example, the multiplication on a monoid corresponds to the element $xy \in F(\{x, y\})$ on the free monoid on two generators, and the identity element corresponds to the element $1 \in F(0) = \{*\}$. Since elements of $U(F(n))$ correspond to maps $F(1) \to F(n)$, these elements should correspond to putting a *comonoid* structure on $F(1)$ in, and hence make $F(1)$ a monoid object of $T_{\mathbf{Mon}}$. Then a product preserving functor X to **Set** sends makes the $X(1) = U(X)$ a monoid object in **Set**, and hence an object of **Mon**.

Since we have discused **Mon**, let us study the case of groups more closely. Let $T_{\mathbf{Grp}}$ be defined as above as the opposite of free groups $\{e\}, F(x_1)$, $F(x_1, x_2), \ldots$. There is an obvious functor **Grp**. The following distinguished elements

$$e \in \{1\}, \qquad x_1^{-1} \in F(x_1), \qquad x_1 x_2 \in F(x_1, x_2)$$

define maps $F(1) \to F(i)$ for $i = 0, 1, 2$. Then associativity and identity satisfy

$$\begin{array}{ccc} F(1) & \longrightarrow & F(2) \\ \downarrow & & \downarrow \\ F(2) & \longrightarrow & F(3) \end{array} \qquad\qquad \begin{array}{ccccc} & & F(1) & & \\ & \swarrow^{\cong} & \downarrow & \searrow^{\cong} & \\ F(1)\coprod\{e\} & \longleftarrow & F(2) & \longrightarrow & \{e\}\coprod F(1) \end{array}$$

and the inverse law is the fact that

$$\begin{array}{ccc} F(1) & \longrightarrow & F(2) \\ \downarrow & \searrow & \downarrow{\scriptstyle (x_1, x_1^{-1})} \\ & F(0) & \searrow \\ F(2) & \xrightarrow[(x_1^{-1}, x_1)]{} & F(1) \end{array}$$

commutes. In each case, the commutativity is really just expressing an equation about the distinguished elements: for instance, the natural coproduct maps in the associativity condition send $x_1 \mapsto x_1 x_2$ and $x_2 \mapsto x_3$, or $x_1 \mapsto x_1$ and $x_2 \mapsto x_2 x_3$. Therefore that diagram says that

$$(x_1 x_2) x_3 = x_1 (x_2 x_3) \qquad \text{in } F(x_1, x_2, x_3).$$

The identity condition says that $(x_1)1 = 1(x_1) = x_1$ in $F(x_1)$. The inversion condition says that $x_1 x^{-1} = x_1^{-1} x_1 = 1 \in F(1)$. This shows that $F(1)$ is a cogroup in $T_{\mathbf{Grp}}^{\mathrm{op}}$, and hence any product preserving functor $T_{\mathbf{Grp}} \to \mathbf{Set}$ sends

$F(1)$ to a group object in **Set**, i.e. a group. This gives a functor $\mathrm{Alg}(T_{\mathbf{Grp}}) \to \mathbf{Grp}$.

To check that we have an equivalence, as explained before we simply need to check that the composite sends a group G to itself. Explicitly, the cogroup structure on $F(1)$ makes $U(G) \cong \mathrm{Hom}(F(1), G)$ into a group object. To see that the multiplications coincide, given $g, g' \in U(G)$, we consider them as maps $F(1) \to U(G)$, then their product corresponds to the precomposition

$$F(1) \to F(2) = F(1) \sqcup F(1) \xrightarrow{(g,g')} G,$$

which sends $x_1 \mapsto x_1 x_2 \mapsto gg'$, hence the group structure on G coincides with that induced by $F(1)$, completing the proof.

Note that if we instead take free abelian groups instead of free groups, everything works identically except that we have an abelian group object, i.e. in $F(2)$ we have the additional commutativity condition:

$$\begin{array}{ccc} F(2) & \xrightarrow[\cong]{\qquad} & F(2) \\ & \searrow \quad \swarrow & \\ & F(1) & \end{array}$$

Here the top map is the canonical braiding automorphism from the coproduct structure $F(2) = F(1) \coprod F(1)$ flipping the generators. With little modification, everything works for (commutative) magmas, (commutative) semigroups, (commutative) monoids.

Example 1.2.5 The case $\mathbf{C} = \mathbf{Ring}$ (of commutative rings) is only slightly more complicated because there is a second binary operation. To construct it, start with $T_{\mathbf{Ring}}$ as above, written as the opposite of the free rings $\mathbb{Z}, \mathbb{Z}[x_1], \mathbb{Z}[x_1, x_2], \ldots$. Then the multiplication structure comes from the elements $x_1 x_2$ and addition is from $x_1 + x_2$, and the distributivity and commutativity and all other axioms holds from these ones. Given a product preserving $X \colon T_{\mathbf{Ring}} \to \mathbf{Set}$, we equip $X(1)$ with a product $X(1) \times X(1) = X(2) \to X(1)$ which is given by X applied to the map $\mathbb{Z}[x_1] \to \mathbb{Z}[x_1, x_2]$ for multiplication, and likewise for addition. This works as before. Note that we recover R-algebras by replacing T with the opposite of $R, R[x_1], R[x_1, x_2], \ldots$.

Example 1.2.6 A structurally different example is $\mathbf{C} = \mathrm{Mod}(R)$ for R a fixed ring, because there is no obvious object like group/monoid object construction which recovers the bilinear map $R \times M \to M$ for an R-module. We start as before by taking $T^{\mathrm{op}} = 0, R^1, R^2, \ldots$ as the free algebras. The syntax of an abelian group is encoded in $0, R^1, R^2$ just as before via the elements

$$0 \in 0, \qquad -1 \in R^1, \qquad (1,1) \in R^2$$

making R^1 canonically an abelian group object in T, and hence if $X \in \mathrm{Alg}(T)$ then $U(X)$ is an abelian group. For each $r \in R$, there is the module map

$r\colon R \to R$ which in turn induces $\phi_r\colon U(X) \to U(X)$, so that in particular we have a set map $R \times U(X) \to U(X)$.

To verify the axioms, note that the diagram

$$\begin{array}{ccc} R^1 & \xrightarrow{r\cdot -} & R^1 \\ \downarrow & & \downarrow \\ R^2 & \xrightarrow[r \coprod r]{} & R^2 \end{array}$$

commutes, as both composites send $1 \mapsto (r, r)$. This implies that $r\colon R^1 \to R^1$ is a morphism of cogroups, hence ϕ_r is a group homomorphism for each r. Next observe that the two diagrams

$$\begin{array}{ccc} R^1 & & \\ \downarrow & \searrow^{(r+s)\cdot -} & \\ R^2 & \xrightarrow[(r\cdot -, s\cdot -)]{} & R^1 \end{array} \qquad \begin{array}{ccc} R^1 & & \\ {\scriptstyle r\cdot -}\downarrow & \searrow^{sr\cdot -} & \\ R^1 & \xrightarrow[s\cdot -]{} & R^1 \end{array}$$

commute, as the first sends $1 \mapsto (1, 1) \mapsto r + s$ and the second sends $1 \mapsto r \mapsto sr$ both ways. This converts to the equations $\phi_{r+s}(x) = \phi_r(x) + \phi_s(x)$ and $\phi_{sr}(x) = \phi_s(\phi_r(x))$. Hence M is an R-module and we have the equivalence as usual. This technique of encoding the R-multiplication works for anything with an action, e.g. like G-sets.

1.3 Detour: Final functors

Here we briefly pause the discussion of algebraic categories to explain the generally useful concept of finality of functors. Specifically, we will discuss the connected components functor, (co)final functors and their behavior with respect to colimits, and then sifted colimits and reflexive coequalizers. The slogan is: final functors are those along which restriction doesn't change colimits.

