HCHA Notes

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This is a notes document following D. Cisinksi's *Higher Categories and Homotopical Algebra* [HCHA], with the goal of building up to modern higher category theory, which in this context means the study of quasicategories, i.e. $(\infty, 1)$-categories. Sections correspond to chapters in the text.

1 Ch. 1: Review

[HCHA, Chapter 1] contains information which is, at the time of me writing this, well-known to me. Thus, I will instead put other pre-reqs and interesting asides in this section.

1.1 Pushout-product

1.1.1 Generalities

Let $\mathcal{C}$ be a closed monoidal category with unit $\mathbb{I}$ and tensor product $\otimes$, and assume $\mathcal{C}$ has all pushouts and pullbacks. In what follows we will construct a closed monoidal category structure on $\mathrm{Arr}(\mathcal{C})$, the category of morphisms.

Remark 1.1.1 I don't actually know if what I'm saying works in this generality and is true, but it looks to be true. The case we will use is the Cartesian closed structure on a presheaf category $\mathrm{PSh}(A)$, and I will prove all results I actually need. Also, if the category is not literally closed monoidal, but that some functors $(-) \otimes Y$ have right adjoints, then we can still use the results here. This will apply for $\mathcal{C} = \mathbf{Top}$.

Given morphisms $a \colon A \to A'$ and $b \colon B \to B'$ in the category, we define a morphism

$$a \square b \colon A' \otimes B \underset{A \otimes B}{\amalg} A \otimes B' \to A' \otimes B'$$

by the canonical map from the pushout. Let's denote the domain by $P_{a,b}$, and call both the domain and the map itself the *pushout-product* of a and b. Dually, given a map $b \colon B \to B'$ and $f \colon X \to Y$, we will write

$$f^{\blacksquare b} \colon X^{B'} \to X^B \underset{Y^B}{\times} Y^{B'}$$

for the canonical map into the pullback, where we are denoting the right adjoint of the functor $Z \mapsto B \otimes Z$ by $Z \mapsto Z^B$.

Theorem 1.1.2 *The arrow category* $\mathrm{Arr}(\mathcal{C})$ *is canonically a closed monoidal category with the product defined as*

$$(a, b) \mapsto a \,\square\, b,$$

tensor unit $1_{\mathbb{I}}$*, and the closed structure given by an adjunction*

$$\mathrm{Hom}_{\mathrm{Arr}(\mathcal{C})}(a \,\square\, b, f) \cong \mathrm{Hom}_{\mathrm{Arr}(\mathcal{C})}(a, f^{\blacksquare b}),$$

where all coherences are induced by universal properties of limits and colimits.

The two things we care about are the adjunction formula and associativity of the pushout product. Keeping notation as before, the morphisms in the arrow category are pairs of maps making the relevant diagram commute; in other words they are of the form

$$\mathrm{Hom}_{\mathrm{Arr}(\mathcal{C})}(a, b) = \mathcal{C}(A, B) \underset{\mathcal{C}(A,B')}{\times} \mathcal{C}(A', B'),$$

the pullback taken in the category of sets. To show the adjunction, observe that

$$\mathcal{C}(P_{a,b}, Z) = \mathcal{C}(A, Z^{B'}) \underset{\mathcal{C}(A,Z^B)}{\times} \mathcal{C}(A', Z^B)$$

for any Z, by the closed structure in $\mathcal{C}$. We have the giant commutative diagram

$$\begin{array}{ccccc}
\mathcal{C}(A, X^{B'}) & \longrightarrow & \mathcal{C}(A, Y^{B'}) & = & \mathcal{C}(A, Y^{B'}) \\
\downarrow & & \downarrow & & \| \\
\mathcal{C}(A, X^B) & \longrightarrow & \mathcal{C}(A, Y^B) & \longleftarrow & \mathcal{C}(A, Y^{B'}) \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{C}(A', X^B) & \longrightarrow & \mathcal{C}(A', Y^B) & \longleftarrow & \mathcal{C}(A', Y^{B'})
\end{array}$$

in sets; by Fubini's theorem the diagram has a limit, computed either by taking pullbacks vertically then horizontally, or horizontally then vertically. If we take pullbacks of each column first, and then take their pullback, we get exactly the result of the above formula for $\mathcal{C}(P_{a,b}, -)$ plugged into the formula for morphisms $a \,\square\, b \to f$ in the arrow category. Going the other direction, since $\mathcal{C}(A, -)$ and $(A', -)$ preserve limits, we get

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Arr}(\mathcal{C})}(a \,\square\, b, f) &= \mathcal{C}(A, X^{B'}) \underset{\mathcal{C}(A', X^B \times_{Y^B} Y^{B'})}{\times} \mathcal{C}(A', X^B \underset{Y^B}{\times} Y^{B'}) \\
&= \mathrm{Hom}_{\mathrm{Arr}(\mathcal{C})}(a, f^{\blacksquare b})
\end{aligned}$$

since $X^{B'} \times_{Y^{B'}} Y^{B'} = X^{B'}$. This is the adjunction formula.

Let us also remark that if $(\mathcal{C}, \otimes)$ is symmetric, then $(\mathrm{Arr}(\mathcal{C}), \square)$ is symmetric as well. Indeed, applying the symmetry to all objects and morphisms in the pushout defining $P_{a,b}$, we obtain $P_{b,a}$, and then $P_{a,b} \to A' \otimes B'$ becomes $P_{b,a} \to B' \otimes A'$.

To show that the pushout-product is associative, we will construct another 3x3 commutative diagram, and this time take pushouts. It will be obvious that the

diagram is commutative, and we are using the coherence theorem to suppress associativities. We change notation slightly, instead considering three maps

$$U \xrightarrow{i} V, \quad W \xrightarrow{j} X, \quad Y \xrightarrow{k} Z.$$

We want to show $(i \,\square\, j) \,\square\, k \cong i \,\square\, (j \,\square\, k)$.

$$\begin{array}{ccccc} V \otimes W \otimes Z & \longleftarrow & U \otimes W \otimes Z & \longrightarrow & U \otimes X \otimes Z \\ \uparrow & & \uparrow & & \uparrow \\ V \otimes W \otimes Y & \longleftarrow & U \otimes W \otimes Y & \longrightarrow & U \otimes X \otimes Y \\ \downarrow & & \downarrow & & \| \\ V \otimes X \otimes Y & \longleftarrow & U \otimes X \otimes Y & = & U \otimes X \otimes Y \end{array}$$

The trick is that along the rows, the last variable is constant, but along the columns the first variable is constant. Since $P \otimes (-)$ preserves colimits for fixed P, along one dimension the pushout is computed to be

$$V \otimes P_{j,k} \xleftarrow{i \otimes P_{j,k}} U \otimes P_{j,k} \xrightarrow{U \otimes j \square k} U \otimes X \otimes Z,$$

where in the third variable we used that pushing out along the identity can be cancelled. By definition, the pushout of the above span is $P_{i,j\square k}$. Moreover, the universal property gives us a natural map $P_{i,j\square k} \to V \otimes X \otimes Z$, which agrees with $i \,\square\, (j \,\square\, k)$. On the other hand, if we start in the opposite direction we obtain the span

$$P_{i,j} \otimes Z \xleftarrow{P_{i,j} \otimes k} P_{i,j} \otimes Y \xrightarrow{i \square j \otimes Y} V \otimes X \otimes Y,$$

and then pushing out again gives $P_{i\square j,k}$, once again equipped with the natural map $(i \,\square\, j) \,\square\, k \colon P_{i\square j,k} \to V \otimes X \otimes Z$. By Fubini's theorem, we get a natural isomorphism

$$i \,\square\, (j \,\square\, k) \cong (i \,\square\, j) \,\square\, k$$

in the arrow category, describing the monoidal associator.

Remark 1.1.3 We will want to use this result in the case of topological spaces (which are not necessarily in a 'nice category'), so it is helpful to track where exponential objects are being used in the proof of associativity. Basically, we have used that the functors

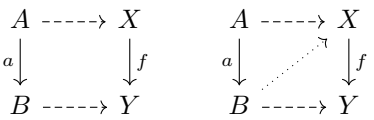
$$V \otimes (-), \quad U \otimes (-), \quad Z \otimes (-), \quad Y \otimes (-)$$

all preserve colimits, so that the associativity relation can be applied to i, j, k in the case that the domain/codomain of both i and k have exponentials. Note that when $\otimes$ is symmetric, this means we have associativity whenever two of the maps have both domain and codomain exponentiable.

1.1.2 Applications

This section is misplaced, and contains content relevant to Chapter 2 and 3. But in any case these are all direct applications of the preceding construction.

First of all, suppose we have two maps $a\colon A \to B$ and $f\colon X \to Y$.

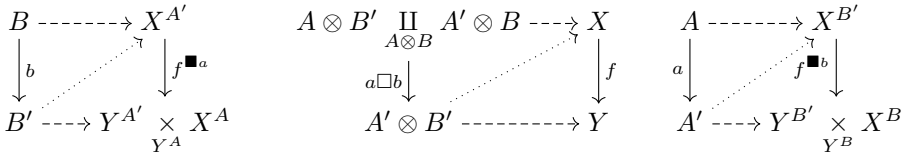


Above, the *set of lifting problems* from a to f is the set of pairs of morphisms in $\mathcal{C}$ making the square on the left commute. In other words, it is $\mathrm{Hom}_{\mathrm{Arr}(\mathcal{C})}(a, f)$. One a lifting problem (u, v) is fixed, a *solution* or *filler* is a map $h\colon B \to X$ such that $ha = u$ and $fh = v$. Note that there is a canonical map

$$\mathcal{C}(B, X) \to \mathcal{C}(A, X) \underset{\mathcal{C}(A,Y)}{\times} \mathcal{C}(B, Y) = \mathrm{Hom}_{\mathcal{C}}(a, f)$$

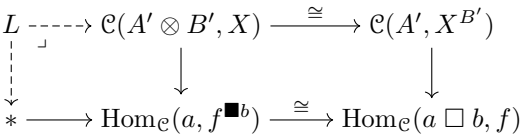
to the set of lifting problems, and the set of solutions to a lifting problem is the preimage of a point. Now let $\mathcal{C}$ be symmetric monoidal.

Proposition 1.1.4 *Fix morphisms $A \xrightarrow{a} A', B \xrightarrow{b} B'$ and $X \xrightarrow{f} Y$ in $\mathcal{C}$. There is a natural 1-1 correspondence*



between the problems shown in each of the three diagrams. Moreover, once such a lifting problem is fixed, there is natural 1-1 correspondence between solutions.

Proof Let us do the middle and right-hand squares, as the other follows from using the symmetry isomorphism on $\mathcal{C}$. The correspondence is exactly the adjunction described in the previous section. Note that the diagram



commutes, and the right-hand square is a pullback for trivial reasons, so the pasting lemma gives the 1-1 correspondence between solutions. □

Corollary 1.1.5 *Fix morphisms $A \xrightarrow{a} A', B \xrightarrow{b} B'$ and $X \xrightarrow{f} Y$ in $\mathcal{C}$.*

1. $a \,\square\, b \in l(f) \iff a \in l(f^{\blacksquare b}) \iff b \in l(f^{\blacksquare a})$.
2. *Let S be a saturated class of morphisms and J any class of morphisms. Then there is an equality*

$$\{a : a \,\square\, b \in S \text{ for all } b \in J\} = l(\{f^{\blacksquare b} : b \in J, f \in r(S)\}),$$

so in particular this class is saturated.

Proof (1) Prop. 1.1.4. (2) Let C be the class, then since $S = l(r(S))$,

$$\begin{aligned} a \in C &\iff a \in l(f^b) \quad \forall b \in J, f \in r(S) \\ &\iff a \in l(\{f^{\blacksquare b} : b \in J, f \in r(S)\}) \end{aligned}$$

as desired. □

Remark 1.1.6 Edit: The proof above is wrong, because I assumed $S = l(r(S))$ which is of course not true in general. But it holds e.g. if S admits the small object argument, or S is the left side of a weak factorization system.

Now we will quickly discuss some applications of this to some easy corollaries in [HCHA] which are stated there without proof. So, now we are working in the category of simplicial sets.

