A Proof of Jennings' Theorem

Vihaan Dheer

1 Introduction

The goal of this short expository paper is to provide a complete proof of a theorem of Jennings in representation theory and discuss some applications thereof. Jennings' theores provides quite a bit of information on the structure of finite p-groups over a field of characteristic p.

Most importantly, we will get a explicit formulas for the Loewy length of the group algebra, the dimensions of the radical/Socle layers, and we will be able to deduce that the radical series coincides with the Socle series in this case.

Computing the group ring kG for $\operatorname{char} k = p > 0$ and G a finite p-group is of interest for various reasons. It is a situation where we can deduce more information than the general case; for example, we know that $\operatorname{Rad}(kG) = IG$, where IG is the augmentation ideal. If G is cyclic, then we have a complete description of the simple and indecomposable projectives.

Also, in many cases even if G is not a p-group, if p divides the order of $|G|$ then G can be split up into a p-group over k and and a group whose ring is semisimple. This is the case e.g. when $G = H \rtimes K$, for H a Sylow p-subgroup, or equivalently when G has a normal Sylow p-subgroup.

1.1 Background

Having motivated Jennings' theorem, let us give a brief historical overview of its proof. The result was first published by S. A. Jennings in 1941 [Jen41] as a special result of his thesis, who defined the *Jennings series* of a group and proved the theorem in its original form. Jennings' proof was somewhat complicated. It is notable that in a footnote to the original paper, Jennings pointed out that Zassenhaus had around the same time proven that the Jennings series also arises out of the Lie algebra associated to the group ring.

D. G. Quillen, in 1968, expanded on that idea and gave a much shorter proof using the natural Lie algebra structure which gives rise to the Jennings series [Qui68]. His argument is essentially the one we use here, but we follow more closely the presentation of Benson in his textbook [Ben91]. Quillen's proof uses a slightly more specific notion of a *p-restricted* Lie algebra, which can be associated to a certain lower central series of the group.

The key idea is that the group commutator provides a Lie algebra commutator, and there is a natural pth power operation in the algbera. The proof works because p-restricted Lie algebras satisfy a Poincaré-Birkhoff-Witt (PBW) theorem, and we will use it to show that the Jennings series coincides with the series of dimension subgroups.

The paper will proceed as follows: In Sec. 2 we will state and prove Jennings theorem. Deviating from the presentation in [Ben91], we will begin with a more general discussion of restricted Lie algebras because they are of interest in their own right (especially in the theory of group schemes in geometry). Then in Sec. 3 we prove the important corollary of Jennings theorem, and cinclude wtih with some examples.

2 Jennings' Theorem

2.1 Restricted Lie Algebras

For this section, fix a field k of characteristic $p > 0$. If $\mathfrak{g}$ is a Lie algebra, then for $x \in \mathfrak{g}$ we use the standard notation $\operatorname{ad} x \colon \mathfrak{g} \to \mathfrak{g}$ for the map $y \mapsto [x, y]$. Our main object of study in this section is a p-restricted Lie algebra, or just restricted Lie algebra for short. The main reference for this subsection is [SF88, Ch. 2.1-2.5].

Definition 1 A *p-restricted Lie algebra* over k is a Lie algebra $\mathfrak{g}$ over k equipped with a map

$$\mathfrak{g} \to \mathfrak{g} \qquad X \mapsto X^{[p]}$$

called the *p-mapping* satisfying the following properties:

1. $\operatorname{ad}(x^{[p]}) = \operatorname{ad}(x)^p$ for all $x \in \mathfrak{g}$.
2. $(\alpha x)^{[p]} = \alpha^p x^{[p]}$ for all $x \in \mathfrak{g}$ and $\alpha \in k$.
3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{j=1}^{p-1} s_j(x, y)$ for all $x, y \in \mathfrak{g}$.