Let C be a (usually small) category. Let $\sim$ be the equivalence relation on $\mathrm{Obj}(C)$ generated by $c \sim c'$ if there exists a map $c \to c'$. We write $\pi_0(C) \coloneqq \mathrm{Obj}(C)/\sim$ for the quotient. So, $\pi_0(C) = \varnothing$ iff $C = \varnothing$; C is *connected* if $\pi_0(C) \cong *$ is a one-point set.

Here are some fun facts about π_0. If $F\colon C \to D$ is a functor, the composite $\mathrm{Obj}(C) \xrightarrow{D} \mathrm{Obj}(D) \twoheadrightarrow \pi_0(D)$ clearly identifies c and c' if a map $c \to c'$ exists, so we get an induced map $F_*\colon \pi_0(C) \to \pi_0(D)$ given by $F_*[c] = [F_*c]$. Then clearly $F_*G_* = (FG)_*$, so this defines a functor

$$\pi_0\colon \mathbf{Cat} \to \mathbf{Set}$$

called the *connected components functor*. Note that if $\alpha\colon F \to G$ is a natural transformation, there are maps $Fc \to Gc$, hence $F_*[c] = [Fc] = [Gc] = G_*[c]$, i.e. $F_* = G_*$. In particular, if $C \simeq C'$ is an equivalence - or e.g. if an adjunction

exists - then $\pi_0(C) \cong \pi_0(C')$. Hence the map

$$\mathrm{Fun}(C, D) \xrightarrow{\pi_0} \mathbf{Set}(\pi_0(C), \pi_0(D))$$

in fact factors through $\pi_0(\mathrm{Fun}(C, D)) \to \mathbf{Set}(\pi_0(C), \pi_0(D))$.

Finally, π_0 has a right-adjoint, namely $\mathrm{Disc}\colon \mathbf{Set} \to \mathbf{Cat}$ sending a set to the discrete category on that set of objects. This is easy to see because a functor $F\colon C \to \mathrm{Disc}(S)$ is a function $\mathrm{Obj}(C) \to S$ but it must send all maps to identities, so if $c \to c'$ exists then c and c' have the same image. In particular $\mathrm{Fun}(C, \mathrm{Disc}(S)) = \mathbf{Set}(\pi_0(C), S)$. It's also easy to see that this is part of a quadruple adjoint

$$\pi_0 \dashv \mathrm{Disc} \dashv \mathrm{forget} \dashv \mathrm{Codisc}$$

where Codisc sends a set to the category having objects those elements and a unique morphism between each object.

Example 1.3.1 Most of the usual familiar categories are connected, because they have initial or terminal objects, and the equivalence of such an object clearly implies connectedness. A standard disconnected category is the diagram category for an n-fold product (when $n > 1$).

Next, let $F\colon C \to D$ be a functor. Recall that for $d \in D$, the coslice category d/F or $d \downarrow F$ has objects consisting of pairs $(c, d \xrightarrow{\alpha} F(c))$, and its morphisms $(c, \alpha) \to (c', \alpha')$ are just maps $u\colon c \to c'$ in C such that $u_*\alpha = \alpha'$. The slice category F/d or $F \downarrow d$ is defined similarly.

Fixing F, the coslice constructions defines a strict presheaf

$$(-)/F\colon D^{\mathrm{op}} \to \mathbf{Cat}$$

of categories. Indeed, a map $f\colon d \to d'$ in D induces $f^*\colon d'/F \to d/F$ given by $f^*(c, \alpha) = (c, \alpha f)$, and obviously preserves morphisms by associativity. For the same reason $(fg)^* = g^*f^*$, and we can define a presheaf $\pi_0(-/F)$ of sets.

Definition 1.3.2 A functor $F\colon I \to J$ of small categories is *final* if j/F is connected for all j. In other words, the presheaf $\pi_0(-/F)$ is terminal. The functor F is *cofinal* if F/j is connected for all j.

Observe that if we have a natural transformation $\eta\colon F \to G$, there is a morphism of presheaves $\eta_*\colon (-)/F \to (-)/G$ defined on components $c/F \to c/G$ by sending $(c, \alpha) \mapsto (c, \eta_c\alpha)$, so the coslice construction actually defines a functors

$$\mathrm{Fun}(C, D) \to \mathrm{PSh}(D, \mathbf{Cat}) \xrightarrow{\pi_{0,*}} \mathrm{PSh}(D),$$

and in particular finality only depends on the isomorphism type of F.

Here are some basic observations about this definition. First, we will not worrying about cofinality because F/j is the same as j/F^{op} for $F^{\mathrm{op}}\colon I^{\mathrm{op}} \to J^{\mathrm{op}}$ the opposite functor. Next, the connectedness condition needs only to be checked on an isomorphism class of each object: by functoriality, $\pi_0(j/F) \cong \pi_0(j'/F)$ if $j \cong j'$.

Warning The natural map $d/F \to d/D$ is *not* an embedding, as an object of d/F includes the data of a C-object (and is not just a map in D). So, e.g., the unique functor $\{\bullet \quad \bullet\} \to \{\bullet\}$ is not final.

Let us now discuss the relation between final functors and colimits. Fix a functor $F\colon I \to J$ and suppose we have a diagram $X\colon J \to \mathcal{C}$. Precomposition $F^*\colon : \mathcal{C}^J \to \mathcal{C}^I$ induces a natural family of maps

$$F^*\colon \mathrm{Hom}_{\mathcal{C}^J}(X, \Delta_c) \longrightarrow \mathrm{Hom}_{\mathcal{C}^I}(XF, \Delta_c)$$

for each $c \in \mathcal{C}$. In particular, if η is a universal cocone then the restriction $F^*(\eta)$ is a cocone from XF and so there is a natural map $\varinjlim_i X_{F(i)} \to \varinjlim_j X_j$, provided the colimits exist. The question of whether this is an isomorphism or not is the same as whether

$$F^* = F_c^*\colon \mathrm{Cocone}(X, c) \to \mathrm{Cocone}(XF, c)$$

is a natural isomorphism. Even stronger, if this is a natural isomorphism then the existence of one colimit guarantees the other's existence and that they are equal. Our major theorem is the following.

Theorem 1.3.3 *Let $F\colon I \to J$ be a functor of small categories. The following are equivalent:*

1. *F is a final functor.*
2. *For every category $\mathcal{C}$ and diagram $X\colon J \to \mathcal{C}$, the natural transformation*
$$F^* = F_c^*\colon \mathrm{Hom}_{\mathcal{C}^J}(X, \Delta_c) \to \mathrm{Hom}_{\mathcal{C}^I}(XF, \Delta_c)$$
is an isomorphism.
3. *For each diagram $X\colon J \to \mathbf{Set}$, the natural map*
$$\varinjlim_{i \in I} X_{F(i)} \to \varinjlim_{j \in J} X_j$$
is a bijection.
4. *For every category $\mathcal{C}$ and diagram $X\colon J \to \mathcal{C}$, if either of the colimits $\varinjlim_i X_{F(i)}$ or $\varinjlim_j X_j$ exist in $\mathcal{C}$, then the other one exists and the natural map*
$$\varinjlim_{i \in I} X_{F(i)} \to \varinjlim_{j \in J} X_j$$
is an isomorphism in $\mathcal{C}$.