Theorem 1.1.7 ([HCHA, 3.1.4-3.1.7])

1. *For any anodyne extension $i\colon K \to L$ and monomorphism $j\colon U \to V$, the induced inclusion*
$$K \times V \cup L \times U \to L \times V$$
is an anodyne extension.
2. *For $j\colon U \to V$ a monomorphism and $p\colon X \to Y$ a Kan fibration, the canonical map*
$$(j^*, p_*)\colon \underline{\mathrm{Hom}}(V, X) \to \underline{\mathrm{Hom}}(U, X) \underset{\underline{\mathrm{Hom}}(U,Y)}{\times} \underline{\mathrm{Hom}}(V, Y)$$
is a Kan fibration.
3. *For $i\colon K \to L$ an anodyne extension and $p\colon X \to Y$ a Kan fibration, the canonical map*
$$(i^*, p_*)\colon \underline{\mathrm{Hom}}(L, X) \to \underline{\mathrm{Hom}}(K, X) \underset{\underline{\mathrm{Hom}}(L,Y)}{\times} \underline{\mathrm{Hom}}(K, Y)$$
is a trivial fibration.

Proof (1) This map is $i \,\square\, j$, so fix the map j. The class

$$\{K \xrightarrow{i} L : i \,\square\, j \text{ is anodyne}\}$$

is saturated because the class of anodyne extensions is saturated, so it suffices to show that the class contains all anodyne extensions. By [HCHA, Prop. 3.1.2], we may show it contains inclusions of the form $(\{\epsilon\} \to \Delta^1) \,\square\, (\partial\Delta^n \to \Delta^n)$. But indeed,

$$\big((\{\epsilon\} \to \Delta^1) \,\square\, (\partial\Delta^n \to \Delta^n)\big) \,\square\, j \cong (\{\epsilon\} \to \Delta^1) \,\square\, \big((\partial\Delta^n \to \Delta^n) \,\square\, j\big),$$

and $(\partial\Delta^n \to \Delta^n) \,\square\, j$ is a monomorphism, so by [HCHA, Def. 2.4.11(1)], the above maps are anodyne.

(2) The map (j^*, p_*) is what we have called $p^{\blacksquare j}$. A Kan fibration has right lifting with respect to horn inclusions, which by [HCHA, Prop 3.1.2] is the same

has having right lifting with respect to anodyne extensions. For any anodyne map i,

$$p^{\blacksquare j} \in r(i) \iff i \in l(p^{\blacksquare j}) \iff i \,\square\, j \in l(p) \iff p \in r(i \,\square\, j).$$

But we have just shown that the $i \,\square\, j$ is anodyne.

(3) Just as above, given $j\colon U \to V$ a monomorphism,

$$p^{\blacksquare i} \in r(j) \iff p \in r(i \,\square\, j),$$

and $i \,\square\, j$ is anodyne. □

Corollary 1.1.8 *For any simplicial set W and Kan complex X, the simplicial set $\underline{\mathrm{Hom}}(W, X)$ is a Kan complex.*

Proof Using (2) above, take $p = X \to *$ and $j = \varnothing \to W$, then the corresponding map is

$$\underline{\mathrm{Hom}}(W, X) \to \underline{\mathrm{Hom}}(\varnothing, X) \times_{\underline{\mathrm{Hom}}(\varnothing,*)} \underline{\mathrm{Hom}}(W, *) = *,$$

and it's a Kan fibration. □

Example 1.1.9 Suppose $p\colon X \to Y$ is a trivial fibration. Since $\varnothing \hookrightarrow \Delta^n$ is a monomorphism, $p_n\colon X_n \to Y_n$ is surjective for all $n \geq 0$. In particular, taking 0-simplices in (3) we recover the statement that that p has the right lifting property against anodyne maps.

Remark 1.1.10 Any vertex inclusion $\Delta^0 \to \Delta^n$ is anodyne.

1. $\{n\} \to \Delta^n$ *is anodyne*, because it is a strong deformation retract ([HCHA, Prop. 2.4.32]). Indeed, a natural transformation from the identity functor $[n] \to [n]$ to the constant functor is given by the inclusions $k \to n$.
2. *If monomorphisms $f, g\colon X \to Y$ are homotopic and one of them is anodyne, then the other is anodyne.* We may assume there is a homotopy $h\colon \Delta^1 \times X \to Y$ from f to g. Since the first two horn inclusions are $(0)\colon \{0\} \to \Delta^1$ and $(1)\colon \{1\} \to \Delta^1$, these are anodyne. Since $\varnothing \to X$ is mono, it follows that

   $$(0) \,\square\, (\varnothing \to X), (1) \,\square\, (\varnothing \to X)\colon \Delta^0 \times X \to \Delta^1 \times X$$

   are anodyne too. If f is as well, then both $(0) \times X$ and $f = h \circ ((0) \times X)$ are absolute weak equivalences so h is as well, and thus $((1) \times X)h = g$ is. The same argument works in the other direction too.
3. There is a path $\Delta^1 \to \Delta^n$ from any k to n, namely the map (kn) sending $0 \mapsto k, 1 \mapsto n$. So we conclude by (2).

Note: The annoying part of this argument was (2), but it would have been trivial if we knew [HCHA, Thm. 3.1.29], that trivial cofibrations were anodyne. Indeed, clearly maps homotopic to weak equivalences are weak equivalences.

Remark 1.1.11 It's *not* true that Kan fibrations are surjective on n-simplices (or 0-simplices) in general (though this only admits trivial counterexamples). However the only obstruction is surjectivity on 0-simplices: Indeed, this implies $f\colon X \to Y$ has right lifting against $\varnothing \to \Delta^0$, and it also has right lifting against $\Delta^0 \to \Delta^n$ by Remark 1.1.10, thus it has right lifting against $\varnothing \to \Delta^n$. (Note that we only needed to use the easier part (1) of the remark).

Now let's prove some analogous results for ∞-categories, i.e. so we're replacing anodyne extensions and Kan fibrations with inner fibrations and inner anodyne extensions. Recall that an inner fibration is something that has right lifting against the maps $\Lambda^n_k \to \Delta^n$ for $0 < k < n$, and an inner anodyne map is something that has left lefting against inner fibrations. In particular, an ∞-category is a simplicial set X for which $X \to *$ is an inner fibration.

Proposition 1.1.12 *For an inner anodyne extension $i\colon K \to L$ and any monomorphism $j\colon U \to V$, the induced inclusion $i \square j$ is an inner anodyne extension.*

Proof We already know that $\{i : i \square j \text{ is inner anodyne}\}$ is saturated. Fixing $n \geq 0$ and letting $a = \Lambda^2_1 \to \Delta^2$, $b = \partial\Delta^n \hookrightarrow \Delta^n$, we need to show that $a \square b$ is in this class. But

$$(a \square b) \square j \cong a \square (b \square j)$$

is inner anodyne by [HCHA, Prop. 3.2.3], because $b \square j$ is a monomorphism, completing the proof. □

Proposition 1.1.13 *Let $p\colon X \to Y$ be a morphism. The following are equivalent:*

i. p is an inner fibration.

ii. For any inner anodyne extension $i\colon K \to L$, the induced map

$$(i^*, p_*)\colon \underline{\mathrm{Hom}}(L, X) \to \underline{\mathrm{Hom}}(K, X) \underset{\underline{\mathrm{Hom}}(L,Y)}{\times} \underline{\mathrm{Hom}}(K, Y)$$

is a trivial fibration.

iii. The canonical map

$$\underline{\mathrm{Hom}}(\Delta^2, X) \to \underline{\mathrm{Hom}}(\Lambda^2_1, X) \underset{\underline{\mathrm{Hom}}(\Delta^2_1,Y)}{\times} \underline{\mathrm{Hom}}(\Delta^2, Y)$$

is a trivial fibration.

iv. For any monomorphism $j\colon U \to V$, the canonical map

$$(j^*, p_*)\colon \underline{\mathrm{Hom}}(V, X) \to \underline{\mathrm{Hom}}(U, X) \underset{\underline{\mathrm{Hom}}(U,Y)}{\times} \underline{\mathrm{Hom}}(V, Y)$$

is an inner fibration.

Proof $(i) \Rightarrow (ii)$ For j mono, $p^{\blacksquare i} \in r(j) \leftrightarrow i \square j \in r(p)$, the latter being true by the above proposition. $(ii) \Rightarrow (iii)$ is the special case of the inner anodyne map $\Lambda^2_1 \to \Delta^2$.

$(iii) \Rightarrow (iv)$ Let i be mono and write $a = \Lambda^2_1 \to \Delta^2$. Then $i \square j$ is also mono, and

$$\begin{aligned} p^{\blacksquare a} \in r(i \square j) &\iff p \in r(a \square (i \square j)) \\ &\iff p \in r((a \square i) \square j) \\ &\iff p^{\blacksquare j} \in r(a \square i), \end{aligned}$$

so the desired map has right lifting against $a \square i$ for all monomorphisms i, but the saturation of that class is the inner anodyne maps by [HCHA, Prop. 3.2.2].

$(iv) \Rightarrow (i)$ Take $j = 1 \colon \Delta^0 \to \Delta^0$. □

Corollary 1.1.14 *A simplicial set X is an ∞-category if and only if the canonical map*

$$\underline{\mathrm{Hom}}(\Delta^2, X) \to \underline{\mathrm{Hom}}(\Lambda^2_1, X)$$

is a trivial fibration.

Proof Apply Prop. 1.1.13 to $X \to \Delta^0$ and use $(i) \iff (iii)$. □

Corollary 1.1.15 *If A is a simplicial set and X is an ∞-category, then $\underline{\mathrm{Hom}}(A, X)$ is an ∞-category.*

Proof Apply Prop. 1.1.13, $(i) \Rightarrow (iv)$ to $p = X \to \Delta^0$ and $j = \varnothing \to A$. □

2 Ch. 2: Model categories

2.1 Background

Model categories are categories equipped with three distinguished subclasses of morphisms satisfying some axioms. They provide a natural and well-behaved setting to perform categorical localization, that is to formally invert morphisms.

In general, given a category $\mathcal{C}$ and a class $W \subset \mathrm{mor}(\mathcal{C})$ of morphisms, we can always define a localiaztion $\mathcal{C} \to \mathcal{C}[W^{-1}]$ satisfying the right universal property, i.e. that for any category $\mathcal{D}$ the induced map

$$\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}^W(\mathcal{C}, \mathcal{D})$$

is an equivalence of categories, where Fun^W means the full subcategory spanned by those functors which send each morphism in W to an invertible morphism. In other words, the localization is a representing object for this functor $\mathrm{Fun}^W(\mathcal{C}, -)$, so that any functor sending W to invertible elements factors up to isomorphism through $\mathcal{C}[W^{-1}]$.

Example 2.1.1 If M is a monoid and BM denotes the corresponding one-object category with morphisms M, then $BM[M^{-1}]$ is the same as $B(M^{gp})$, where M^{gp} is the group completion of M.

In general, we cannot say much about a localization; e.g. it is not obvious even in small examples when two nonisomorphic objects have isomorphic images under $\mathcal{C} \to \mathcal{C}[W^{-1}]$. If we add additional structure to the category, a *model structure*, we can get a precise description of the morphisms in the localization.

There are two main motivating examples for model categories outside of ∞-category theory. Firstly, the construction of the derived category (i.e. the localization by quasi-isomorphisms) of chain complexes for an abelian category. The classical idea is to pass through the construction the construction of the chain homotopy category. To compute derived functors one uses projective (or injective) resolutions, because the notions of homotopy equivalence and quasi-isomorphisms (weak equivalence) agree on complexes of projective objects.

This scenario is parallel to that of the homotopy theory of CW complexes. The standard homotopy invariants (homotopy and (co)homology) respect weak homotopy equivalences, so they would descend to functors from $\mathbf{Top}[\mathrm{whe}^{-1}]$. But, there are many constructions which are invariant under homotopy equivalence, but not under weak homotopy equivalence. In this case we might want to look at its derived functor, which is the universal way of making it invariant under weak homotopy equivalence.

(*Note: What's in the following paragraph is not true.*) For example, the functor assigning to a pair the pointed quotient space naturally respects homotopy,

and hence homotopy equivalence. But in general, if two pairs are weakly homotopy equivalent, the quotient spaces may not be. The claim *is* true for CW pairs though. So, we get a map on the derived categories which respects weak equivalence, which is computed by taking a CW approximation of pairs. In this example, there is a functorial model directly from pairs which is the mapping cone of the inclusion, i.e. $(X, A) \mapsto X \cup_A CA$. As we can see, if we start out with a CW pair (X, A), then $(X \cup CA, CA)$ is a CW complex with CA a contractible subcomplex, so $(X \cup_A CA)/CA = X \cup_A * = X/A$, and $(X \cup CA)/CA \simeq X \cup CA$.

Just as in topology, we will do something similar in the case of simplicial sets. This additional quasi-derived structure will allow us to construct infinity-categories and study their homotopy theory in a well-behaved way.