Here the term $s_j(x, y)$ is uniquely determined by the relation

$$(\operatorname{ad}(tx + y)(x))^{p-1} = \sum_{j=1}^{p-1} j s_j(x, y)$$

where powers of ad are interpreted as composition, and t is a formal variable. A morphism of restricted Lie algebras is one which preserves the map $x \mapsto x^{[p]}$.

For example, if $p = 2$ then $s_1(x, y) = [y, x]$, and if $p = 3$ then the relation gives

$$t[x, [y, x]] + [y, [y, x]] = 2ts_2(x, y) + s_1(x, y).$$

Like this one can see how to derive a general formula, but this is not useful.

Example 2 Recall that if A is an associated algebra, then A is naturally a Lie algebra with $[x, y] = xy - yx$. The map $x \mapsto x^{[p]} = x^p$ makes A into a

p-restricted Lie algebra, and the properties follow from the relation

$$\operatorname{ad}(x)^m(y) = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} x^i y x^{m-i}$$

for any m. In particular, $\mathfrak{gl}_n(k)$ is a p-restricted Lie algebra.

Instead of relying on the formula for $(x+y)^{[p]}$, we need the following more important property: If $\mathfrak{g}$ is a p-restricted Lie algebra, then for $x, y \in \mathfrak{g}$, $s_i(x,y) \in \langle x,y\rangle^p$ where $\langle x,y\rangle$ is the subalgebra generated by x and y. The proof is an easy induction.

We move on to the *restricted universal enveloping algebra*. If $\mathfrak{g}$ is a restricted Lie algebra, the restricted enveloping algebra $u(\mathfrak{g})$ should have the universal property that for A an associative k-algebra, restricted Lie algebra homomorphisms $\mathfrak{g} \to A$ correspond to algebra homomorphisms $u(\mathfrak{g}) \to A$.

It is easy to see how to construct this: namely take the standard enveloping algebra $U(\mathfrak{g})$, let I be the two-sided ideal generated by elements $x^{[p]} - x^p$ for all $x \in \mathfrak{g}$, and set $u(\mathfrak{g}) \coloneqq U(\mathfrak{g})$. Then the universal property of $U(\mathfrak{g})$ trivially implies the desired one for $u(\mathfrak{g})$. The main result of this section is the following:

Theorem 3 ([SF88, Thm. 5.1]) *Let $\mathfrak{g}$ be a restricted finite-dimensional Lie algebra, and suppose $e_1, \dots, e_n$ is a basis for $\mathfrak{g}$.*

1. *The collection of distinct elements*

$$e_1^{j_1} \cdots e_n^{j_n} \in u(\mathfrak{g}) \qquad 0 \le j_i < p \tag{$*$}$$

   is a basis for $u(\mathfrak{g})$.

2. *In particular, $\mathfrak{g} \subset u(\mathfrak{g})$, and $\dim_k(u(\mathfrak{g})) = p^n$.*

Remark 4 For the proof of Jennings' theorem, we will not need the full strength of this result (namely we don't need PBW), and instead only the easier direction that the collection $(*)$ *spans* $u(\mathfrak{g})$, so that $\dim_k(u(\mathfrak{g})) \le p^n$.

Proof (of Thm. 3) Observe first that (2) is an immediate consequence: indeed, the basis for $\mathfrak{g}$ is a subset of the basis for $u(\mathfrak{g})$ by taking all j_i to be 0, except one, to be 1. Hence $\mathfrak{g} \hookrightarrow u(\mathfrak{g})$ is injective. The dimension statement is clear by counting the number of elements in $(*)$.

For (1), it is easy to see that the elements $(*)$ span $u(\mathfrak{g})$: indeed, the statement would be trivial if we could reorder the e_i and if the powers j_i could be arbitrary large. The former part is easy to handle because we can use commutators to reorder the basis, which reduce the degree by one (so it recursively follows). To address the powers, note that if $j > p$ then

$$x^j = x^p x^{j-p} = x^{[p]} x^{j-p} \qquad \text{in } u(\mathfrak{g}),$$

so we can always reduce the degree to express elements using powers strictly less than p. This is all we need to deduce Jennings' theorem.