The text preceding the theorem statement already explains (2) $\Leftrightarrow$ (4), and (4) $\Rightarrow$ (3) is obvious. We will now demonstrate (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). Begin with no assumptions, and consider some cocone $\eta\colon XF \to c$. For each j, we associate to this a function

$$f_\eta(j)\colon \mathrm{Obj}(j/F) \to \mathrm{Hom}_{\mathcal{C}}(X_j, c)$$

given by $(i, u\colon j \to F(i)) \mapsto \big(X_j \xrightarrow{u_*} X_{F(i)} \xrightarrow{\eta_i} c\big)$. Given a map $v\colon (i,u) \to (i',u')$ in j/F, the diagram

$$\begin{array}{ccc} X_j & \xrightarrow{u_*} & X_{F(i)} \\ {\scriptstyle u'_*}\downarrow & \swarrow^{v_*} & \downarrow{\scriptstyle \eta_i} \\ X_{F(i')} & \xrightarrow[\eta_{i'}]{} & c \end{array}$$

commutes because each of the two triangles does, so $f_\eta(j)(i,u) = f_\eta(j)(i',u')$, which means that $f_\eta(j)$ descends to a map

$$f_\eta(j)\colon \pi_0(j/F) \to \mathrm{Hom}_{\mathcal{C}}(X_j, c).$$

By the functorality of $(-)/F$, a map $v\colon j \to j'$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Obj}(j'/F) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_{j'}, c) \\ \downarrow & & \downarrow \\ \mathrm{Obj}(j/F) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(X_j, c) \end{array}$$

so the equation descends to π_0 and we see that $f_\eta\colon \pi_0(-/F) \to \mathrm{Hom}_{\mathcal{C}}(X, c)$ is a morphism of presheaves. Hence the association $\eta \mapsto f_\eta$ induces a function

$$\mathrm{Hom}_{\mathcal{C}^I}(XF, \Delta_c) \to \mathrm{Hom}_{\widehat{J}}\big(\pi_0(-/F), \mathrm{Hom}_{\mathcal{C}}(X, c)\big) \qquad (*)$$

Claim: The function $(*)$ is bijective. *Proof* Injectivity is easy: In the coslice $F(i)/F$ there is always the canonical element (i, Id), so given a cocone $\varepsilon\colon XF \to c$ we see that

$$f_\varepsilon(F(i))[i, \mathrm{Id}] = X_{F(i)} \xrightarrow{\mathrm{Id}} X_{F(i)} \xrightarrow{\varepsilon_i} c,$$

so in other words we can recover the legs of the cocone from f_ε. For surjectivity, suppose we have a map $g\colon \pi_0(-/F) \to \mathrm{Hom}(X, c)$ and just as above define

$$\varepsilon_i \coloneqq g_{F(i)}([i, \mathrm{Id}]).$$

We first prove that ε is a cocone $XF \to c$. Given $u\colon i \to i'$, naturality of g in Fu implies that

$$\begin{aligned} \varepsilon_i^* \circ (Fu)_* &= (Fu)^* \circ g_{F(i')}[i', \mathrm{Id}_{F(i')}] \\ &= g_{F(i)} \circ (Fu)^*[i', \mathrm{Id}_{F(i')}] \\ &= g_{F(i)}[i', Fu], \end{aligned}$$

but $u\colon (i, \mathrm{Id}_{F(i)}) \to (i', Fu)$ is a morphism in $F(i)/F$, so in particular

$$g_{F(i)}[i', Fu] = g_{F(i)}[i, \mathrm{Id}_{F(i)}] = \varepsilon_i.$$

It remains to show that $f_\varepsilon = g$. For some $v\colon j \to F(i)$ we use the same naturality condition with respect to v to obtain

$$\begin{aligned} f_\varepsilon(j)[i, v] &= \varepsilon_i \circ v_* \\ &= (g_{F(i)}[i, \mathrm{Id}]) \circ v_* \end{aligned}$$

$$\begin{aligned} &= v^* \circ g_{F(i)}[i, \mathrm{Id}] \\ &= g_j \circ v^*[i, \mathrm{Id}] \\ &= g_j([i, v]), \end{aligned}$$

hence $f_\varepsilon(j) = g_j$ as desired.

With the natural isomorphism $(*)$ in hand, we can easily finish the proof of Thm. 1.3.3. Recall one more fact, that there is a canonical identification

$$\mathrm{Hom}_{\widehat{\mathcal{J}}}(\Delta_*, \mathrm{Hom}_{\mathcal{C}}(X, c)) \cong \mathrm{Hom}_{\mathcal{C}^J}(X, \Delta_c)$$

since an element of the left side is a family $g_j \in \mathrm{Hom}_{\mathcal{C}}(X_j, c)$ of elements such that $f_j = f_{j'} u_*$ for each $u\colon j \to j'$ in J.

Proof (of Thm. 1.3.3) (1) $\Rightarrow$ (2): If F is final, then $\pi_0(-/F) \cong \Delta_*$ is the terminal presheaf, so by the above remark we get a chain of natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}^I}(XF, \Delta_c) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathcal{J}}}(\Delta_*, \mathrm{Hom}(X, c)) = \mathrm{Hom}_{\mathcal{C}^J}(X, \Delta_c),$$

and we just need to show it is an inverse of the canonical map F_c^*. One direction suffices: starting with a cocone $\varepsilon\colon XF \to c$, the composite above extends it to a cocone $\eta\colon X \to c$ whose $F(i)$ component satisfies

$$\eta_{F(i)} = f_\varepsilon(F(i))(*) = f_\varepsilon(F(i))[i, \mathrm{Id}] = \varepsilon_i,$$

so in particular $F^*(\eta)_i = \varepsilon_i$, i.e. $F^*\eta = \varepsilon$ as desired.

(3) $\Rightarrow$ (1) If this holds in **Set**, then note that it also holds in $\widehat{K} = \mathrm{PSh}(K)$ for any small K. Indeed, (co)limits are computed objectwise so the diagram

$$\begin{array}{ccc} \left(\varinjlim_i X_{F(i)}\right)_k & \longrightarrow & \left(\varinjlim_j X_j\right)_k \\ \downarrow\cong & & \downarrow\cong \\ \varinjlim_i X_{F(i)}(k) & \xrightarrow{\cong} & \varinjlim_j X_j(k) \end{array}$$

commutes in **Set**, and the top arrows are the components of the colimit comparison map. Therefore the condition in (2) follows with $\mathcal{C} = \widehat{K}$.

Given that assumption, we have bijections

$$\begin{aligned} \mathrm{Hom}_{\widehat{\mathcal{J}}}(\Delta_*, \mathrm{Hom}_{\mathcal{C}}(X, c)) &\cong \mathrm{Hom}_{\mathcal{C}^J}(X, \Delta_c) \cong \mathrm{Hom}_{\mathcal{C}^I}(XF, \Delta_c) \\ &\cong \mathrm{Hom}_{\widehat{\mathcal{J}}}(\pi_0(-/F), \mathrm{Hom}_{\mathcal{C}}(X, c)) \end{aligned}$$

all natural in c. Consider the special case $\mathcal{C} = \widehat{J}$, let $Y \in \widehat{J}$ be any presheaf and set $c = Y \in \mathcal{C}$. For any $j \in J$ the Yoneda lemma gives the natural isomorphism

$$Y(j) \cong \mathrm{Hom}_{\mathcal{C}}(\mathrm{Hom}_J(-, j), Y),$$

in particular taking $\mathbf{y}\colon J \to \widehat{J}$ to be the Yoneda embedding, taking $X = \mathbf{y}$ we conclude

$$\mathrm{Hom}_{\mathcal{C}}(X, c) = \mathrm{Hom}_{\widehat{\mathcal{J}}}(\mathbf{y}, Y) \cong Y$$

as functors. As part of Yoneda this is still natural in Y, so the above equation shows

$$\mathrm{Hom}_{\widehat{J}}(\Delta_*, -) \cong \mathrm{Hom}_{\widehat{J}}(\pi_0(-/F), -)$$

as functors, so $\Delta_* \cong \pi_0(-/F)$ by the Yoneda lemma. □

1.4 Sifted colimits

Let us finally look at some example of final functors. Finality is useful in practice because it makes an a priori difficult question about colimits easily checkable by inspection of the diagram category.