2.2 Factorization Systems

We shorten 'left lifting property' to llp and likewise for rlp. Recall that for $F \subset \mathrm{mor}(\mathcal{C})$, $l(F)$ means the set of morphisms with the llp with respect to every morphism in F.

A class of morphisms $F \subset \mathrm{mor}(\mathcal{C})$ is *saturated* if it is stable under retracts, pushouts, and transfinite compositions. Likewise, F is *cosaturated* if $F^{\mathrm{op}} := F \subset \mathrm{mor}(\mathcal{C}^{\mathrm{op}})$ is saturated.

It is easy to show the following facts about the classes $l(F)$ and $r(F)$, given classes F, F' of morphisms in $\mathcal{C}$:

1. $F \subset r(F')$ iff $F' \subset l(F)$.
2. If $F \subset F'$, then $l(F') \subset l(F)$ and $r(F') \subset r(F)$.
3. $r(F) = r(l(r(F)))$ and $l(F) = l(r(l(F)))$.
4. $r(F)$ in $\mathcal{C}$ equals $l(F)$ in $\mathcal{C}^{\mathrm{op}}$, and vice versa.
5. $l(F)$ is saturated, and $r(F)$ is cosaturated.

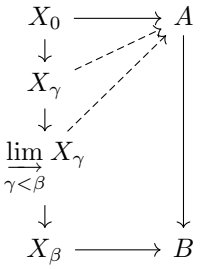
Let us see why some of these hold. (1) is how we defined l and r, and (2) is clear because each element of $l(F')$ or $r(F')$ has to satisfy more conditions than those in $l(F)$ and $r(F)$. For (3), note that $r(F) \subset r(F)$ means by (1) that $F \subset l(r(F))$, and by (2) $r(l(r(F))) \subset r(F)$. On the other hand $l(r(F)) \subset l(r(F))$ means by (1) that $r(F) \subset r(l(r(F)))$. The other part is similar. (4) is the definition of r and l.

This leaves (5), and by (4) we only have to show the first part. The stability under retracts and pushouts follows literally immediately from the definition. We prove stability under transfinite composition: Let $\alpha > 0$ be some ordinal

and consider a functor $X_\bullet \colon \alpha \to \mathcal{C}$. Suppose we are given a commuting square

$$\begin{array}{ccc} X_0 & \longrightarrow & A \\ \downarrow & & \downarrow \\ \varinjlim_{\beta<\alpha} X_\beta & \longrightarrow & B \end{array}$$

then we want to find a diagonal lift. For each $\beta < \alpha$ we construct by transfinite induction a map $X_\beta \to A$ fitting into the diagram in between X_0 and $\varinjlim_\beta X_\beta$, and extending all previous maps along the map to X_β. For $\beta = 0$ just take the top of the square. For $\beta > 0$, consider the diagram:



The inductive hypothesis gives the maps $X_\gamma \to A$ for all $\gamma < \beta$, and since they agree on previous ordinals we get a map from the colimit to A. The lower left map has left lifting by assumption, so we get a lift $X_\beta \to A$. This completes the induction, and likewise we get an induced map $\varinjlim_\beta X_\beta \to A$ giving a lift.

Definition 2.2.1 A *weak factorization system* in $\mathcal{C}$ is pair of two classes (A, B) of morphisms both stable under retracts, satisfying $A \subset l(B)$, and that any morphism $f \colon X \to Y$ in $\mathcal{C}$ factors into pi with $i \in A$ and $p \in B$.

Lemma 2.2.2 *Suppose $f \colon X \to Y$ can be factored into $f = pi$. If $f \in r(i)$, then f is a retract of p. Likewise, if $f \in l(p)$, then f is a retract of i.*

Proof Write the factorization as a commutative square with top row 1_X, then the corresponding lift mapping to X provides the remaining data for a retract with bottom row 1_Y. □

Corollary 2.2.3 *Let (A, B) be a pair of classes of morphisms of $\mathcal{C}$ such that each morphism in $\mathcal{C}$ factors as pi for $i \in A$ and $p \in B$. Then (A, B) is a weak factorization system if and only if $A = l(B)$, if and only if $B = r(A)$.*

Proof ($\Rightarrow$) By definition, $A \subset l(B)$, yet the retract lemma implies that $l(B) \subset A$. ($\Leftarrow$) Conversely, if $A = l(B)$, then $B \subset r(A)$ and the retract lemma says $B = r(A)$, so A is saturated. But then B is also closed under retracts because a retract in the opposite category is a retract in the category itself. □

Remark 2.2.4 The corollary means that any weak factorization system (A, B) in $\mathcal{C}$ is of the form $(l(B), B)$ and $(A, r(A))$. But,

2.3 Review of MC results

The very basics are below. Everything here is elementary category theory except perhaps for the small object argument, which basically let's us find factorizations which are annoying to construct explicitly:

1. In a weak factorization system (A, B), we have $A = l(B)$, $r(A) = B$, and every morphism admits a factorization $f = pi$ with $p \in B$ and $i \in A$.
2. Small object argument: A is a small category and I is a small set of morphisms of presheaves, then $(l(r(I)), r(I))$ is a *functorial* weak factorization system in $\widehat{A}$. Hence in any category of presheaves, the monomorphisms form the left part of a weak factorization system.
3. Model catgeory: Locally small, finitely bicomplete, $\mathcal{C}$, equipped with Cof, Fib, and W, such that $(Cof, Fib \cap W)$ and $(Cof \cap W, Fib)$ are weak factorization systems. Also satisfies 2-out-of-3 property for commutative triangles of weak equivalences.
4. If $\mathcal{C}$ is a model category, then $\mathcal{C}^{\mathrm{op}}$, $\mathcal{C}/X$ and $X/\mathcal{C}$ are all model categories for any object X. In general, a class of weak equivalences is a class containing isomorphisms and satisfying 2-out-of-3. Fibrant means $X \to * \in Fib$, cofibrant means $\varnothing \to X \in Cof$.
5. (Ken Brown's lemma) Let $\mathcal{C}$ be a model category, $\mathcal{D}$ have weak equivalences, and $F\colon \mathcal{C} \to \mathcal{D}$ send trivial cofibrations between cofibrant objects to weak equivalences. Then it sends all weak equivalences between cofibrant objects to weak equivalences.
6. $\mathrm{ho}(\mathcal{C})$ means the localization of a category equipped with a class of morphisms W. Model categories let us give an explicit description of the localization at the weak equivalences using homotopy-like equivalence relations.

Next, we discuss the basic results regarding the homotopy category/homotopy theory of model categories. Fix a model category $\mathcal{C}$.

1. A cylinder of A is a factorization of the codiagonal into $A \amalg A \xrightarrow{(\partial_0, \partial_1)} IA \xrightarrow{\sigma} A$, where (∂_0, ∂_1) is a cofibration and σ is a weak equivalence (so $\sigma\partial_i = 1_A$). A cocylinder is a factorization of the diagonal $X \xrightarrow{s} X^I \xrightarrow{(d^0, d^1)} X \times X$ so that s is a weak equivalence and (d^0, d^1) is a fibration.
2. For $f_i\colon A \to X$, a left homotopy $f_0 \Rightarrow f_1$ is a cylinder IA equipped with a morphism $h\colon IA \to X$ such that $h\partial_i = f_i$. Right homotopy defined using cocylinders similarly.
3. Let A be cofibrant and X be fibrant. The following are equivalent for $f_0, f_1\colon$ there exists a left homotopy f_0 to f_1, or there exists a right homotopy f_0 to f_1, or for every cylinder of A there's a left homotopy using that cylinder, or for every cocylinder there's a right homotopy using that

cocylinder. Then defining $f_0 \sim f_1$ in any of these cases (for A cofibrant and X fibrant), $\sim$ is an equivalence relation.

4. We write
$$[A, X] = \mathrm{Hom}_{\mathcal{C}}(A, X) \,/\, \sim$$
for the homotopy classes of maps from a cofibrant object to a fibrant object. Writing $\mathcal{C}_c$ (resp. $\mathcal{C}_f$) for the full subcategory of cofibrant (resp. fibrant) objects, this defines a functor $[-,-]\colon \mathcal{C}_c^{\mathrm{op}} \times \mathcal{C}_f^{\mathrm{op}} \to \mathbf{Set}$. $\mathcal{C}_c$ and $\mathcal{C}_f$ are naturally equipped with classes of weak equivalences (from $W \subset \mathrm{Mor}(\mathcal{C})$). The natural inclusions $\mathcal{C}_c \to \mathcal{C}$ and $\mathcal{C}_f \to \mathcal{C}$ induces equivalences
$$\mathrm{ho}(\mathcal{C}_f) \xrightarrow{\sim} \mathrm{ho}(\mathcal{C}) \qquad \mathrm{ho}(\mathcal{C}_c) \xrightarrow{\sim} \mathrm{ho}(\mathcal{C})$$
on the homotopy categories (see Remark 2.3.1). The inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ of bifibrant (i.e. fibrant-cofibrant) objects additionally induces an equivalence of categories $\mathrm{ho}(\mathcal{C}_{cf}) \xrightarrow{\sim} \mathrm{ho}(\mathcal{C})$.

5. The functor $[-,-]$ is compatible with weak equivalences and hence descends to a functor on the homotopy categories
$$[-,-]\colon \mathrm{ho}(\mathcal{C}_c)^{\mathrm{op}} \times \mathrm{ho}(\mathcal{C}_f) \to \mathbf{Set}.$$
Moreover, $\sim$ is compatible with composition on $\mathcal{C}_{cf}$, and hence we get a quotient category $\pi(\mathcal{C}_{cf})$ by this equivalence relation. Then canonically
$$\mathrm{Hom}_{\pi(\mathcal{C}_{cf})}(A, X) = [A, X],$$
and the inclusion $\mathcal{C}_{cf} \to \mathcal{C}$ induces $\pi(\mathcal{C}_{cf}) \xrightarrow{\sim} \mathrm{ho}(\mathcal{C})$.

6. Similarly to the above,
$$[A, X] \cong \mathrm{Hom}_{\mathrm{ho}(\mathcal{C})}(A, X)$$
whenever A is cofibrant and X is fibrant.

7. If $\mathcal{C}$ is a model category, then $\mathrm{ho}(\mathcal{C})$ is locally small, and f is a weak equivalence if and only if $\gamma(f)$ is.

Remark 2.3.1 There is no proof given for [HCHA, Thm. 2.2.15], which contains the equivalences of categories in (4) above. This is easy in the case of functorial factorizations: e.g. for $\mathcal{C}_c$, the assumption gives a functor $F\colon \mathcal{C} \to \mathcal{C}_c$ equipped with a natural morphism $F(X) \to X$ that is a weak equivalence for all X. This descends to $\mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{C}_c)$, and one composite is the identity, while $F \to \mathrm{Id}$ descends to an isomorphism from the other composite to the identity.

Next, we come to derived functors on a model category $\mathcal{C}$, which are just Kan extensions. This will also involve the concept of Quillen adjunctions, which will induce a derived adjunction on the homotopy category. Often, we will be interested in model structures on the diagram categories $\mathcal{C}^I$ so we can study the derived functors of (co)limit functors $\mathcal{C}^I \to \mathcal{C}$, i.e. 'homotopy (co)limits'.

Now fix a model catgeory $\mathcal{C}$ with a localization functor $\gamma\colon \mathcal{C} \to \mathrm{ho}(\mathcal{C})$. Recall that for a category $\mathcal{D}$ we have an induced map

$$\gamma^*\colon \mathrm{Fun}\big(\mathrm{ho}(\mathcal{C}), \mathcal{D}\big) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

on functor categories. A *left Kan extension* of a functor $F\colon \mathcal{C} \to \mathcal{D}$ along γ, denoted $\mathrm{Lan}_\gamma F\colon \mathrm{ho}(\mathcal{C}) \to \mathcal{D}$, is a partial left adjoint of γ^* at F. In other words, it is equipped with a natural map inducing a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathrm{ho}(\mathcal{C}),\mathcal{D})}(\mathrm{Lan}_\gamma F, G) \cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G \circ \gamma)$$

for any $G\colon \mathrm{ho}(\mathcal{C}) \to \mathcal{D}$. The natural map is $b\colon F \to \mathrm{Lan}_\gamma F \circ \gamma$. Right Kan extensions are defined dually.

1. In the context of the model category $\mathcal{C}$ and a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *left derived functor* of F is a *right* Kan extension of F along γ, and we write

   $$\mathbf{L}F \coloneqq \mathrm{Ran}_\gamma F\colon \mathrm{ho}(\mathcal{C}) \to \mathcal{D}, \quad a\colon \mathbf{L}F \circ \gamma \to F$$

   for the functor and associated maps. Likewise, we define $\mathbf{R}F$ and $b\colon F \to \mathbf{R}F \circ \gamma$ using the left Kan extension along γ.