On the other hand, by the classical PBW theorem, we know that the terms $(*)$ are linearly independent as elements of $U(\mathfrak{g})$ because they are part of the canonical PBW basis. So, suppose that a linear combination of these elements lies in I; we need to show the combination is zero. Indeed, this is immediate from degree considerations and another application of the PBW theorem, because the e_i^j with $j \geq p$ are part of the PBW basis in $U(\mathfrak{g})$, so any expression involving them can not have e_j degree lower than p. On the other hand the elements $(*)$ all do, so both sides must be zero. □

For our purposes, this is all we need about restricted Lie algebras, but we comment that much can be said about them. We refer the reader to [SF88, Ch. 2]. These objetcs play a fundamental role in the theory of algebraic groups as well.

2.2 The Proof of Jennings' Theorem

In this section we state and prove Jennings' theorem. Fix a field k of characteristic $p > 0$, and let G be a finite p-group, say of order p^n. Recall that in this case, ([Web16, Prop. 6.3.3]) we have $\mathrm{Rad}(kG) = IG$, and so $\mathrm{Rad}^i(kG) = (IG)^i$. We consider the following:

Definition 5 The *dimension subgroups* of G are

$$F_r(G) \coloneqq \{g \in G : g - 1 \in \mathrm{Rad}^r(kG)\}$$

for $r \geq 1$.

Recall that IG is spanned by the elements $g - 1$ for $g \neq 1$, so $F_1(G) = G$. These are indeed subgroups because clearly for $g, h \in G$, we have

$$gh - 1 = (g - 1) + (h - 1) + (g - 1)(h - 1), \tag{1}$$

so the result is clear. Moreover, $F_r(G) \trianglelefteq G$, because for h arbitrary we have

$$h(g - 1)h^{-1} = hgh^{-1} - 1,$$

which is in $\mathrm{Rad}^r(kG)$ because it is a two-sided ideal. The fact that the $\mathrm{Rad}^r(kG)$ form a decreasing sequence implies that $\{F_r(G)\}$ is also a decreasing sequence, and if $r \geq \ell$, the Loewy length, then $F_r(G) = \{1\}$. So the dimension subgroups form a decreasing filtration of G. We also have $(g-1)^p = g^p - 1$, so the pth power of an element from $F_r(G)$ is in $F_{pr}(G)$. In addition, note that

$$[g, h] - 1 = \big((g - 1)(h - 1) - (h - 1)(g - 1)\big)g^{-1}h^{-1}$$

where we are using the convention that $[g, h] = ghg^{-1}h^{-1}$. Again because $\mathrm{Rad}^r(kG)$ is a two-sided ideal, this implies that $[F_r(G), F_s(G)] \subset F_{r+s}(G)$.

Finally, recall that the *Frattini subgroup* $\Phi(G)$ of a group G is the intersection of its maximal subgroups, and in the case of G a p-group we know that $\Phi(G) =$

$G^p[G,G]$. We claim that $\Phi(G) = F_2(G)$. Indeed, $\mathrm{Rad}^2(kG) = \mathrm{Rad}(IG)$, so we have to show

$$\Phi(G) = \bigcap_{\substack{M \subsetneq IG \\ \text{maximal}}} \{g \in G : g - 1 \in M\}.$$

First Eq. (1) shows that $\{g \in G : g-1 \in M\}$ is still a subgroup because M is a module, and in fact it is maximal; if $H \supsetneq \{g \in G : g - 1 \in M\}$, then $M \subsetneq IH$ beacuse the latter has basis $\{g - 1 : g \in H - \{1\}\}$ over k. Hence $IH = IG$, so $H = G$; this proves the $\subset$ direction. On the other hand, if $H < G$ is maximal, then IH is contained in a maximal ideal M, so

$$H = \{g \in G : g - 1 \in IH\} \subset \{g \in G : g - 1 \in M\}.$$

But if some $g - 1 \in M$ and $g \notin H$, then $\langle H, g\rangle = G$, but since IH is an ideal, IH contains $I\langle H, g\rangle = IG$, a contradiction. This proves our claim. The following is what we have proven so far.