Proposition 1.4.1 *A category J is filtered if and only if for all finite categories I the diagonal functor $\Delta\colon J \to J^I$ is final.*

Proof Note that given an object $D \in J^I$ (i.e. a finite D-diagram in J), an object (j, η) of D/Δ is the same as cocone $\eta\colon D \to \Delta_j$, so in particular nonemptiness of D/Δ for all D and finite I is equivalent to every finite diagram having a cocone. It remains to show that if J is filtered and $D \in J^I$, then the (nonempty) category D/Δ is connected.

Suppose (j, η) and (j', η') are two objects. Take a cocone from the finite collection

$$\{\eta_i\colon D_i \to j\} \cup \{\eta_i'\colon D_i \to j'\}$$

to some object k, which provides maps $u\colon j \to k$ and $v\colon j' \to k$ such that $u\eta_i = v\eta_i'$ for all $i \in I$. Thus $(k, u_*\eta) = (k, v_*\eta')$ in D/Δ, and u, v provide maps from $(j, \eta), (j', \eta')$ into this object respectively. □

We will demonstrate a similar result in the case of sifted categories, but the proof will be quite different. To do this, let us recall Fubini's theorem.

Lemma 1.4.2 *Let $F\colon I \times J \to \mathcal{C}$ be a small diagram and consider its adjunct $\widetilde{F}\colon I \to \mathcal{C}^J$. If either the limit $\varprojlim F$ exists or the limits $\varprojlim_j \widetilde{F}(i)$ exist for each i and*

$$\varprojlim_{i \in I} \left(\varprojlim_{j \in J} \widetilde{F}(i) \right)_i = \varprojlim_{i \in I} \varprojlim_{j \in J} F(i, j)$$

exists, then the other one exists and they canonically coincide.

Proof Note that the equality above always holds and is unambiguous since limits in a presheaf category are computed objectwise. Since hom functors jointly create limits, we may reduce the case $\mathcal{C} = \mathbf{Set}$, in which case both limits will exist. On one hand, the constant functor is left adjoint to

$$\varprojlim_{I \times J}\colon \mathbf{Set}^{I \times J} \to \mathbf{Set},$$

and on the other hand, we have the two adjunctions

$$\Delta\colon \mathbf{Set} \leftrightarrows \mathbf{Set}^J : \varprojlim_J \qquad \Delta\colon \mathbf{Set}^J \leftrightarrows (\mathbf{Set}^J)^I : \varprojlim_I,$$

so composition of adjoints gives the required natural isomorphism under the isomorphism $(\mathbf{Set}^J)^I \cong \mathbf{Set}^{I \times J}$. □

The main result in this section will be the following structure theorem of sifted colimits. This will mostly be easy since we have the formalism of final functors now.

Theorem 1.4.3 *Let $J \neq \varnothing$ be a small category. The following are equivalent:*

1. *J is a sifted category.*
2. *The diagonal functor $\Delta\colon J \to J \times J$ is final.*
3. *J-colimits commute with finite products in any cartesian closed category that admits J-shaped colimits.*

Proof (3) $\Rightarrow$ (1) is the special case of **Set**. Now fix a cartesian closed category $\mathcal{C}$ with J-colimits, and let $X_\bullet, Y_\bullet\colon J \to \mathcal{C}$ two diagrams. First observe by Fubini that we have natural isomorphisms

$$\begin{aligned}\varinjlim_{(j,j')\in J\times J}(X_j \times Y_{j'}) &\cong \varinjlim_{j\in J}\varinjlim_{j'\in J}(X_j \times Y_{j'})\\ &\cong \varinjlim_{j\in J}\big(X_j \times \varinjlim_{j'\in J} Y_{j'}\big)\\ &\cong \varinjlim_{j\in J} X_j \times \varinjlim_{j\in J} Y_j,\end{aligned}$$

and restriction along the diagonal functor provides the map

$$\varinjlim_{j\in J}\big(X_j \times Y_j\big) \longrightarrow \varinjlim_{(j,j')\in J\times J}(X_j \times Y_{j'}) \cong \varinjlim_{j\in J} X_j \times \varinjlim_{j\in J} Y_j \qquad (*)$$

where the full composition coincides with the canonical comparison map between J-colimits and the binary product. If condition (2) holds, then $(*)$ is an isomorphism and J-colimits commute with binary products in $\mathcal{C}$. If we let $c \coloneqq \varinjlim_{j\in J} 1$ where $1 \in \mathcal{C}$ is the terminal object, then $c \times c \cong c$ by the diagonal map. This implies $c = 1$, because for each object d the diagonal map

$$\mathrm{Hom}(d, c) \xrightarrow{\sim} \mathrm{Hom}(d, c) \times \mathrm{Hom}(d, c)$$

is an isomorphism in **Set**, which means $\mathrm{Hom}(d, c) = *$, i.e. c is terminal. This shows $\varinjlim_J$ preserves the terminal object, and hence all finite products.

Conversely, specialize to **Set**, and assume (1). Then the composite $(*)$ is an isomorphism, so the first map is as well. This proves the finality colimit condition for diagrams $J \times J \to \mathbf{Set}$ which are given as a product of two diagrams $J \to \mathbf{Set}$ composed with the product functor $\mathbf{Set} \times \mathbf{Set} \to \mathbf{Set}$. Let us call such a diagram *separable*.

Claim: Each diagram $F\colon J \times J \to \mathbf{Set}$ is a colimit in $\mathbf{Set}^{J\times J}$ of separable diagrams.

Assuming this is proven, let $F \in \mathbf{Set}^{J\times J}$ be a diagram and write $F = \varinjlim_{i\in I} G_i$ for some $G\colon I \to \mathbf{Set}^{J\times J}$ so that each $G(i)$ is separable. Then each of the

natural maps

$$\lambda_i \colon \varinjlim_{j \in J} G_i(j,j) \xrightarrow{\sim} \varinjlim_{(j,j') \in J \times J} G_i(j,j')$$

is an isomorphism, and hence the canonical map

$$\lambda \colon \varinjlim_{j \in J} G_\bullet(j,j) \xrightarrow{\sim} \varinjlim_{(j,j') \in J \times J} G_\bullet(j,j')$$

is an isomorphism in the functor category $\mathbf{Set}^I$. Since $\varinjlim_I \colon \mathbf{Set}^I \to \mathbf{Set}$ preserves colimits and sends $G_\bullet(j,j')$ to $F(j,j')$ for each $(j,j') \in J \times J$, the canonical diagram

$$\begin{array}{ccc} \varinjlim_{i \in I} \varinjlim_j G_i(j,j) & \xrightarrow{\cong} & \varinjlim_{i \in I} \varinjlim_{j,j'} G_i(j,j') \\ \cong \downarrow & & \downarrow \cong \\ \varinjlim_j F(j,j) & \dashrightarrow & \varinjlim_{j,j'} F(j,j') \end{array}$$

commutes giving the result.

Proof (of claim): Since each diagram is a colimit over the category of elements $\int_{(J \times J)^{op}} F$ of representables, it suffices to show that the each of the (co)representable functors $\mathrm{Hom}_{J \times J}((j,j'),-)$ are separable diagrams. But by definition,

$$\mathrm{Hom}_{J \times J}((j,j'),(k,k')) = \mathrm{Hom}_J(j,k) \times \mathrm{Hom}_J(j',k'),$$

so in particular $\mathrm{Hom}_{J \times J}((j,j'),-) = \mathrm{Hom}_J(j,-) \times \mathrm{Hom}_J(j',-)$ is the product in **Set** of two J-shaped diagrams. □

Remark 1.4.4 Note that even though we have come up with some equivalent conditions for sifted-ness and filtered-ness that make them look formally analagous, our proofs in each case were completely different. E.g., that $\Delta \colon J \to J^I$ being final is equivalent to J being filtered for finite I was essentially reformulating the definition, but for siftedness we had to use that **Set** is cartesian closed, the Yoneda lemma in a clever way.