2. If F sends trivial cofibrations between cofibrant objects to isomorphisms, then $\mathbf{L}F$ exists and we can explicitly compute it using $\mathcal{C}_c$. Indeed, by Ken Brown, $F|_{\mathcal{C}_c}\colon \mathcal{C}_c \to \mathcal{D}$ sends weak equivalences to isomorphisms, so we get a factorization

   $$F|_{\mathcal{C}_c} = \mathcal{C}_c \to \mathrm{ho}(\mathcal{C}_c) \xrightarrow{F_c} \mathcal{D}.$$

   Let $a'_X\colon QX \to X$ be a weak equivalence from a cofibrant object for each X, which uniquely specifies an inverse Q for the equivalence $\mathrm{ho}(\mathcal{C}_c) \xrightarrow{\sim} \mathrm{ho}(\mathcal{C})$. Now define

   $$\mathbf{L}F(Y) = F_c(QY),$$

   and $a_X\colon \mathbf{L}F(\gamma(X)) = F_c(Q\gamma(X)) \to F(X)$. Then $(\mathbf{L}F, a)$ is a left derived functor. By construction, for any $G\colon \mathcal{D} \to \mathcal{E}$, $(G\mathbf{L}F, Ga)$ is a left derived functor of GF.

3. Now let $\mathcal{C}$ and $\mathcal{C}'$ be two model categories. If $F\colon \mathcal{C} \to \mathcal{C}'$ sends trivial cofibrations between cofibrant objects to weak equivalences (e.g. it preserves trivial cofibrations), Then $\gamma' F$ sends trivial cofibrations between cofibrant objects to isomorphisms, so the above constructions gives

   $$\mathbf{L}F \coloneqq \mathbf{L}(\gamma' F)\colon \mathrm{ho}(\mathcal{C}) \to \mathrm{ho}(\mathcal{C}'),$$

   the *total left derived functor of* F. Note that if $F\colon \mathcal{C} \to \mathcal{C}'$ and $F'\colon \mathcal{C}' \to \mathcal{C}''$ both preserve cofibrant objects and trivial cofibrations then $(\mathbf{L}F' \circ \mathbf{L}F) \xrightarrow{\sim} \mathbf{L}(F' \circ F)$.

4. If $\mathcal{C}, C'$ are model categories, an adjunction $F \dashv G$ between them is a *Quillen adjunction* if F preserves cofibrations and G preserves fibrations. It is easy to see that this is equivalent to requiring F to preserve cofibrations and trivial cofibrations, or G to preserve fibrations and trivial fibrations.

5. A Quillen adjunction $F \dashv G$ naturally induces an adjunction

$$\mathbf{L}F \colon \mathrm{ho}(\mathcal{C}) \leftrightharpoons \mathrm{ho}(\mathcal{C}') : \mathbf{R}G$$

of derived functors.

We next turn to homotopy limits and colimits, which are constructed by putting canonical model structures on diagram categories.

1. If $\mathcal{C}$ is a model category and I is a small category, then $\mathcal{C}^I$ is equipped with weak equivalences if we define a weak equivalence to be one which is so pointwise. We likewise declare fibrations to be those maps which are pointwise fibrations, and $Cof = l(W \cap Fib)$. If these data define a model structure on $\mathcal{C}^I$, then we say the projective model structure and exists, and call this structure the projective model structure.

2. The projective model structure exists in these cases: when I is a small discrete category; when $I = \{\bullet \to \bullet\}$; when $I = \{\bullet \leftarrow \bullet \to \bullet\}$; finally, when I is a well-ordered set with initial element $0 \in I$ (assuming $\mathcal{C}$ has small colimits).

3. If for some I the projective model structure on $\mathcal{C}^I$ exists (and so that we have I-shaped limits in $\mathcal{C}$), then the adjunction

$$\varinjlim_I \colon \mathcal{C}^I \rightleftarrows \mathcal{C} : \delta$$

is Quillen because δ by definition preserves both weak equivalences and fibrations. Hence we get a derived adjunction

$$\mathrm{Hom}_{\mathrm{ho}(\mathcal{C})}(\mathbf{L}\varinjlim_I F, X) \cong \mathrm{Hom}_{\mathrm{ho}(\mathcal{C}^I)}(F, \delta(X)).$$

This has the following property: if $f \colon X \to Y$ is a natural transformation of cofibrant diagrams in $\mathcal{C}^I$ which is a pointwise weak equivalence, then the induced map $\varinjlim_I X \to \varinjlim_I Y$ is a weak equivalence.

4. In general, if $F \colon I \to \mathcal{C}$ is a diagram (in $\mathcal{C}$), $X \in \mathrm{Ob}(\mathcal{C})$, then a cocone $F \to X$ exhibits X as the *homotopy colimit* of F if the natural map $(\mathbf{L}\varinjlim_I)F \to X$ is an isomorphism in $\mathrm{ho}(\mathcal{C})$. In general, if F is cofibrant then $\varinjlim_I F$ is already the homotopy colimit (by definition of derived functor), so in general to compute them we can just choose a cofibrant replacement.

5. A commutative square is homotopy cocartesian if, as a cocone, it exhibits the bottom right base as the homotopy pushout of the upper left hand maps. Then homotopy cocartesian squares are invariant under transposition; and if two opposite sides are weak equivalences then it is homotopy cocartesian.

6. Suppose the square

$$\begin{array}{ccc} X & \xrightarrow{x} & X' \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow{y} & Y' \end{array}$$

is cocartesian. Then the pushout of a weak equivalence along a cofibration is a weak equivalence, provided that X and X' are cofibrant objects.

7. If in the above cocartesian square, we don't necessarily assume that x is a weak equivalence, then the square is homotopy cocartesian.

2.4 Review of Cisinski model structures

3 Homotopy theory of Infinity Categories

3.1 Review of anodyne maps

This section should contain a quick review of the results in 3.1 and 3.2. The main result of this section was the existence of Ex^∞, and the fact that in the Kan-Quillen (KQ) model structure, the fibrations are the Kan fibrations, and the trivial cofibrations are the anodyne maps.

Example 3.1.1 Any monomorphism $\Delta^m \to \Delta^n$ is an anodyne extension. Indeed,

$$\begin{array}{ccc} \Delta^m & \longrightarrow & \Delta^n \\ & \searrow & \downarrow \\ & & \Delta^0 \end{array}$$

commutes so it suffices to show that $\Delta^n \to \Delta^0$ is a weak equivalence. This follows from Remark 1.1.10, because the notion of $\Delta^1 \times (-)$-homotopy applies since Δ^0 is fibrant.

Let X be a simplicial set and $f\colon x \to y$ a 1-morphism. We can freely invert f to obtain another simplicial set $X[f^{-1}]$ as follows: attach a 1-cell from y to x freely, i.e. form the pushout

$$\begin{array}{ccc} \partial\Delta^1 & \xrightarrow{(y,x)} & X \\ \downarrow & \ulcorner & \downarrow \\ \Delta^1 & \xrightarrow{g} & X[g] \end{array} \qquad \begin{array}{ccc} \partial\Delta^2 & \longrightarrow & X[g] \\ \downarrow & \ulcorner & \downarrow \\ \Delta^2 & \longrightarrow & X[gf=1] \end{array}$$

on the left, and then force g to be a left inverse for f as on the right. Thus this comes with a distinguished left inverse and a distinguished 2-cell witnessing commutativity. Now we can reverse this construction and define

$$X[f^{-1}] \coloneqq (X[gf=1])[fh=1],$$

equipped with a natural map $X \to X[f^{-1}]$.

Lemma 3.1.2 *The map $X \to X[f^{-1}]$ is an anodyne extension.*

Proof By iterating and applying the $X \mapsto X^{\mathrm{op}}$, we reduce to showing $X \to X[gf=1]$ is anodyne. It is mono because $\partial\Delta^1 \to \Delta^1$ and $\partial\Delta^2 \to \Delta^2$ are. Moreover, we can also construct $X[f^{-1}]$ by the sequence of pushouts:

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{f} & X \\ {\scriptstyle (01)}\downarrow & \ulcorner & \downarrow \\ \Delta^2 & \xrightarrow{s} & X' \end{array} \qquad \begin{array}{ccc} \Delta^1 & \xrightarrow{s\circ(02)} & X' \\ \downarrow & \ulcorner & \downarrow \\ \Delta^0 & \longrightarrow & X[gf=1] \end{array}$$

Here $s|_{(02)}$ is injective because s is free on that side, so $X' \to X[gf]$ is a weak equivalence since $\Delta^1 \to \Delta^0$ is. (01) is anodyne so $X \to X'$ is too, thus $X \to X[gf=1]$ is. □

Definition 3.1.3 Let $J = \Delta^1[(0 \to 1)^{-1}]$ where $0 \to 1$ is the nondegenerate 1-cell in Δ^1. It is equipped with the canonical maps $0, 1 \colon \Delta^0 \to \Delta^1 \to J$, and with these $(-) \times J$ becomes an exact cylinder on **sSet**. The *categorical anodyne extensions* is the smallest class of J-anodyne extensions containing $\Lambda^n_k \to \Delta^n$ for $n \geq 2$ and $0 < k < n$.

A map $J \to X$ is the data of a 1-morphism $\Delta^1 \to X$ along with a 'proof that it is invertible', namely a left and right inverse and 2-cells witnessing this as such. So, the fact that $\Delta^1 \to J$ is anodyne is another proof of the fact that all 1-morphisms in a Kan complex are invertible. Thus also a map $J \times X \to Y$ specifies an invertible 1-morphism in $\underline{\mathrm{Hom}}(X, Y)$, i.e. it is the data of a natural isomorphism between two functors $X \to Y$. Thus the notion of J-homotopy is natural isomorphisms between functors.

4 Topology

In this section, I will discuss some ideas from topology which relate to the infinity categorical ideas discussed above. Mostly the content is *not* taken from Cisinski. The goal is to try to invent model categorical ideas from basic notions in topology. We put some motivation here, and then study cofibrations and fibrations in detail afterwards.

Let $p\colon E \to B$ be a *Serre fibration*: Recall that this means p has the homotopy lifting property with respect to the disk inclusions $\partial D^n \to D^n$ for $n \geq 0$, or in other words it has right lifting against the maps

$$\partial D^n \times I \cup D^n \times 0 \to D^n \times I = (\partial D^n \to D^n) \,\square\, (0 \to I).$$

Hence we could call *Serre anodyne maps* those maps which have left lifting against all Serre fibrations, which is the weak saturation of the maps shown above. Now, consider the class C of morphisms f such that $f \,\square\, (0 \to I)$ is Serre anodyne, or equivalently those maps such that all Serre fibrations have homotopy lifting against them.

We know from our theory that this is saturated, and contains the boundary inclusions $\partial D^n \to D^n$. Hence it contains all inclusions of CW complexes. More generally, it also contains inclusions $A \to X$ where X is obtained from attaching cells to A, i.e. relative CW complex inclusions. Note also that Serre anodyne maps are generated by the inclusions $D^n \times 0 \to D^n \times I$, because there is a homeomorphism of pairs

$$(D^n \times I, D^n \times 0) \cong (D^n \times I, D^n \times 0 \cup \partial D^n \times I),$$

so we just need to check homotopy lifting against the disks $\varnothing \to D^n$ to conclude that we have homotopy lifting against all relative CW inclusions.

Now let's consider p to be a *(Hurewicz) fibration*, meaning that it has homotopy lifting against all spaces $\varnothing \to X$. In particular it is Serre, and we can consider the same setup as before, namely Hurewicz anodyne maps being generated by $\{X \times 0 \to X \times I\}$, and the class C of f with $f \,\square\, (0 \to I)$ Hurewicz anodyne.

Let's see what we can conclude about this without doing any work.

1. Let's compute $r(\{\varnothing \to X : X\})$. If a map $f\colon X \to Y$ is in this class, then it is a split epimorphism, because $\mathrm{Id}\colon Y \to Y$ admits a lift to $Y \to X$ whose composition with f is the identity. Conversely, if f has a splitting s, then every map $W \to Y$ admits a lift as $W \to Y \xrightarrow{s} X$. Thus this class is the class of split epimorphisms.