Proposition 6 *Let G be a p-group and k a field of characteristic p.*

1. *The dimension subgroups of G form a descreasing filtration*

$$G = F_1(G) \supset F_2(G) \supset F_3(G) \supset \cdots$$

   of subgroups with length bounded by the Loewy length of kG.

2. *Each $F_r(G)$ is a normal subgroup of G, and $F_2(G) = \Phi(G)$.*

3. *We have the containments*

$$F_r(G)^p \subset F_{rp}(G) \qquad [F_r(G), F_s(G)] \subset F_{r+s}(G)$$

   for all r, s.

The properties of the dimension subgroups motivate what is called the *Jennings series of G*: Namely, it is the fastest descending central series

$$G = \Gamma_1(G) \supset \Gamma_2(G) \supset \Gamma_3(G) \supset \cdots$$

such that $\Gamma_r(G)^p \subset \Gamma_{rp}$ for all r, and $[\Gamma_r(G), \Gamma_s(G)] \subset \Gamma_{r+s}(G)$ for all r, s. Such a fastest descending series exists because (1) G is nilpotent, being a p-group, and (2) we can explicitly construct $\Gamma_r(G)$ by taking $\Gamma_1(G) = G$, and then taking $\Gamma_r(G)$ to be generated by the subgroups

$$[\Gamma_s(G), \Gamma_t(G)], s + t \geq r \qquad \Gamma_s(G)^p, ps \geq r.$$

Note that in particular $[G, \Gamma_r(G)] \subset \Gamma_{r+1}(G)$, so this is a central series, and clearly it is the fastest descending one because the rth term of any series satisfying the two properties must contain $\Gamma_r(G)$. Since the dimension subgroups form such a series, $\Gamma_r(G) \subset F_r(G)$ for all r, so the Jennings series is shorter than the Loewy length.

Let us consider $\Gamma_2(G)$; this is generated by the commutator $[G, G]$ and the power G^p, because all other generating subgroups in the conditions are inside

one of these. Both are normal, so this is none other than $G^p[G,G] = \Phi(G)$, hence the dimension series and Jennings series agree for $r = 1, 2$. Since the series is central, successive quotients are abelian. Moreover, the defining condition means that

$$\Gamma_r(G)^p \subset \Gamma_{r+1}(G),$$

(because $p \geq 2$) so the pth power of element of $\Gamma_r(G)/\Gamma_{r+1}(G)$ is the identity. In particular the quotients are elementary abelian p-groups, and as $\mathbb{Z}$-modules they are annihilated by $p\mathbb{Z}$, hence they are naturally $\mathbb{F}_p$-vector spaces. In what follows we will set

$$n_r \coloneqq \dim_{\mathbb{F}_p}\big(\Gamma_r(G)/\Gamma_{r+1}(G)\big) \qquad r \geq 1,$$

and so obviously $p^{n_r} = [\Gamma_r(G) : \Gamma_{r+1}(G)]$. We can also just tensor with k to make this an n_r-dimensional vector space over k. Doing this, define the *Jennings algebra of G over k* to the graded vector space

$$\mathrm{Jen}_*(G) \coloneqq \bigoplus_{r\geq 1} k \otimes_{\mathbb{F}_p} \big(\Gamma_r(G)/\Gamma_{r+1}(G)\big)$$

over k. The plan is to apply the machinery from the previous section to this object; hence we first establish that it is a graded p-restricted Lie algebra. We need the Hall-Witt identity:

Lemma 7 (Hall-Witt) *Let H be a group. Then we have*

$$\big[[x,y],z^y\big]\big[[y,z],x^z\big]\big[[z,x],y^x\big] = 1$$

for all $x, y, z \in H$.