Remark 1.4.5 The main conclusion of Thm. 1.4.3 is often stated as a nonempty C is sifted iff $\mathrm{cospan}(c_0,c_1)$ is (nonempty and) connected. This is equivalent because if Δ is the diagonal functor, then $(c_0,c_1)/\Delta \cong \mathrm{cospan}(c_0,c_1)$. Indeed, objects in either are objects $d \in C$ with two morphisms $c_0 \to d$ and $c_1 \to d$, and morphisms are morphisms $d \to d'$ in C commuting with both maps. As a corollary, we obtain that a small category C with binary coproducts is sifted, because given $c_0, c_1 \in C$, the cospan

$$c_0 \to c_0 \sqcup c_1 \leftarrow c_1$$

is an initial object of $\mathrm{cospan}(c_0,c_1)$.

Filtered colimits provide an important class of sifted colimits. The other one are *reflexive coequalizers*, which we turn to now. A pair $u,v \colon a \to b$ of parallel

maps in a category C is *reflexive* if they have a common section, i.e. there is an $s\colon b \to a$ such that $us = vs = 1_b$. A reflexive coequalizer is a coequalizer of a reflexive diagram.

Let us show that a reflexive coequalizer is the same thing as a colimit of a diagram with the following indexing category: There are two objects $\{a, b\}$, and the non-identity maps are $u, v\colon a \to b$, $s\colon b \to a$, and the composites $s \circ u, s \circ v\colon a \to a$, subject to the relation that $us = vs = 1_b$.

This category is closed under composition because composing su and sv on either side by any other specified morphism results in some identity map. We denote this by I_{ref}. Let I_{eq} be the subcategory with the same objects but without the maps s, su, or sv, so that coequalizer diagrams are those with source I_{eq}.

Lemma 1.4.6 *The inclusion $\iota\colon I_{\mathrm{eq}} \to I_{\mathrm{ref}}$ is a final functor.*

Proof The maps u, v are already present in I_{eq} and hence are connected to $1_a \in a/\iota$. The other two objects of a/ι are (a, sv) and (a, su), but using u (or v) connects them to the above ones, as

$$u\colon (a, su) \to (b, u) \qquad u\colon (a, sv) \to (b, v)$$

are morphisms in a/ι. In b/ι, we have the morphism $u\colon (a, s) \to (b, 1)$ connecting the only two objects. □

Now, note that a diagram $X\colon I_{\mathrm{ref}} \to \mathcal{C}$ is by definition a reflexive pair equipped with a choice of common section. Thm. 1.3.3 says that the colimit of X exists iff the coequalizer of the pair of morphisms exists, and in this case they have the same value. The main utility of this will be the following result.

Theorem 1.4.7

1. *The category I_{ref} is sifted, so reflexive coequalizers are sifted colimits.*
2. *Let $\mathcal{C}$ be a category and a, b objects whose coproduct exists, and let $f, g\colon a \to b$ be parallel morphisms. Of the diagrams*

$$a \underset{g}{\overset{f}{\rightrightarrows}} b \qquad \text{and} \qquad a \amalg b \underset{(g,1_b)}{\overset{(f,1_b)}{\rightrightarrows}} b,$$

   the right-hand diagram is a reflexive pair, and the colimit of one exists iff the colimit of the other does. In this case, the map $b \to \mathrm{Coeq}(f, q)$ exhibits the latter as the coequalizer of the right-hand diagram.
3. *All reflexive pairs in* **Set** *are isomorphic to pairs of the form in (2).*

Proof (1) Due to the obvious isomorphism of categories $\mathrm{cospan}(a, b) \cong \mathrm{cospan}(b, a)$, we need to check that $\mathrm{cospan}(a, b)$, $\mathrm{cospan}(a, a)$, and $\mathrm{cospan}(b, b)$ are connected. For the first, the three cospans are $(1_a, s)$, (sv, s), and (su, s), and they are connected by $sv\colon (1, s) \to (sv, s)$ and $su\colon (1, s) \to (su, s)$.

$\mathrm{cospan}(b, b)$ is trivially connected because the only cospans are pairs of the same morphism, and I_{ref} is connected. For $\mathrm{cospan}(a, a)$, the above fact implies that all pairs of the same map are in the same connected component. Note that we need not worry about order swaps, because the swapping isomorphism induces a bijection on π_0 which preserves this class. Next we have the three cospans $(sv, 1), (su, 1), (su, sv)$, yet

$$su\colon (sv, 1) \to (sv, su) \qquad sv\colon (su, 1) \to (sv, su)$$

are morphisms since $susv = sv$ and $svsu = su$, so they are connected. They trivially lie in the same class as (u, v) due to $s\colon (u, v) \to (su, sv)$, and finally $s\colon (u, v) \to (1, 1)$ by the section condition.

(2) Because of the maps $(f, 1_b), (g, 1_b)$, it's clear that a colimit of one of the diagrams (equipped with the map from b) would be a cocone for the other diagram. The common section is the coproduct map $b \to a \amalg b$. Since only colimits are involved, we can show the corresponding result in **Set**, that the canonical map

$$\mathrm{Eq}(f, g) \to \mathrm{Eq}((f, 1_a), (g, 1_a)) \to a \rightrightarrows a \times b$$

to the equalizer diagram is an isomorphism. Indeed,

$$\begin{aligned} \mathrm{Eq}(f, g) &= \{x \in a : f(x) = g(x)\} \\ &= \{x \in a : (f, 1)(x) = (g, 1)(x)\} \\ &= \mathrm{Eq}((f, 1), (g, 1)) \end{aligned}$$

as desired.

(3) If $f, g\colon a \to b$ is a reflexive pair in **Set** with common section $s\colon b \to a$, write $b' = b \setminus s(a)$ so that we have a bijection $b \cong a \sqcup b'$, and the restriction of the corresponding maps f, g along $a \to a \sqcup b'$ is the identity, so the pair is isomorphic to the following pair

$$a \sqcup b' \underset{(g|_{b'}, 1_a)}{\overset{(f|_{b'}, 1_a)}{\rightrightarrows}} a$$

completing the proof. □

In the remainder of this section we will discus some closure properties of colimits (and hence limits). This will be a review of elementary facts. More precisely, we will prove the following list of properties:

Proposition 1.4.8 *Let $\mathcal{C}$ be a category.*

1. *If $\mathcal{C}$ has binary coproducts then it has n-ary coproducts for $n \geq 1$. In particular binary coproducts + initial object = finite coproducts.*
2. *If $\mathcal{C}$ has filtered colimits and finite coproducts then it has arbitrary coproducts.*

3. *If $\mathcal{C}$ has reflexive coequalizers and binary coproducts then it has coequalizers. In particular if $\mathcal{C}$ has reflexive coequalizers and (finite) coproducts, then it has (finite) colimits.*
4. *If $\mathcal{C}$ has filtered colimits and finite colimits then it is cocomplete.*
5. *If $\mathcal{C}$ has sifted colimits and finite coproducts then it is cocomplete.*

Proof (1) This is standard: $n = 1$ is trivial, and if it has $(n-1)$-ary coproducts, then let $c_1, \dots, c_n$ be in $\mathcal{C}$. Writing $c = c_1 \amalg \cdots \amalg c_{n-1}$, we claim $c \amalg c_n$ equipped with the maps $c_i \to c \to c \amalg c_n$ along with $c_n \to c \amalg c_n$ form the n-ary coproduct. Indeed, this is checked by verifying the corresponding statement about products in **Set**.