2. On the other hand, we also know that $p\colon E \to B$ is a fibration iff it has homotopy lifting against $\{\varnothing \to X\}$, i.e. iff $p^{\blacksquare(0\to I)} = E^I \to B^I \times_B E$ is in the class from (1). In particular, a map p is a fibration iff $E^I \to B^I \times_B E$ admits a section. Such a section is called a *path-lifting function*, and if we view the codomain of the fibration as pairs $(\widetilde{x}, \gamma)$ such that $p(\widetilde{x}) = \gamma(0)$, then the section gives a path $\widetilde{\gamma} = s(\widetilde{x}, \gamma)$ in E starting at $\widetilde{x}$ and lifting γ.

This is like an internal version of the defining property: we trade off having to check homotopy lifting against all spaces to just checking against I, but now we have to show it's continuous.

3. The above hints at an idea for constructing fibrations. Let $f\colon X \to Y$ be any map, and consider the corresponding map $p\colon X^I \times_Y Y \to Y$

Warning Consider an inclusion $i\colon A \hookrightarrow X$. Then $i \,\square\, (0 \to I)$ is a monomorphism $X \times 0 \cup_{A\times 0} A \times I \to X \times I$, and its image is the subspace $X \times 0 \cup A \times I$ in $X \times I$. But, this may not be a topological embedding. In the case where i is closed embedding, then $i \,\square\, (0 \to I)$ is obviously a closed embedding, because the subspace $X \times 0 \cup A \times I$ has a cover by the closed sets $X \times 0, A \times I$.

Thus, when we say that the homotopy lifting property with respect to I is equivalent to right lifting for $i \,\square\, (0 \to I)$, we really mean $i \,\square\, (0 \to I)$ and not necessarily the corresponding subspace. In concrete examples, our spaces are closed embeddings, so there's no issue identifying them.

Remark 4.0.1 By a more complicated argument given in Prop. A.18 of Hatcher, it is actually true that the homotopy extension property is satisfied if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$, as a subspace. But we will avoid this result.

4.1 Cofibrations

4.1.1 Intro

A cofibration satisfies the 'co'-homotopy lifting property with respect to all spaces.

Definition 4.1.1 A map $i\colon A \to X$ is a *cofibration* if it satisfies the homotopy extension property with respect to all spaces, i.e. for each space Y, homotopy $h\colon A \times I \to Y$ and map $\widetilde{h_0}\colon X \to Y$ with $\widetilde{h_0} i = h_0$, there is a homotopy $\widetilde{h}\colon X \times I \to Y$ with $\widetilde{h} i = h$ and starting at $\widetilde{h_0}$.

Here are the obviously equivalent conditions:

1. $i \,\square\, (0 \to I) \in l(\{Y \to * : Y\})$.
2. $i \,\square\, (0 \to I)$ is a split monomorphism, i.e. it admits a retraction.
3. $i \in l(\{ev_0\colon Y^I \to Y : Y\})$.

In particular the fibrations form a saturated class. Let us remark that closed embeddings are also a saturated class: closure under pushouts is known to us and closure under transfinite compositions follows from the definition of the weak topology. Closure under retracts is clear because if $A \subset X$ is closed and $B \to Y$ is a retract of the inclusion, then Y is a retract of X and $Y \cap A = B$. In particular B is closed in Y. It follows that closed fibrations also form a saturated class.

By the explanation earlier, if i is closed, then (2) is equivalent to the subspace $X \times 0 \cup A \times I \subset X \times I$ being a retract. In general, it is only a priori clear that (2) implies the retract property.

Example 4.1.2 Let us show that the inclusion $\partial I \to I$ is a cofibration, for which we simply need to give a retraction $I^2 \to \partial I \times I \cup I \times 0$. This is easy: use the point $(1/2, 2)$ lying above I^2 to project any point onto one of the three sides excluding the top edge. Note therefore that for any X the map

$$X \times \partial I \times I \cup X \times I \times 0 \to X \times I \times I = (X \times \partial I \to X \times I) \,\square\, (0 \to I)$$

admits a retraction, as justified by the fact that $X \times \partial I \subset X \times I$ is closed. Hence $X \times \partial I \to X \times I$ is a cofibration. In general the product of a closed fibration with a space will be a closed fibration.

Remark 4.1.3 This is in some sense the most important example, and we will derive a lot from it. It says that if we have two homotopies $f_t, g_t \colon X \to Y$ and a homotopy $h_t \colon X \to Y$ from f_0 to g_0, then this extends to a homotopy from f_t to g_t continuous in t.

The most important tool for dealing with fibrations are mapping cylinders. Given a map $f \colon X \to Y$, its *mapping cylinder* M_f is the pushout of $X \times 1 \to X \times I$ along f. Since $X \times 1 \subset X \times I$ is closed, $Y \subset M_f$ is canonically a closed subspace. Since $X \times 0 \subset X \times I \sqcup Y$ is closed and saturated with respect to the quotient map, we identify $X = X \times 0 \subset M_f$ as a closed subspace as well.

Induced by the map $f \times I \colon X \times I \to Y$ and $\mathrm{Id} \colon Y \to Y$ we get a canonical retraction of the mapping cylinder M_f onto Y. By definition, $X \hookrightarrow M_f \xrightarrow{r} Y$ is the identity.

Lemma 4.1.4 *Let $f \colon X \to Y$ be a map.*

1. *The canonical retraction $r \colon M_f \to Y$ is a strong deformation retact.*
2. *The canonical inclusion $i \colon X \hookrightarrow M_f$ is a (closed) cofibration.*

Proof (1) We'll use the fact that $(-) \times I$ preserves colimits. Define $r_t \colon M_f \times I \to M_f$ as follows: on $Y \times I$ it is the projection to Y, and on $X \times I \times I$ we define

$$r_t([x, s]) = [x, s - st + t].$$

On $X \times I$, $r_t([x, 1]) = [x, 1] = f(x)$ so this is well-defined and continuous. Now, $r_0([x, s]) = [x, s]$, $r_1([x, s]) = [x, 1] = f(x)$, and $r_t(y) = y$ for $y \in Y$.

(2) Consider the commutative diagram

$$\begin{array}{ccccc}
\varnothing & \longrightarrow & X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & X \sqcup X & \xrightarrow{1 \sqcup f} & X \sqcup Y \\
 & & {\scriptstyle (i_0, i_0)}\downarrow & & \downarrow{\scriptstyle \bar{i}} \\
 & & X \times I & \longrightarrow & M_f
\end{array}$$

The top left square and outer perimeter of the upper square are pushouts, so the top right square is, and the outer perimeter of the right-hand rectangle is a pushout, so the bottom square is. $(i_0, i_1) \cong X \times \partial I \to X \times I$ is a cofibration by Example 4.1.2, hence $\bar{i}$ is too.

$X \to X \sqcup Y$ is a cofibration for trivial reasons: a homotopy $h_t \colon X \to Z$ alongside a lift $(a, b) \colon X \sqcup Y \to Z$ of h_0 induce $(h_t, b) \colon X \sqcup Y \to Z$. Thus the composite $X \to X \sqcup Y \to M_f$, the canonical inclusion, is a cofibration. □

Remark 4.1.5 The mapping cylinder is obviously functorial: given a morphism (a, b) of arrows $f \to g$, the canonical map $X \times I \sqcup Y \to W \times I \sqcup Z$ induces a map $(a, b)_* \colon M_f \to M_g$, which strictly respects compositions and identities. It is also clear from their definition that both the canonical inclusion i and the canonical retraction r is natural with respect to such maps. In other words, they are natural transformations

$$\mathrm{Pr}_1 \xrightarrow{i} M_{(-)} \xrightarrow{r} \mathrm{Pr}_2$$

for the projection functors $\mathrm{Pr}_1, \mathrm{Pr}_2 \colon \mathrm{Arr}(\mathbf{Top}) \to \mathbf{Top}$. Moreover, this composite gives (on arrows) the identity, and this setup is a *functorial factorization*.

A goal of this section is to establish some results which show that cofibrations have especially nice properties. The main takeway of Lemma 4.1.4 is that every map is, up to homotopy equivalence, a cofibration.

Given a space X, the mapping cylinder of the map $X \to *$ is just the cone CX, with $X \subset CX$ embedded at 0. Hence CX is contractible, and the canonical deformation retract pushes the whole space onto the vertex of CX.

Corollary 4.1.6 *A CW pair inclusion $A \to X$ is a (closed) cofibration.*

Proof By Lemma 4.1.4 and a homeomorphism $(CS^{n-1}, S^{n-1}) \cong (D^n, \partial D^n)$, the inclusions $\partial D^n \to D^n$ are cofibrations. Since cofibrations form a saturated class we are done. □

Proposition 4.1.7 *Every cofibration is a topological embedding.*

Proof If $i \colon A \to X$ is a cofibration, then we can define a homotopy to the mapping cylinder

$$g \colon A \times I \to M_f \qquad (a, t) \mapsto [a, 1 - t],$$

well-defined since it factors through $\bar{i} \colon A \times I \to M_i$. This starts at the map $a \mapsto [a, 1] = i(a)$, hence the inclusion $X \to M_f$ provides an initial lift, so we get an extension $\widetilde{g} \colon X \times I \to M_f$ such that $\widetilde{g}_t(i(a)) = [a, 1 - t]$ and $\widetilde{g}_0$ is the inclusion. In particular, $\widetilde{g}_1(i(a)) = a$, so $\widetilde{g}_1|_{i(A)} \colon i(A) \to A$ is a continuous and two-sided inverse of the map $i \colon A \to i(A)$. □

This section is trying to make the case that a non-cofibrant closed embedding is somewhat pathological (though we will give an example of this), and accordingly that cofibrant closed embeddings have good properties. The next is an intuitive such property:

Proposition 4.1.8 *Let $A \hookrightarrow X$ be a cofibration with A contractible. Then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

Proof Lift a contraction $A \times I \to A \to X$ to a homotopy $h_t\colon X \to X$, such that $h_t(A) \subset A, h_0 = 1_X$, and $h_1(A) = a_0$. Hence $h_1 = \widetilde{h}_1 q$ for some $\widetilde{h}_1\colon X/A \to X$, and write $\overline{h_t}\colon X/A \to X/A$ for the homotopy on quotients, continuous since $(-) \times I$ preserves pushouts. Then

$$X/A \xrightarrow{\widetilde{h}_1} X \xrightarrow{q} X/A$$

is the map $h_1 \sim h_0 = 1_X$. Likewise,

$$X \xrightarrow{q} X/A \xrightarrow{\widetilde{h}_1} X$$

is the map $\overline{h_1} \sim \overline{h_0} = 1_{X/A}$. □

Example 4.1.9 We can use mapping cylinders to show uniqueness of CW approximations. Indeed, if $w_i\colon K_i \to X$ for $i = 1, 2$ are two CW approximations, then the inclusion $K_2 \to M_{w_2}$ is a weak equivalence because $K_w \to M_{w_2} \to X$ is and $M_{w_2} \to X$ is. Hence (M_{w_2}, K_2) is n-connected for all $n \geq 0$, so the map $K_1 \to X \to M_{w_1}$ is homotopic to a map $K_1 \to K_2 \to M_{w_2}$. Again, the composite is a weak equivalence and the second map is too, so the first map is a homotopy equivalence.

4.1.2 Homotopy cofibers

Let (X, A) be a pair. If $p = [A \times 1] \in CA$ is the tip of the cone over A, then p is a closed point contained in the interior of $CA \subset X \cup_A CA$, so the natural map

$$H_k(X \cup_A CA - p, CA - p) \to H_k(X \cup_A CA, CA)$$

is an isomorphism. But $CA - p = A \times [0, 1)$, and $X \cup_A CA - p = X \cup_A (A \times [0, 1))$, so the canonical deformation retraction $A \times [0, 1) \to A$ extends to a deformation retraction

$$X \cup_A (A \times [0, 1)) \to X \cup_A A = X,$$

and in particular the inclusion $(X, A) \to (X \cup_A CA - p, CA - p)$ is a homotopy equivalence of pairs. Hence $(X, A) \hookrightarrow (X \cup_A CA, CA)$ induces an isomorphism on homology.

Now suppose the inclusion $i\colon A \hookrightarrow X$ is a cofibration, then taking the pushout of the inclusion $A \to CA$ along i, the inclusion $CA \hookrightarrow X \cup_A CA$ is a cofibration. By the pasting lemma, we have the canonical identification $(X \cup_A CA)/CA =$

X/A, and thus the quotient map $X \cup_A CA \to X/A$ is a homotopy equivalence. We have a commutative diagram

$$\begin{array}{ccc} (X, A) & \longrightarrow & (X \cup_A CA, CA) \\ \downarrow & & \downarrow \\ (X/A, A/A) & = & (X/A, A/A) \end{array}$$

of pairs. The right hand map induces an isomorphism on relative homology because $X \cup_A CA \to X/A$ does and CA is contractible (by naturality of the LES), so the left hand quotient map is an isomorphism on relative homology. We have shown:

Proposition 4.1.10 *If (X, A) is a pair and $A \hookrightarrow X$ is a cofibration, then the map*

$$H_n(X, A) \to H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$$

induced by the quotient map is an isomorphism.