Proof Just expand out each of the terms using the definitions, and everything cancels to produce the identity. □

Now we observe that $[\Gamma_{r+1}(G), \Gamma_{s+1}(G)] \subset \Gamma_{r+s+2} \subset \Gamma_{r+s+1}$, so it is well-defined to declare the commutator on $\mathrm{Jen}_*(G)$ to be the group commutator (extended by making elements of k central). Then elements of the same degree commute with other, and the bilinearity and anti-symmetry on $[-,-]$ is clear from the definition. The Jacobi identity also follows from the Hall-Witt identity, making $\mathrm{Jen}_*(G)$ a Lie algebra over k (because the extra conjugation can be moved to the outside of the commutator, which has strictly higher degree hence is zero in the quotient).

Likewise, since $\Gamma_{r+1}(G)^p \subset \Gamma_{pr+p}(G) \subset \Gamma_{pr+1}(G)$, the map sending a group element of degree r to its pth power in degree pr is well-defined. This is in fact a (graded) p-mapping in the sense of the previous section. The first two axioms are straightforward, and the third is proven by induction on the minimum degree of x and y, but we don't actually need it, as it was not used substantively in proving the properties of the restricted universal enveloping algebra. In any case, this structure makes $\mathrm{Jen}_*(G)$ into a graded p-restricted Lie algebra.

The way we use this construction is by taking the associated graded of kG, filtered by the radical:

$$\operatorname{gr}_* kG \coloneqq \bigoplus_{i \geq 0} \operatorname{Rad}^i(kG) / \operatorname{Rad}^{i+1}(kG).$$

This has a well-defined multiplication because $\operatorname{Rad}^i(kG) \cdot \operatorname{Rad}^j(kG) \subset \operatorname{Rad}^{i+j}(kG)$. Because $\Gamma_r(G) \subset F_r(G)$, we have maps $\Gamma_r(G) \to \operatorname{Rad}^r(G)$ given by $g \mapsto g-1$, and the same thing in dimension $r+1$ means that it is well-defined on the quotients. This gives a k-linear map $\operatorname{Jen}_*(G) \to \operatorname{gr}_* kG$.

Due to our computations of the commutator and pth powers at the start of this section, the map takes the commutator and p-mapping of $\operatorname{Jen}_*(G)$ to the algebra commutator $xy - yx$ and power $x \mapsto x^p$ in $\operatorname{gr}_* kG$. By the universal property of the p-restricted universal enveloping algebra of the previous section, this gives us a natural algebra homomorphism

$$\psi \colon u(\operatorname{Jen}_*(G)) \to \operatorname{gr}_* kG.$$

Since $\operatorname{gr}_*(kG)$ is generated by its degree one component, and by our observation that the first and second dimension subgroups and Jennings subgroups coincide, ψ is an isomorphism in (algebra, not graded) degree 1, hence is surjective. By the boundedness of the Jennings series, $\operatorname{Jen}_*(G)$ is finite-dimensional; we can choose a basis for $\operatorname{Jen}_*(G)$ like

$$x_1, \ldots, x_{n_1}, \ldots, x_{n_1+n_2}, \ldots, x_m$$

where $m = \sum_i n_i$. Notice that

$$p^m = \prod_r p^{n_r} = \prod_r [\Gamma_r(G) : \Gamma_{r+1}(G)] = |G|.$$

Hence $\dim_k(u(\operatorname{Jen}_*(G))) = |G|$ by Thm. 3. On the other hand,

$$\begin{aligned} \dim_k \operatorname{gr}_* kG &= \sum_{i \geq 0} \left(\dim_k \operatorname{Rad}^i(kG) - \dim_k \operatorname{Rad}^{i+1}(kG) \right) \\ &= \dim_k \operatorname{Rad}^0(kG) = \dim_k(kG) = |G|. \end{aligned}$$