(2) Let $\{c_i : i \in I\}$ where I is a small index set. Let $\mathcal{P}_{\text{fin}}(I)$ be the poset of finite subsets of I under inclusion, which we note is a filtered category because the union of two finite subsets is finite. For each $S \in \mathcal{P}_{\text{fin}}(I)$ choose a coproduct $c^{(S)}$ equipped with maps $\mu_s^{(S)} \colon c_s \to c^{(S)}$ for each $s \in S$. If $S \subset S'$, the maps $c_s \to c^{(S')}$ for $s \in S$ induce a map $c^{(S)} \to c^{(S)}$, so the correspondence

$$\mathcal{P}_{\text{fin}}(I) \to \mathcal{C} \qquad S \mapsto c^{(S)}$$

defines a filtered diagram, whose colimit c exists in $\mathcal{C}$. We claim that the canonical maps $c_i \to c^{(\{i\})} \to c$ exhibit c as the coproduct of $\{c_i : i \in I\}$. Note that we may assume $c^{(\{i\})}$ was chosen to be c_i.

To check the universal property, let $f_i \colon c_i \to d$ be maps in $\mathcal{C}$, then for each finite S this induces a map $f^{(S)} \colon c^{(S)} \to d$. By construction of the diagram $f^{(S')}$ agrees with $f^{(S)}$ after composing with $c^{(S)} \to c^{(S')}$ if $S \subset S'$. Thus we obtain a map $c \to d$, which commutes with canonical $c^{(S)} \to c$, so in particular all $c_i = c^{(\{i\})} \to d$ which are f_i. Uniqueness follows from the uniqueness on $c^{(S)}$ for all finite S.

(3) The second statement comes from the fact that coequalizers + (finite) coproducts = (finite) colimits. The first is from Thm. 1.4.7, which shows that to compute a coequalizer $a \rightrightarrows b$, we can take the corresponding reflexive coequalizer $a \amalg b \rightrightarrows b$.

(4) Finite colimits give coequalizers and finite coproducts, which by (2) implies coequalizers + coproducts, i.e. colimits.

(5) Sifted colimits implies reflexive coequalizers by Thm. 1.4.7 and filtered colimits, and reflexive coequalizers and finite coproducts implies finite colimits by (3), so we conclude by (4). □

This adds to the our list of dualities between siftedness/finite coproducts and filteredness/finite colimits. We will soon see that algebras over a Lawvere theory and essentially algebraic theories fit into this duality as well.

2 Locally finitely presentable categories

In this section we will complete our study of algebraic categories, generalize them to *essentially algebraic categories*, and define locally finitely presentable (LFP) categories. We will then prove Gabriel-Ulmer duality, showing that LFP categories and essentially algebraic categories are actually the same thing.

2.1 Fundamental results on presheaves

From the way we defined algebraic categories it was obvious that they had sifted colimits and small limits, and that sifted/filtered colimits behaved similarly to how they behave in **Set**. This was because these facts were reflected by the embedding $\mathrm{Alg}(T) \hookrightarrow \mathrm{Fun}(T, \mathbf{Set})$.

It is a feature of all of our examples that algebraic categories have all small colimits and they admit free-forgetful adjunctions. It is less trivial, though true, to show this for a general $\mathrm{Alg}(T)$ because colimits aren't computed in $\mathrm{Alg}(T)$, but rather are computed through a reflector (i.e. $\mathrm{Alg}(T)$ is a reflective subcategory of the functor category).

There is also a more general class of objects which are almost algebraic, but still have many of the properties we have discussed; the primary examples are categories, simplicial sets, and other similar 'essentially algebraic' structures which are built (1) from more than just one set, and (2) have equations which are partially defined: E.g. a the associativity law on a small category is not an equation on all morphisms, only on composable ones.

Towards this generalization, we will prove a few fundamental results about presheaves. This will let us characterize more closely categories of algebras.

Theorem 2.1.1 *Let C be an essentially small category with finite products, and $F\colon C \to \mathbf{Set}$ a functor. The following are equivalent:*

1. *F preserves finite products.*
2. *The category $\int^C F$ has binary products.*
3. *The category of elements $\int_{C^{\mathrm{op}}} F$ is sifted.*
4. *As an element of $\mathrm{Fun}(C, \mathbf{Set})$, F is a sifted colimit of representables.*

Some definitions are in order: For our purposes, the *category of elements* of X is defined for a presheaf $X\colon A^{\mathrm{op}} \to \mathbf{Set}$, denoted by $\int_A X$, and its objects are pairs (a, x) where $a \in \mathrm{Obj}(A)$ and $x \in X_a$. Its morphisms are morphisms $u\colon a \to b$ of A such that $u^*(y) = x$. The Yoneda lemma says that $\int_A X \cong \mathbf{y} \downarrow X$ where $\mathbf{y}\colon A \to \mathrm{PSh}(A)$ is the Yoneda embedding, and it also says that the projection functor $\int_A X \to \mathrm{PSh}(A)$ sending $(a, x) \mapsto \mathbf{y}_a$ exhibits X as a colimit of representables (see appendix).

When working with a functor $F\colon C \to \mathbf{Set}$, the more natural notion is the category $\int^C F$ with objects (c, x) for $x \in F_c$, and morphisms $(c, x) \to (c', y)$ given by $u\colon c \to c'$ in C for which $u_*(x) = y$. Unwinding the definitions, we see that $\int^C F = (\int_{C^{\mathrm{op}}} F)^{\mathrm{op}}$, but we still call $\int_{C^{\mathrm{op}}} F$ the category of elements of F, because it is that category over which F is colimit in $\mathrm{Fun}(C, \mathbf{Set})$.

Proof (of Thm. 2.1.1) (2) ⇒ (3) is by definition and Remark 1.4.5. (3) ⇒ (4) is because F is a colimit over $\int_{C^{\mathrm{op}}} F$ of the functor $(a, x) \mapsto \mathbf{y}_a$. (4) ⇒ (1) We have basically seen this proof already: If $F = \varinjlim_{j\in J} X_j$ for J sifted, each X_j is representable, then since the representables preserve limits we have

$$\begin{aligned} F(c \times c') &= \left(\varinjlim_{j\in J} X_j\right)(c \times c') = \varinjlim_{j\in J} X_j(c \times c') \\ &= \varinjlim_{j\in J} X_j(c) \times X_j(c') = \varinjlim_{j\in J} X_j(c) \times \varinjlim_{j\in J} X_j(c') \\ &= F(c) \times F(c'), \end{aligned}$$

because finite products commute with sifted colimits in **Set**. The same proof works for the terminal object.

(1) ⇒ (2): The product of (c, x) and (d, y) is $(c \times d, (x, y))$ with the canonical projections, where we are identifying (x, y) and its image under the inverse isomorphism $F(c) \times F(d) \xrightarrow{\sim} F(c \times d)$. Indeed, maps $u\colon (b, w) \to (c, x)$ and $v\colon (b, w) \to (d, y)$ correspond to maps $b \to c \times d$ such that

$$(u, v)_*(w) = (u_*(w), v_*(w)) = (x, y),$$

as desired. □

Let us generalize the argument at the end of the last theorem using the following lemma. The application we have in mind is the observation that, by definition, the category $\int^C F$ is the the comma category $* \downarrow F$ where $*$ is viewed as a functor $\mathbb{1} \to \mathbf{Set}$.