One easy application: recall the definition of the (unbased) suspension $\Sigma X \coloneqq CX/X$, which is functorial because the mapping cylinder, and hence the cone, are. By mapping cylinders, $X \to CX$ is a cofibration, hence the canonical map $H_n(CX, X) \xrightarrow{\sim} \widetilde{H}_n(\Sigma X)$ is an isomorphism. Note that the boundary map $H_n(CX, X) \to \widetilde{H}_{n-1}(X)$ is an isomorphism for all n, giving the *suspension isomorphism* $\widetilde{H}_n(X) \xrightarrow{\sim} \widetilde{H}_{n+1}(\Sigma X)$ on homology. Since the diagram

$$\begin{array}{ccccc} \widetilde{H}_n(X) & \xleftarrow{\cong} & H_{n+1}(CX, X) & \xrightarrow{\cong} & \widetilde{H}_{n+1}(\Sigma X) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle Cf_*} & & \downarrow{\scriptstyle \Sigma f_*} \\ \widetilde{H}_n(Y) & \xleftarrow{\cong} & H_{n+1}(CY, Y) & \xrightarrow{\cong} & \widetilde{H}_{n+1}(\Sigma Y) \end{array}$$

commutes for a map $f\colon X \to Y$, the suspension isomorphism is natural. We can use this to give a trivial proof of $\widetilde{H}_n(S^n) = \mathbb{Z}[-n]$ along with explicit generating cocycles.

So in general, for a cofibration $i\colon A \to X$ there is a long exact sequence

$$\cdots \to \widetilde{H}_n(A) \to \widetilde{H}_n(X) \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1}(A) \to \cdots$$

in homology. We are now going to generalize this sequence significantly. Even better, we will find a way to realize this sequence geometrically. First, fix a map $f\colon X \to Y$. We define the *homotopy cofiber* of this map as the cone C_f, which is by definition

$$C_f \coloneqq M_f/A.$$

The sequence $X \xrightarrow{f} Y \to C_f$ is called the *cofiber sequence* associated to f, where the second map is $Y \hookrightarrow M_f \twoheadrightarrow C_f$. There is another useful definition of the cofiber sequence as follows: let C^0X be opposite sided cone, i.e. $(X \times I)/(X \times 0)$, which is canonically homeomorphic to CX by passing $X \times I \xrightarrow{\sim}$

$X \times I$ given by $(x,t) \mapsto (x, 1-t)$ to quotients. Then there is a commutative diagram:

$$\begin{array}{ccccc} & & X \times 1 & \longrightarrow & Y \\ & & \downarrow & \ulcorner & \downarrow \\ X \times 0 & \longrightarrow & X \times I & \longrightarrow & M_f \\ \downarrow & \ulcorner & \downarrow & & \downarrow \\ * & \longrightarrow & C^0 X & \longrightarrow & C_f \end{array}$$

Since the left square and bottom perimeter are pushouts, the bottom right square is, and since the top square is a pushout, the right-hand perimeter is. Thus $C_f = C^0 X \cup_X Y$, or alternatively $Y \cup_f CX$.

We have already seen that since $i \colon X \to M_f$ is a cofibration, the quotient map

$$C_i = M_f \cup_X CX \to C_i / CX = M_f / X = C_f$$

is a homotopy equivalence by contractibility of CX, so the cofiber sequence for f is homotopy equivalent to the one in the diagram:

$$\begin{array}{ccccc} X & \hookrightarrow & M_f & \longrightarrow & C_i \\ \| & & \simeq\downarrow & & \simeq\downarrow \\ X & \xrightarrow{f} & Y & \longrightarrow & C_f \end{array}$$

Moreover, the composite $M_f \to C_i \to C_f$ is the quotient map by X, so we get a commutative diagram/long exact sequence

$$\begin{array}{ccccccc} \widetilde{H}_n(X) & \xrightarrow{i_*} & \widetilde{H_n}(M_f) & \longrightarrow & H_n(M_f, X) & \xrightarrow{\partial} & \widetilde{H}_{n-1}(X) \\ \| & & \cong\downarrow & & \cong\downarrow & & \| \\ \widetilde{H}_n(X) & \dashrightarrow^{f_*} & \widetilde{H}_n(Y) & \dashrightarrow & \widetilde{H}_n(C_f) & \dashrightarrow^{\partial} & \widetilde{H}_{n-1}(X) \end{array}$$

as in the bottom row, associated to *any map* f. In particular, C_f is acyclic (has no reduced homology) if and only if f induces an isomorphism on all homology. Now let us find a geometric interpretation of the boundary map $\widetilde{H}_n(C_f) \to \widetilde{H}_{n-1}(X)$.

Observe that since $X \to CX$ is a closed embedding, $Y \to C_f$ is also a closed embedding. By the pasting lemma, $C_f / Y = CX/X = \Sigma X$, so there is a natural quotient map $C_f \to \Sigma X$.

Lemma 4.1.11 *The canonical quotient map $C_f \to \Sigma X$ induces the boundary map of the long exact sequence in homology associated to $X \to Y \to C_f$, under the identification $\widetilde{H}_{n-1}(X) \cong \widetilde{H}_n(\Sigma X)$.*

Proof The pushout square identifying C_f as the pushout $C^0 X_X Y$ shows that the quotient map $M_f \to C_f$ commutes with the embeddings of Y, i.e. it is a map of pairs $(M_f, Y) \to (C_f, Y)$. By the pasting lemma, M_f / Y is canonically

CX, so we have a commutative diagram:

$$\begin{array}{ccc} M_f & \longrightarrow & CX \\ \downarrow & & \downarrow \\ C_f & \longrightarrow & \Sigma X \end{array}$$

The top preserves the embeddings of X, and hence this extends to a commutative diagram of pairs of the form:

$$\begin{array}{ccc} (M_f, X) & \longrightarrow & (CX, X) \\ \downarrow & & \downarrow \\ (C_f, X/X) & \longrightarrow & (\Sigma X, X/X) \end{array}$$

It follows that the diagram

$$\begin{array}{ccccc} & & \xrightarrow{\partial} & & \widetilde{H}_{n-1}(X) \\ & & & \overset{\cong}{\nearrow} & \\ H_n(M_f, X) & \longrightarrow & H_n(CX, X) & & \\ \overset{\cong}{\downarrow} & & \downarrow & \overset{\Sigma}{\searrow} & \\ \widetilde{H}_n(C_f) & \longrightarrow & \widetilde{H}_n(\Sigma X) & & \end{array}$$

in homology commutes. The top arrow comes from naturality in the LES for the pair $(M_f, X) \to (CX, X)$, so going up the left side and to $\widetilde{H}_{n-1}(X)$ is the boundary map in the homology cofiber sequence, giving the result. □

The lemma shows the following: the sequence of maps

$$X \xrightarrow{f} Y \to C_f \to \Sigma X$$

induce maps

$$\widetilde{H}_n(X) \to \widetilde{H}_n(Y) \to \widetilde{H}_n(C_f) \to \widetilde{H}_n(\Sigma X) \xrightarrow{\sim} \widetilde{H}_{n-1}(X)$$

which fits into the canonical long exact sequence. Moreover, we can iterate this construction to produce the *Puppe sequence*,

$$X \to Y \to C_f \to \Sigma X \to \Sigma Y \to \Sigma C_f \to \Sigma^2 X \to \cdots,$$

where each consecutive four terms are obtained by applying the suspension functor to the previous four terms. Applying $\widetilde{H}_0$ to this, and using the suspension isomorphisms $\widetilde{H}_0(\Sigma^n Z) \cong \widetilde{H}_n(Z)$, this gives a geometric presentation of the long exact sequence.

As a final result in this section, let us show that the Puppe sequence gives rise to another long exact sequence of pointed sets. First of all, note that the composite $X \to Y \to C_f$ is null homotopic, because up to homotopy equivalence it is $X \to M_f \to C_f$ where the sequence is the quotient map to M_f/X. Hence the composite

$$[C_f, Z] \to [Y, Z] \to [X, Z] \qquad (*)$$

is null (as pointed sets), for any space Z. Moreover, if $\alpha\colon Y \to Z$ is so that αf is nullhomotopic, then αf extends to $CX \to Z$ and thus this extends to a map $C_f \to Y \cup_X CX \to Z$ whose restriction to Y is α. Thus $(*)$ is an exact sequence.

Next we are going to show that in fact every three term subsequence in the Puppe sequence is homotopy equivalent to a cofiber sequence. The key lemma is the following:

Lemma 4.1.12 *If $i\colon A \to X$ is a cofibration, then there is a canonical quotient map $C_i \to X/A$ which is a homotopy equivalence.*

Proof We have already seen this: in this case the pushout $CA \to C_i$ of i is a cofibration, and thus $C_i \to C_i/CA \cong X/A$ is a homotopy equivalence by contractibility. □

Now let $f\colon X \to Y$. $g\colon Y \to C_f$ is the pushout of the cofibration $X \to CX$ along f, hence is a cofibration. Moreover canonically $C_f/Y = \Sigma X$, hence by the lemma the canonical quotient map $C_g \to \Sigma X$ which is a homotopy equivalence. It is clear that with this map that the diagram

$$\begin{array}{ccccccc} Y & \xrightarrow{g} & C_f & \longrightarrow & C_g & \longrightarrow & \Sigma Y \\ \| & & \| & & \simeq\downarrow & & \| \\ Y & \xrightarrow{g} & C_f & \longrightarrow & \Sigma X & \longrightarrow & \Sigma Y \end{array}$$

commutes since the map is the canonical quotient. Lastly, denoting by h the map $C_f \to C_g$ above, apply this previous result to the upper cofiber sequence and we get

$$\begin{array}{ccccccccc} & & C_f & \longrightarrow & C_g & \longrightarrow & C_h & \longrightarrow & \Sigma C_f \\ & & \| & & \| & & \simeq\downarrow & & \downarrow \\ Y & \xrightarrow{g} & C_f & \xrightarrow{h} & C_g & \longrightarrow & \Sigma Y & \longrightarrow & \Sigma C_f \\ \| & & \| & & \simeq\downarrow & & \| & & \\ Y & \xrightarrow{g} & C_f & \longrightarrow & \Sigma X & \longrightarrow & \Sigma Y & & \end{array}$$

All of this shows the following:

Proposition 4.1.13 *Let $f\colon X \to Y$ be a map.*

1. *The Puppe sequence for f is homotopy equivalent to a sequence of iterated mapping cones of the form*

$$X \xrightarrow{f} Y \xrightarrow{g_1} C_f \xrightarrow{g_2} C_{g_1} \xrightarrow{g_3} C_{g_2} \to \cdots$$

in which g_i is a cofibration for all $i \geq 1$, and (writing $g_0 = f$) g_{i+1} is the standard map to the cofiber of g_i for all i.

2. *Any four consecutive terms in the Puppe sequence are homotopy equivalent to a standard cofiber triangle of the form* $A \xrightarrow{g} B \to C_g \to \Sigma A$ *for some* g.
3. *For each space* Z*, the Puppe sequence induces an exact sequence*

$$\cdots \to [\Sigma Y, Z] \to [\Sigma X, Z] \to [C_f, Z] \to [Y, Z] \to [X, Z]$$

of pointed sets.

Proof We showed (1) for the initial terms of the Puppe sequence which is a cofiber sequence, but then four term sequences afterwards are just the standard cofiber sequences of the suspensions of f. Similarly, we showed (2) in the initial case so it extends to the higher suspensions.

Finally, we showed that for a cofiber sequence $X \to Y \to C_f$, the induced sequence on homotopy sets is exact. But if all three term subsequences of the Puppe sequence are homotopy equivalent to a standard cofiber sequence, then the sequence in (3) is exact at all points. □

In light of the last fact, we call a sequence of spaces with this property a 'coexact sequence', and the Puppe sequence is the coexact sequence associated to the map f. We note that this is very related to cohomology, in that taking Z to be an Eilenberg-MacLane space we obtain a similar long exact sequence in cohomology to our one in homology.

Until now we have studied cofibrations and homotopy cofibers in a pedestrian way; in the next section we will understand this in a more general context.