Hence the surjective map ψ must be an isomorphism. We now state our main result, Jennings' theorem:

Theorem 8 (Jennings) *Let G be a p-group.*

1. *$F_r(G) = \Gamma_r(G)$ for all $r \geq 1$.*
2. *The Poincaré series of the radical layers of kG has the expression*

$$\sum_{i \geq 0} \dim_k \frac{\operatorname{Rad}^i(kG)}{\operatorname{Rad}^{i+1}(kG)} t^i = \prod_{r \geq 1} \left(\frac{1 - t^{pr}}{1 - t^r} \right)^{n_r}$$

where $p^{n_r} = [F_r(G) : F_{r+1}(G)]$.

3. The Loewy length of kG is $\ell(kG) = 1 + \sum_{r\geq 1}(p-1)rn_r$.

Proof The main part of the result is (1), which we can now prove from our work above. We prove it by induction, and we have already established the cases $r = 1, 2$. For greater r, we observe that each element of $F_r(G)$ not in $F_{r+1}(G)$ represents a nonzero element in $\mathrm{gr}_* kG$, so there is a unqiue preimage under ψ. By its definition and the inductive hypothesis, this must come from a unique element of $\Gamma_r(G)/\Gamma_{r+1}(G)$, so in particular the element we started with is in the image of the inclusion $\Gamma_r(G) \hookrightarrow F_r(G)$, which proves (1).

For (2), Thm. 3 moreover implies that the elements

$$x_1^{j_1} \cdots x_m^{j_m} \qquad 0 \leq j_i < p$$

are a basis for the restricted enveloping algebra. The graded isomorphism ψ implies that this corresponds to a k-basis for $\mathrm{gr}_* kG$, where the n_r elements of the $x_1, \ldots, x_m$ for to the rth Jennings group induce a basis for the rth graded component

$$\mathrm{gr}_r kG = \mathrm{Rad}^r(kG)/\mathrm{Rad}^{r+1}(kG).$$

Counting the combinations of numbers j_i, and then considering permutations, gives the following series

$$\sum_i \dim_k(\mathrm{Rad}^i(kG)/\mathrm{Rad}^{i+1}(kG))t^i = \prod_r (1 + t^r + t^{2r} + \cdots + t^{(p-1)r})^{n_r}$$

and the expression in (2) is just the geometric series representation of the factors.

(3) Finally, the Loewy length is just the earliest i for which these radical quotients give 0. In other words, it is one more than the degree of the Poincaré polynomial above. This degree is obviously $\sum_r (p-1)rn_r$ by inpection because the top degree term in each factor is $t^{pr}/t^r = t^{(p-1)r}$, raised to the n_r. □

3 Examples and Applications

We will conclude by giving the primary application of Jennings theorem, which is the following result, and then a small example.

Theorem 9 *Let k be a field of characteristic $p > 0$ and G a p-group. Then the radical series of kG is equal to the Socle series of kG.*

Proof Letting ℓ be the Loewy length as in Thm. 8, the Poincaré polynomial of Jennings' theorem is symmetric in the sense that

$$\dim_k(\mathrm{Rad}^i(kG)/\mathrm{Rad}^{i+1}(kG)) = \dim_k(\mathrm{Rad}(J^{\ell-i-1})/\mathrm{Rad}^{\ell-i}(kG)).$$

by an easy computation with the polynomial. By [Web16, Ch. 6, Exer. (7)] we know

$$\mathrm{Soc}^{n+1}(kG^*)/\mathrm{Soc}^n(kG^*) \cong (\mathrm{Rad}^n(kG)/\mathrm{Rad}^{n+1}(kG))^*,$$

and by [Web16, Prop. 8.5.2] this gives us:

$$\dim(\mathrm{Rad}^n(kG)/\mathrm{Rad}^{n+1}(kG)) = \dim(\mathrm{Soc}^{n+1}(kG)/\mathrm{Soc}(kG)).$$