Lemma 2.1.2 *Let $F\colon C \to D \leftarrow E : G$ be a span of categories. Let I be a small category. If C and E have I-shaped limits and G preserves them, then $F \downarrow G$ has I-shaped limits and the projections preserve them.*

Proof Given a diagram $X_\bullet\colon I \to F \downarrow G$, write $X_i = (c_i, e_i, u_i\colon F(c_i) \to G(e_i))$. It is easy to see that $u_\bullet$ is a natural transformation from $I \to F \downarrow G \to C \to D$ to $I \to F \downarrow G \to E \to D$, so write

$$c \coloneqq \varprojlim_{i\in I} c_i \qquad e \coloneqq \varprojlim_{i\in I} e_i \qquad u \coloneqq \varprojlim_{i\in I} u_i,$$

and let

$$\beta = F(c) \to \varprojlim_i F(c_i) \xrightarrow{u} \varprojlim_i G(c_i) \xrightarrow{\sim} G(c),$$

where the first and last maps are the canonical comparison map and the inverse of the canonical comparison map respectively. Thus (c, e, β) is an element of

the comma category, and the projections $c \to c_i$ and $e \to e_i$ induce morphisms $(c, e, \beta) \to X_i$ because the diagram

$$\begin{array}{ccccc} F(c) & \longrightarrow & \varprojlim_i F(c_i) & \xrightarrow{u} & \varprojlim_i G(c_i) & \longleftarrow & G(c) \\ & \searrow & \downarrow & & \downarrow & \swarrow & \\ & & F(c_i) & \longrightarrow & G(c_i) & & \end{array}$$

commutes. These form a cone because the condition is checked levelwise in C and E. We claim that these maps exhibit (c, e, β) as the limit, and this will suffice since the projections to C and E of this cone are the limit cones $(c, c \to c_i)$ and $(e, e \to e_i)$ respectively.

Indeed, take a cone $(f_i, g_i) \colon (a, b, p) \to (c_i, e_i, u_i)$. Then (f_i) and (g_i) are also cones, so there are unique maps $f \colon a \to c$ and $g \colon b \to e$ making the standard diagrams commute. Existence is clear (again from C and E) once we show that (f, g) is a morphism $(a, b, p) \to (c, e, \beta)$. This is true because the diagram

$$\begin{array}{ccccccc} & & F(a) & \xrightarrow{p} & G(b) & & \\ & \swarrow & \downarrow & & \downarrow & \searrow & \\ F(c) & \longrightarrow & \varprojlim_i F(c_i) & \xrightarrow{u} & \varprojlim_i G(e_i) & \longleftarrow & G(e) \end{array}$$

commutes: the triangles are by naturality of the canonical comparison map, and the the square is from standard properties of the functoriality of limits. Finally, uniqueness is trivial because the commutativity condition implies that a candidate pair (f', g') implies that both f' and g' are candidate maps $a \to c$ and $b \to e$, hence by uniqueness of those limits $f' = f$ and $g = g'$. □

Theorem 2.1.3 *Let C be an essentially small category with finite limits, and $F \colon C \to \mathbf{Set}$ a functor. The following are equivalent:*

1. *F preserves finite limits.*
2. *The category $\int^C F$ has finite limits.*
3. *The category of elements $\int_{C^{\mathrm{op}}} F$ is filtered.*
4. *As an element of $\mathrm{Fun}(C, \mathbf{Set})$, F is a filtered colimit of representables.*

Proof (1) $\Rightarrow$ (2) is due to Lemma 2.1.2 and the fact that $\int^C F = (* \downarrow F)$. (2) $\Rightarrow$ (3) is because $\int_{C^{\mathrm{op}}} F = (\int^C F)^{\mathrm{op}}$ has finite colimits, hence cocones for all finite diagrams. (3) $\Rightarrow$ (4) comes again from the fact that F is a colimit over $\int_{C^{\mathrm{op}}} F$ of representables, and (4) $\Rightarrow$ (1) is the same as in Thm. 2.1.1, except we use the fact that finite limits commute with filtered colimits in **Set**. □

To apply this, let us introduce a generalization of Lawvere theories.

Definition 2.1.4

1. An *essentially algebraic theory* (EAT) is an essentially small category C with finite limits. A morphism of EATs is a finite-limit preserving functor between them.
2. A *model* of an essentially algebraic theory C is a limit-preserving functor $C \to \mathbf{Set}$. The full subcategory of $\mathrm{Fun}(C, \mathbf{Set})$ consisting of models is denoted $\mathrm{Lex}(C)$.

Though it is not immediately obvious, algebraic categories (i.e. categories of algebras over a Lawvere theory) are all categories of models of an EAT. We will see some examples in detail after we develop the basic theory. The obvious similarity between $\mathrm{Alg}(T)$ and $\mathrm{Lex}(C)$ means that many of their properties are shared. A major difference, however, is that C is not required to have a generating object like T was, so there is no canonical forgetful functor $\mathrm{Lex}(C) \to \mathbf{Set}$.

Proposition 2.1.5 *Let C be an EAT and $U\colon \mathrm{Lex}(C) \to \mathrm{Fun}(C, \mathbf{Set})$ the inclusion. Then $\mathrm{Lex}(C)$ has small limits and filtered colimits, and U creates both of them. In particular, filtered colimits commute with finite limits in $\mathrm{Lex}(C)$.*

Proof The result follows from the fact that $\mathrm{Lex}(C)$ is closed (as a full subcategory) under limits and filtered colimits. The proof is the same as in Prop. 1.2.3, except using commutativity of filtered colimits and finite limits. This shows that $\mathrm{Lex}(C)$ has the (co)limits and U creates them.

To conclude, since the evaluation functors $\mathrm{Fun}(C, \mathbf{Set})$ at all objects of C jointly create (i.e. preserve and jointly reflect) all (co)limits, filtered colimits commute with finite limits in $\mathrm{Fun}(C, \mathbf{Set})$. It follows by U creating these (co)limits that the same holds in $\mathrm{Lex}(C)$. □

Theorem 2.1.6

1. *Let T be a Lawvere theory. Then $\mathrm{Alg}(T)$ is a (replete) reflective subcategory of $\mathrm{Fun}(T, \mathbf{Set})$ and in particular it is cocomplete.*
2. *Let C be an EAT. Then $\mathrm{Lex}(C)$ is a (replete) reflective subcategory of $\mathrm{Fun}(C, \mathbf{Set})$ and in particular it is cocomplete.*

Proof Let C be an EAT, and observe that for each $c \in C$, the representable functor $\mathbf{y}(c) = \mathrm{Hom}_C(c, -)$ preserves limits, so we get an embedding $\mathbf{y}\colon C^{\mathrm{op}} \to \mathrm{Lex}(C)$. By Thm. 2.1.3, for each $F \in \mathrm{Lex}(C)$ the category $\int_{C^{\mathrm{op}}} F$ is filtered and $\mathrm{Lex}(C)$ has filtered colimits. □

3 Appendix

3.1 Review: Abstract Nonsense

Let us review some general abstract nonsense in this section that is nice to have. First of all, let us discuss an alternative way to talk about adjunctions.

Definition 3.1.1 If $F\colon D \to C$ is a functor and $c \in C$, a *universal morphism from c to F* is an initial object of the comma category c/F. Spelled out, this means it is an object $d \in D$ equipped with a map $u\colon c \to Fd$ such that for any $d' \in D$ and $v\colon c \to Fd'$, there is a unique map $f\colon d \to d'$ such that $Ff \circ u = v$.

We also call a universal morphism from c to F a *partial left adjoint of F at c*, for the following reasons. First, clearly if a universal morphism (d, u) exists, then the pair is determined up to unique isomorphism by virtue of being an initial object. Second, if (d, u) is a universal morphism then we may rewrite the definition by saying that

$$\mathrm{Hom}_D(d, d') \xrightarrow{\sim} \mathrm{Hom}_C(c, F(d'))$$

sending $f \mapsto Ff \circ u$ is an isomorphism. Hence, if F actually did have a left adjoint $G\colon C \to D$, then $G(c) \cong d$ with corresponding morphism $c \to FGc$ given by a component of the unit.