4.1.3 Homotopy pushouts: Setup

In this section we are going to realize all of the constructions in the previous sections as specific 'homotopy colimits', and from this deduce some homotopy invariance properties for free. The initial observation is the fact that pushouts *do not* behave well with respect to homotopy.

For example, consider the following two diagrams

$$\begin{array}{ccc} \partial D^n & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ D^n & \longrightarrow & S^{n+1} \end{array} \qquad\qquad \begin{array}{ccc} \partial D^n & \longrightarrow & * \\ {\scriptstyle *}\downarrow & \ulcorner & \downarrow \\ D^n & \longrightarrow & * \end{array}$$

Here, on the right, $\partial D^n \to D^n$ is a constant map. Then both diagams are pushouts, and the inclusion of the boundary of D^n is homotopic to the constant map, however the two answers we get, S^{n+1} and $*$, are *not* homotopy equivalent.

To solve this problem, we want to come up with a construction of a homotopy pushout, in a way that it *is* homotopy invariant in any relevant sense. Unfortunately, the answer cannot be to take pushouts the usual homotopy category

hTop of spaces. Many pushouts don't actually exist, and those that do are often trivial (like in the case above).

Example 4.1.14 Consider the pushout of $* \leftarrow S^1 \rightarrow S^1$ in **hTop** for a degree 2 map $S^1 \rightarrow S^1$, and suppose it exists as some space P. Then for a path connected spaces X, so that $[*, X] = *$, we have

$$[P, X] = \{[\gamma] \in [S^1, X] : [z \mapsto \gamma(z^2)] = *\}.$$

Note that if X furthermore has abelian π_1, then the canonical surjection $[S^1, X] \rightarrow \pi_1(X)$ is a bijection, so $[P, X]$ is the set of two-torsion elements in $\pi_1(X)$. Taking $X = K(G, n)$ for $n \geq 1$ and G abelian, we see $H^n(P; G)$ is in bijection with the two-torsion elements of G.

Therefore $H^1(P; \mathbb{Z}) = H^2(P; \mathbb{Z}) = 0$ and $H^1(P; \mathbb{Z}/2) = \mathbb{Z}/2$. But the long exact coefficient sequence associated to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0$ includes the portion

$$\begin{array}{ccccc} H^1(X; \mathbb{Z}) & \longrightarrow & H^1(X; \mathbb{Z}/2) & \longrightarrow & H^2(X; \mathbb{Z}) \\ \| & & \| & & \| \\ 0 & & \mathbb{Z}/2 & & 0 \end{array}$$

which is a contradiction. *Note: Here we heavily used that P has the homotopy type of a CW complex, so in particular this either only works for* $\mathbf{Top}[\text{whe}^{-1}]$ *or in* **hTop** *it shows that the pushout cannot be a CW complex. I know how to extend this argument using the homotopy fiber/Puppe sequence associated to $K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2)$ to show that P is not well-pointed for any choice of basepoint. But, currently, I don't have a proof in the general case.*

The moral of the story is that the correct homotopy pushout should have a universal property which remembers the data of homotopies. For example, if we have $f : A \rightarrow X$, consider the 'homotopy quotient': in spaces, it would have the universal property that maps $Q \rightarrow Y$ correspond to maps $X \rightarrow Y$ such that $A \rightarrow X \rightarrow Y$ is constant.

The starting result for how to work with homotopy colimits is that pushing out along cofibrations is homotopy invariant, in a sense we will make precise shortly. On the other hand, we have showed how to functorially replace any map of topological spaces with a 'homotopy equivalent' closed fibrations. This will let us define homotopy pushouts with the correct homotopy universal property. We need a lemma.

Lemma 4.1.15 *Let $i : A \rightarrow X$ be a cofibration. Then:*

1. *If i is closed, then $i \times Y : A \times Y \rightarrow X \times Y$ is a closed cofibration for any space Y.*
2. *If Y is LCH, then $i \times Y : A \times Y \rightarrow X \times Y$ is a cofibration.*
3. *In fact, for any space Y, $i \times Y$ is a cofibration, which is proved using the more complicated retract criteria that we haven't discussed.*

Proof (1) Example 4.1.2. For (3), we take the product of the retraction corresponding to (X, A) and multiply by Y to obtain a retraction corresponding to $(X \times Y, A \times Y)$. For (2), we have

$$\begin{aligned}(i \times Y) \square (0 \to I) &\cong i \square (\varnothing \to Y) \square (0 \to I) \\ &\cong i \square (0 \to I) \square (\varnothing \to Y) \\ &\cong (i \square (0 \to I)) \times Y.\end{aligned}$$

But if f is a split monomorphism, then so is $f \times Y$. Here we can apply associativity because $\varnothing, Y, 0, I$ are all exponentiable and Remark 1.1.3. □

Lemma 4.1.16 *Let $i\colon A \to X$ be a cofibration. Then*

$$i \square (\partial D^n \hookrightarrow D^n)\colon A \times D^n \cup_{A \times \partial D^n} X \times \partial D^n \to X \times D^n$$

is a cofibration.

Proof We want to check the lifting property with respect to

$$\big(i \square (\partial D^n \to D^n)\big) \square (0 \to I) \cong i \square (\partial D^n \times I \cup D^n \times 0 \hookrightarrow D^n \times I),$$

and the latter map is isomorphic to

$$\begin{aligned}i \square (D^n \times 0 \to D^n \times I) &= i \square \big((\varnothing \to D^n) \square (0 \to I)\big) \\ &= (i \square (\varnothing \to D^n)) \square (0 \to I),\end{aligned}$$

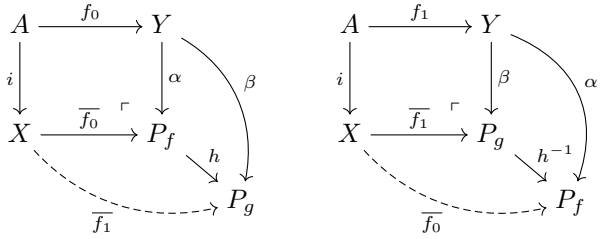
so we reduce to proving that

$$i \square (\varnothing \to D^n) = i \times D^n\colon A \times D^n \to X \times D^n$$

is a cofibration, which is by Lemma 4.1.15(b). □

Now we come to the main result of this section.

Theorem 4.1.17 *Let $i\colon A \to X$ be a cofibration and $f\colon A \times I \to Y$ a homotopy. Suppose we have two pushout squares*



as in the diagram above. Then there are inverse homotopy equivalences $h\colon P_f \to P_g$ and $h^{-1}\colon P_g \to P_f$ such that:

1. *The two right-hand triangles strictly commute, i.e. $h\alpha = \beta$ and $h^{-1}\beta = \alpha$.*

2. *The two left-hand triangles commute up to homotopy, i.e.* $\overline{f_1} \sim h\overline{f_0}$ *and* $\overline{f_0} \sim h^{-1}\overline{f_1}$*, and moreover the two induced homotopies*

$$\overline{f}_1 i = \beta f_1 \sim h\overline{f_0} i = \beta f_0 \qquad \overline{f}_0 i = \alpha f_0 \sim h^{-1}\overline{f_1} i = \alpha f_1$$

by precomposition with i *are given by* $\beta\widetilde{f}$ *and* αf*, where* $\widetilde{f}(-,t) = f(-,1-t)$ *is the reverse homotopy.*

3. *The homotopy equivalences* h *and* h^{-1} *are rel* Y*, in the sense that there are homotopies*

$$H : h^{-1}h \simeq 1_{P_f} \qquad H' \colon hh^{-1} \simeq 1_{P_g}$$

such that $H_t \circ \alpha = \alpha$ *and* $H'_t \circ \beta = \beta$*.*

Proof Though the statement is annoyingly long, for most of the proof we just do the most obvious thing and record the resulting data.

1. $\widetilde{f} \colon f_1 \simeq f_0$, so $\beta\widetilde{f} \colon \beta f_1 = \overline{f_1} i \simeq \beta f_0$, and thus this extends to a homotopy $F \colon \overline{f_1} \simeq F_1$, so $F_1 i = \beta f_0$. Likewise, the homotopy $\alpha f \colon \alpha f_0 = \overline{f_0} i \simeq \alpha f_1$ extends to a homotopy $G \colon \overline{f_0} \simeq G_1$, so $G_1 i = \alpha f_1$. Therefore we get induced maps

$$h \coloneqq \langle F_1, \beta \rangle : P_f \to P_g \qquad h^{-1} \coloneqq \langle G_1, \alpha \rangle : P_g \to P_f,$$

so $h\alpha = \beta$, $h^{-1}\beta = \alpha$, as in Statement 1. Also, $h\overline{f_0} = F_1$ and $h^{-1}\overline{f_1} = G_1$, so F and G provide the homotopies of Statement 2, with the statement about induced homotopies being clear by definition of homotopy extension.

Claim There are homotopies $L \colon \overline{f_0} \simeq h^{-1}h\overline{f_0}$ and $M \colon \overline{f_1} \simeq hh^{-1}\overline{f_1}$ such that $L|_{A\times I}$ is the constant homotopy at αf_0 and $M|_{A\times I}$ is the constant homotopy at βf_1.

2. Assume the claim holds: take the product of the defining pushout square of P_f with I, then the map $L \colon X \times I \to P_f$ along with the constant homotopy $Y \times I \to P_f$ at α agree upon restriction to $A \times I$, and hence define a homotopy $H \colon P_f \times I \to P_f$. By uniqueness, $H_t = \langle L_t, \alpha \rangle$, so $H_0 = 1_{P_f}$ and $H_1 = h^{-1}h$. Moreover, $\alpha H_t = \alpha$ as in Statement 3. Of course, the same thing can be done for M, so we're done.

3. *Proof of Claim:* Note that we have the composite homotopy $E \coloneqq G * h^{-1} f$, namely

$$E = \overline{f_0} \simeq F_1 = h^{-1}\overline{f_1} \simeq h^{-1}G_1 = h^{-1}h\overline{f_0}.$$

The restriction to $A \times I$ is

$$E \circ (i \times I) = (\alpha f) * (h^{-1}\beta\widetilde{f}) = \alpha \circ (f * \widetilde{f}).$$

By definition of the composite homotopy, we can write $f * \widetilde{f} = f \circ (1_A \times \gamma)$ for $[\gamma] \in \pi_1(I, 0) = *$, so there's a homotopy $1_A \times \gamma \sim 1_A \times 0$ rel $A \times \partial I$, and thus a homotopy

$$S \colon E \circ (i \times I) = \alpha \circ (f * \widetilde{f}) \simeq \mathrm{const}_{\alpha f} \quad \text{rel } A \times \partial I.$$

To conclude, we will apply Lemma 4.1.16 which shows that

$$(A \times I) \underset{A\times\partial I}{\times} (X \times \partial I) \to X \times I$$

is a cofibration. For each $t \in I$, we have $S_t \colon A \times I \to P_f$ and the map

$$\langle \overline{f_0}, h^{-1}h\overline{f_0} \rangle = X \times \partial I = X \sqcup X \to P_f,$$

which agree on $A \times \partial I$ because S was rel $A \times \partial I$, and since

$$\overline{f_0} i = \alpha f, \quad h^{-1}h\overline{f_0} i = h^{-1}h\alpha f = h^{-1}\beta f = \alpha f.$$

Thus for each t this glues into a map

$$S'_t \colon (A \times I) \underset{A\times\partial I}{\times} (X \times \partial I) \to P_f,$$

and since $(-) \times I$ preserves pushouts this defines a homotopy. Note that $E \colon X \times I \to P_f$ is an initial lift by definition of S. Then the cofibration property gives a homotopy which ends at a map $T_1 \colon X \times I \to P_f$, satisfying

$$T_1 \circ (i \times \underline{I}) = \mathrm{const}_{\alpha f_0}$$

$T_1|_{X\times 0} = \overline{f_0}$, and $T_1|_{X\times 1} = h^{-1}h\overline{f_0}$, so taking $L = T_1$ proves the claim in this first case, and the other case is identical.

□

Corollary 4.1.18 *If a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle \bar{i}} \\ X & \xrightarrow[\overline{f}]{} & Y \end{array}$$

is cocartesian with i a cofibration and f a homotopy equivalence, then $\overline{f}$ is a homotopy equivalence (and of course $\bar{i}$ is a cofibration).