Finally, remember that:

$$\mathrm{Soc}^n(kG) \supset \mathrm{Rad}^{\ell-n}(kG)$$

this is because $\mathrm{Rad}^n(kG) \cdot \mathrm{Rad}^{\ell-n}(kG) \subset \mathrm{Rad}^{\ell}(kG) = 0$, so $\mathrm{Rad}^{\ell-n}(kG) \subset \mathrm{Soc}^n(kG)$ as the nth socle is the set of elements annihilated by the nth radical. By inducting on n and applying the first equation, we are done. □

Now we come to an example. Let us remark that abelian p-groups are not very interesting because they are just direct products of cyclic p-groups, and we have a very explicit description of the group rings of p-groups and of direct products. The first nonabelian p-group is D_8, which is considered in [Web16, Example 6.4.2]. Recall also that all groups of order p^2 are abelian, so the next examples are nonabelian groups of order 16, of which there are 9, and nonabelian groups of order 27, of which there are 3. Let us consider one of those, namely

$$M_{27} = C_9 \rtimes C_3 = \langle a, b \mid a^9 = b^3 = 1, bab^{-1} = a^4 \rangle .$$

Leave k a fixed field of characteristic 3. We will first compute the Jennings/dimension series of M_{27}. The first one is M_{27} itself, and the second is the Frattini subgroup as we showed

$$\Phi(M_{27}) = M_{27}^3[M_{27}, M_{27}].$$

M_{27}^3 is obviously $\{1, a^3, a^6\}$, and we can see that the derived subgroup is generated by $bab^{-1}a^{-1} = a^4a^{-1} = a^3$, so they are the same and the Frattini subgroup is $\langle a^3 \rangle$.

The first index is $27/3 = 9$, so $n_1 = 2$. So, the only two choices are that the next subgroup is C_3 or 1, corresponding to n_2 of zero or 1. Upon examining our initial formula for the generators of $\Gamma_3(G)$, we see that γ_3 is generated by a^3 and $b^3 = 1$, so $n_2 = 0$ and $\Gamma_3 = \Gamma_2$. The same formula shows that Γ_4 is the trivial group. Therefore the Jennings series is

$$M_{27} \supset \langle a^3 \rangle \supset \langle a^3 \rangle \supset \langle 1 \rangle , n_1 = 2, n_2 = 0, n_3 = 1.$$

Hence, the Poincaré polynomial is given by

$$\prod_{r=1}^{3}(1 + t^r + t^{2r})^{n_r} = (1 + t + t^2)^2(1 + t^3 + t^6)$$
$$= 1 + 2t + 3t^2 + 3t^3 + 3t^4 + 3t^5 + 3t^6 + 3t^7 + 3t^8 + 2t^9 + t^{10}$$

This gives us all of the dimensions of the radical layers (and the Socle layers by the theorem), and also tells us that the Loewy length is 11.

References

[Ben91] D. J. Benson, *Representations and cohomology. I*, Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1991, Basic representation theory of finite groups and associative algebras. MR 1110581

[Jen41] S. A. Jennings, *The structure of the group ring of a p-group over a modular field*, Trans. Amer. Math. Soc. **50** (1941), 175–185. MR 4626

[Qui68] Daniel G. Quillen, *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418. MR 231919

[SF88] Helmut Strade and Rolf Farnsteiner, *Modular Lie algebras and their representations*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 116, Marcel Dekker, Inc., New York, 1988. MR 929682

[Web16] Peter Webb, *A course in finite group representation theory*, Cambridge Studies in Advanced Mathematics, vol. 161, Cambridge University Press, Cambridge, 2016. MR 3617363