Most importantly, we claim that the *existence* of a universal morphism from F to c implies the existence of a left adjoint to F, where a choice of partial adjoint at each c uniquely determines such a left adjoint. To see this, suppose for each c there is a universal morphism $(Gc, \eta_c\colon c \to F(Gc))$. We extend G to a functor as follows.

For any map $f\colon c \to c'$, we have $\eta_{c'} f\colon c \to FGc'$ and hence there is a unique map $Gf\colon Gc \to Gc'$ making the diagram

$$\begin{array}{ccc} c & \xrightarrow{f} & c' \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \eta_{c'}} \\ F(Gc) & \xrightarrow{F(Gf)} & F(Gc') \end{array}$$

commute. To see that G defines a functor, consider the following diagrams:

$$\begin{array}{ccccc} c & \xrightarrow{f} & c' & \xrightarrow{g} & c'' \\ {\scriptstyle \eta_c}\downarrow & & \downarrow{\scriptstyle \eta_{c'}} & & \downarrow{\scriptstyle \eta_{c''}} \\ F(Gc) & \xrightarrow{F(Gf)} & F(Gc') & \xrightarrow{F(Gg)} & F(Gc'') \end{array} \qquad \begin{array}{ccc} c & \xrightarrow{1_c} & c \\ \downarrow{\scriptstyle \eta_c} & & \downarrow{\scriptstyle \eta_c} \\ FGc & \xrightarrow{F(G1_c)} & FGc \end{array}$$

In the first, we see that $Gg \circ Gf$ makes the square defining $G(g \circ f)$ commute, hence $Gg \circ Gf = G(g \circ f)$ by uniqueness. Likewise, 1_{Gc} makes the square for $G1_c$ commute, so $G1_c = 1_{Gc}$. This means that G is a functor, and η is a natural transformation $1 \to FG$ by construction.

Now, whenever we have such a natural transformation, there is an induced map

$$\mathrm{Hom}_D(Gc, d) \to \mathrm{Hom}_C(c, Fd)$$

given by $f \mapsto Ff \circ \eta_c$, which is clearly natural in both c and d. Universality says that this is a bijection for all c, d, and hence this map is a natural isomorphism, showing $G \dashv F$ and supplying us with a canonical adjunction formula.

Example 3.1.2 If I is small and $\mathcal{C}$ is arbitrary, then a colimit of a diagram $F \in \mathcal{C}^I$ is the same thing as a universal map from an object of $\mathcal{C}$ to $\Delta\colon \mathcal{C} \to \mathcal{C}^I$, the constant functor. Therefore, if colimits of all diagrams in $\mathcal{C}^I$ exist, then simply a choice of colimit (with universal cocone) induces a left adjoint functor

$$\varinjlim_I \colon \mathcal{C}^I \to \mathcal{C}.$$

In particular, we get functoriality of colimits and the fact that colimits commute with colimits (at least when all I-shaped colimits exist) for free.

More generally, this is a nice fact to have when searching for left adjoints, because often functoriality is obvious in one direction and we don't want to have to check it in the other. For instance, if $U\colon \mathbf{Ab} \to \mathbf{Set}$ is the forgetful functor, then for each set S we can define the free abelian group $\mathbb{Z}^S$. It has the property maps $\mathbb{Z}^S \to G$ of abelian groups correspondi uniquely to maps $S \to |G|$ to the underlying set under the restriction map to S, so we immediately deduce that this is a left adjoint. Of course, everything can be dualized to right adjoints.

Definition 3.1.3 Let $F\colon A \to B$ be a functor, and consider the precomposition functor

$$F^*\colon \mathrm{Fun}(B, C) \to \mathrm{Fun}(A, C)$$

for some category C. For $X\colon A \to C$, a *left Kan extension of X along F* is a partial left adjoint to F^* at X. If it exists, we denote it by

$$\mathrm{Lan}_F X\colon B \to C,$$

implicitly considered to be equipped with the map $X \to (\mathrm{Lan}_F X) \circ F$.

Intuitively, the left Kan extension of a functor $X\colon A \to C$ along F is the universal solution to the problem of lifting X to a functor $B \to C$ along the restriction by F. Namely, maps out of the 'lift' correspond uniquely to maps out of X. In addition, by the preceding discussion, if the left Kan extensions along F of all functors $A \to C$ exist, then Lan_F provides a left adjoint to F^*.

Example 3.1.4 Let I be a small category and $\mathrm{const}\colon I \to \mathbb{1}$ be the trivial functor. Then precomposition is

$$\mathrm{const}^* = \Delta\colon \mathrm{Fun}(\mathbb{1}, \mathcal{C}) \simeq \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C}),$$

so this reduces to our previous discussion, and the left Kan extension $\mathrm{Lan}_{\mathrm{const}} X$ of a diagram $X\colon I \to \mathcal{C}$ is just a colimits of $\mathcal{C}$.

Example 3.1.5 Let $F\colon A \to B$ be a functor and consider precomposition

$$F^*\colon \mathrm{Fun}(B, A) \to \mathrm{Fun}(A, A).$$

Then $\mathrm{Lan}_F \mathrm{Id}_A$ is a functor $B \to A$ equipped with a natural transformation

$$\eta\colon \mathrm{Id} \to (\mathrm{Lan}_F \mathrm{Id}_A) \circ F$$

such that the map

$$\mathrm{Nat}(\mathrm{Lan}_F \mathrm{Id}_A, G) \to \mathrm{Nat}(\mathrm{Id}, GF)$$

given by $\mu \mapsto (\mu_F) \circ \eta$ is bijective for all G. This condition is in fact satisfied when G is a left adjoint to F, and the converse is true when when the Kan extension is *absolute*, meaning that it is preserved by all functors out of A.

Theorem 3.1.6 *Let A be a small category and $F\colon A \to B$ any functor. Suppose C is a category which has colimits of shape F/b for all b in B. Then the left Kan extension*

$$\mathrm{Lan}_F\colon \mathrm{Fun}(B, C) \to \mathrm{Fun}(A, C)$$

of all functors exists, and it is computed pointwise by the formula

$$(\mathrm{Lan}_F X)(b) = \varinjlim_{(a,u)} X(a)$$

where the colimit is taken over F/b.

Proposition 3.1.7 *Let $F\colon A \to B$ be a fully faithful functor from a small category A, and $X\colon A \to C$ be a functor. If $\mathrm{Lan}_F X$ exists, then the universal map*

$$X \to (\mathrm{Lan}_F X) \circ F$$

is a natural isomorphism.

Theorem 3.1.8 *Let A be a small category and $\mathcal{C}$ a category which admits colimits of shape $\int_A X$ for each X in $\mathrm{PSh}(A)$. Given a functor $u\colon A \to \mathcal{C}$, the functor*

$$u^*\colon \mathcal{C} \to \mathrm{PSh}(A) \qquad c \mapsto \mathrm{Hom}_{\mathcal{C}}(u(-), c)$$

admits a left adjoint $u_! = \mathrm{Lan}_{\mathbf{y}} u$ determined by the fact that

$$u(a) \xrightarrow{\sim} u_!(\mathbf{y}(a))$$

is a natural isomorphism.

Corollary 3.1.9 *If A is a small category, then the Yoneda embedding $\mathbf{y}\colon A \to \mathrm{PSh}(A)$ exhibits $\mathrm{PSh}(A)$ as the free cocompletion of A. In other words, for each cocomplete category $\mathcal{C}$ and functor $u\colon A \to \mathcal{C}$, the restriction functor*

$$\mathbf{y}^*\colon \mathrm{Fun}^c(\mathrm{PSh}(A), \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C})$$

is an equivalence of categories with inverse $u \mapsto \mathrm{Lan}_{\mathbf{y}} u$, where Fun^c denotes colimit-preserving functors. Thus by Thm. 3.1.8, every colimit preserving functor $\mathrm{PSh}(A) \to \mathcal{C}$ has a right adjoint.