Proof Let $g \colon B \to A$ be a homotopy inverse for f, and consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & A \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow{\scriptstyle \bar{i}} & \ulcorner & \downarrow{\scriptstyle j} \\ X & \xrightarrow[\overline{f}]{} & Y & \xrightarrow[\overline{g}]{} & P \end{array}$$

where P is the pushout of g along $\bar{i}$. Then the composite square is a pushout; since $gf \sim 1_A$, there's a homotopy equivalence $h \colon P \to X$ satisfying $h\overline{g}\overline{f} \sim 1_X$. Thus it's a split mono in **hTop**, and $h\overline{g}$ (and thus $\overline{g}$) is a split epi. Since $\bar{i}$ is a cofibration, applying the same argument to the right-hand pushout square, we obtain $h'\overline{f}\overline{g} \sim 1_Y$ so $\overline{f}$ is a split epi, hence an isomorphism. □

One immediate application of these results is to (relative) CW complexes, which are built by pushouts along cofibrations. The most basic form is this:

let Y be a space and consider some cell attaching maps $\varphi_\alpha \colon \partial D^n \to Y$. Let Z be the result of attaching the n-cells, so that we have a pushout diagram

$$\begin{array}{ccc} \bigsqcup_\alpha \partial D^n_\alpha & \xrightarrow{(\varphi_\alpha)_\alpha} & Y \\ \big\downarrow & & \big\downarrow \\ \bigsqcup_\alpha D^n_\alpha & \xrightarrow{(\Phi_\alpha)_\alpha^\ulcorner} & Z \end{array}$$

Our results immediately give us the following information:

1. If $\varphi_\alpha \sim \varphi'_\alpha$ for each α, and Z' is the result of attaching n-cells to Y along the maps φ'_α, then $Z \simeq Z'$ rel Y.
2. Let $h \colon Z \simeq Z'$ be the homotopy rel Y. Statement 2 of the main theorem says not only that $h\Phi_\alpha \sim \Phi'_\alpha$, but also that the restriction to the boundaries are induced from the homotopies of the attaching maps, so in particular
$$h\Phi_\alpha \sim \Phi'_\alpha \colon (D^n_\alpha, \partial D^n_\alpha) \to (Z, Y),$$
i.e. they are homotopic through maps of pairs.
3. Since h is a homotopy equivalence rel A, it induces a homotopy equivalence $\bar{h} \colon Z/Y \simeq Z'/Y$. The pasting lemma also gives canonical identifications
$$\bigvee_\alpha D^n_\alpha/\partial D^n_\alpha = Z/Y \qquad \bigvee_\alpha D^n_\alpha/\partial D^n_\alpha = Z'/Y,$$
and under this identification a consequence of (2) is that $\overline{h}$ is homotopic to the identity map.
4. Since $Y \to Z$ and $Y \to Z'$ are cofibrations, the above gives canonical identifications
$$\bigoplus_\alpha \mathbb{Z} e^n_\alpha \cong \bigoplus_\alpha H_n(D^n_\alpha, \partial D^n_\alpha) = H_n(Z, Y),$$
and likewise for Z', where the middle equality is induced by the maps on homology induced by the characteristic maps Φ_α. On the other hand, h is a homotopy equivalence of pairs $(Z, Y) \simeq (Z', Y)$ and hence induces an isomorphism on relative homology. (2) says that the diagram
$$\begin{array}{ccc} \bigoplus_\alpha H_n(D^n_\alpha, \partial D^n_\alpha) & \xrightarrow{\cong} & H_n(Z, Y) \\ \| & & \big\downarrow{\scriptstyle \cong} h_* \\ \bigoplus_\alpha H_n(D^n_\alpha, \partial D^n_\alpha) & \xrightarrow{\cong} & H_n(Z', Y) \end{array}$$
commutes. In particular, we can identify the cellular (co)chain complexes canonically, not just up to isomorphism.

Example 4.1.19 This shows for instance that the Hopf invariant of a map $S^{2n-1} \to S^n$ depends only on the homotopy class of the map. More generally,

we can use it to show that the homotopy type of a CW complex depends only on its attaching maps up to homotopy, in a sense that we will make precise in the next section.

4.1.4 Homotopy pushouts: Introduction

Based on Thm. 4.1.17 and its corollary, we are in the position to discuss homotopy colimits. First recall that the case of homotopy coproducts is trivial, because they are already homotopy invariant. Indeed, there can be no analogue of Thm. 4.1.17 because no maps are involved in a coproduct diagram. Since $(-) \times I$ preserves coproducts, homotopies of maps $X_i \to Y$ are the same as a homotopy of $\bigsqcup_i X_i \to Y$. Thus in any reasonable sense a homotopy coproduct should coincide with a coproduct.

Consider a span $X \xleftarrow{f} A \xrightarrow{g} Y$. Define a functor

$$\mathrm{hPo}(f,g)\colon \mathbf{Top} \to \mathbf{Top}$$

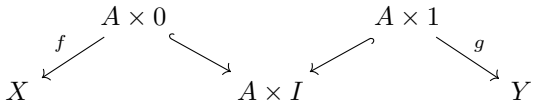
by

$$\mathrm{hPo}(f,g)(Z) = \left\{(\alpha,\beta,H) : \alpha\colon X \to Z, \beta\colon Y \to Z, H\colon \alpha f \simeq \beta g\right\}$$

with the obvious action on morphisms. In other words, $\mathrm{hPo}(f,g)(Z)$ consists of maps completing the span (f,g) to a homotopy commutative square, along with a homotopy recording this commutativity. This is naturally (but not strictly) symmetric, with

$$\mathrm{hPo}(f,g)(Z) \xrightarrow{\sim} \mathrm{hPo}(g,f)(Z) \qquad (\alpha,\beta,H) \mapsto (\beta,\alpha,\overline{H})$$

being the component isomorphism, where $\overline{H}$ means the reverse homotopy. This functor is representable, because the universal property we have defined is clearly the same one as that of the colimit of the diagram



Hence, we also write $\mathrm{hPo}(f,g)$ for the corresponding colimit of this diagram, equipped with the natural maps $X \to \mathrm{hPo}(f,g)$, $Y \to \mathrm{hPo}(f,g)$, and $A \times I \to \mathrm{hPo}(f,g)$. By definition, these canonical maps give a square along with a homotopy recording their commutativity.

Recall from the definition of the mapping cylinder M_f, that a map $M_f \to Z$ is the data of a map $X \to Z$ along with a map $A \to Z$ and a homotopy starting at this map and ending at the composite $A \to X \to Z$. The restriction to $A = A \times 0 \subset M_f$ gives this starting map. Therefore the universal properties in the following pushout sqaures agrees with that of $\mathrm{hPo}(f,g)$, and we have the

pushout squares

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \downarrow & \ulcorner & \downarrow \\ M_f & \longrightarrow & \mathrm{hPo}(g,f) \end{array} \qquad \begin{array}{ccc} A & \hookrightarrow & M_g \\ {\scriptstyle f}\downarrow & \ulcorner & \downarrow \\ X & \hookrightarrow & \mathrm{hPo}(f,g) \end{array}$$

which are canonically identified by reversing the homotopies. The universal property for $\mathrm{hPo}(g,f)$ is meant to be the homotopical analogue of the classical pushout, maps out of which are commutative squares, whereas maps out of $\mathrm{hPo}(g,f)$ are homotopy commutative squares. The central definition is:

Definition 4.1.20 A *homotopy-commutative* square is a square

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle \alpha}\downarrow \\ X & \xrightarrow{\beta} & Z \end{array}$$

in **Top** equipped with a homotopy $F\colon A \times I \to Z$ such that $F_0 = \alpha g$ and $F_1 = \beta f$. a homotopy-commutative square is *homotopy-cocartesian* if the map

$$\mathrm{hPo}(g,f) \to Z$$

induced by the data (α, β, F) is a homotopy equivalence. In this case, we say that (α, β, F) exhibits Z as a *homotopy pushout* of the span (f,g).

Example 4.1.21 Of course, the square

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \longrightarrow & \mathrm{hPo}(g,f) \end{array}$$

commuting with the canonical homotopy $A \times I \to \mathrm{hPo}(g,f)$ is homotopy cocartesian, because the induced map is the identity. This means that a homotopy pushout always exists, and it can be computed by replacing one of maps with the corresponding mapping cylinder inclusion, and taking the regular pushout.

The space Z in the ho-cocartesian square above is obviously unique up to homotopy equivalence, because it is homotopy equivalent to $\mathrm{hPo}(g,f)$. It is unique up to a 'canonical homotopy equivalence' in the sense that there are natural homotopy equivalences to $\mathrm{hPo}(g,f)$, however the choice of a homotopy inverse for one of the maps may be non-unique. But, of course this is unique in the homotopy category, so we can fairly say that the homotopy pushout is unique up to a choice of homotopy equivalence which is canonically determined up to homotopy.

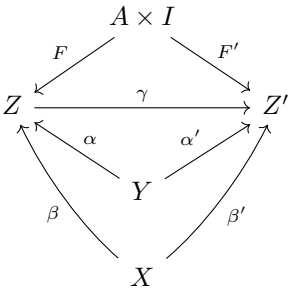
The homotopy pushout has a homotopy universal property. Namely, suppose we have a homotopy cocartesian square with a homotopy F as in Def. 4.1.20.

Then each ho-commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ {\scriptstyle f}\downarrow & F' & \downarrow{\scriptstyle \alpha'} \\ X & \xrightarrow[\beta']{} & Z' \end{array}$$

induces a unique map $\mathrm{hPo}(g,f) \to Z'$ whose compositions with $X \to \mathrm{hPo}(g,f)$, $Y \to \mathrm{hPo}(g,f)$, and $A \times I \to \mathrm{hPo}(g,f)$ recover β', α', and F'. Hence we obtain a map $Z \to Z'$ be precomposing with a homotopy inverse of the homotopy equivalence $\mathrm{hPo}(g,f) \to Z$, so $Z \to Z'$ is unique up to homotopy.

More importantly, the uniqueness is homotopy coherent in that the induced map $\gamma\colon Z \to Z'$ is unique in homotopy class subject to the requirement that the three triangles



commute up to homotopy. This commutativity obviously holds, so we claim that it is unique up to homotopy. In other words, we need to show that if the diagram above commutes up to homotopy for some map $\gamma\colon Z \to Z'$, then in fact the diagram

$$\begin{array}{ccc} \mathrm{hPo}(g,f) & \xrightarrow{(\alpha,\beta,F)} & Z \\ & {\scriptstyle (\alpha',\beta',F')}\searrow & \downarrow{\scriptstyle \gamma} \\ & & Z' \end{array}$$

commutes up to homotopy. We will prove this shortly, so for now we can safely take the universal property to be that the above diagram commutes up to homotopy. Note that if a ho-commutative square satisfies this property, then this property shows that (α, β, F) is a homotopy equivalence.

Lemma 4.1.22 *Let $g\colon A \times I \to Y$ be a homotopy $g_0 \simeq g_1$, and suppose that the square on the left of*

$$\begin{array}{ccc} A & \xrightarrow{g_1} & Y \\ {\scriptstyle f}\downarrow & F \ \ulcorner & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow[\beta]{} & Z \end{array} \qquad\qquad \begin{array}{ccc} A & \xrightarrow{g_0} & Y \\ {\scriptstyle f}\downarrow & (\alpha g)*F \ \ulcorner & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow[\beta]{} & Z \end{array}$$

is a ho-cocartesian square. Then the square on the right is ho-cocartesian.

Proof First of all, this makes sense because $F\colon \alpha g_1 \simeq \beta f$, and $\alpha g\colon \alpha g_0 \simeq \alpha g_1$. □

Note: Include the remaining results from my notes about homotopy pushouts.

4.1.5 Well-pointedness

In this section we study the relation between cofibrations and pointed spaces.

Definition 4.1.23 A based space (X, x_0) is *well-pointed* if $\{x_0\} \to X$ is a cofibration. A space X is *well-pointed* if (X, x_0) is well-pointed for all points $x_0 \in X$.

Note that in our definition, the point basepoint is not necessarily closed, but in most people's definition it is. Our main example will be CW complexes, as we show below.

Example 4.1.24 A simple Hausdorff non-example is the based space $(X, 0)$ where $X = \{0\} \cup \{1/n : n = 1, 2, \dots\} \subset \mathbb{R}$. Indeed, suppose we had a retraction $r \colon X \times I \to X \times 0 \cup 0 \times I$. Then $r(\{1/n\} \times I)$ is a connected subspace containing $\{1/n\}$, hence r is uniquely determined to be the identity on $\{0\} \times I$ and send $\{1/n\} \times I$ to $1/n$. But this is not continuous, since $\lim_{n\to\infty}(1/n, 1) = (0, 1)$, and r fixes $(0, 1)$, but $\lim_{n\to\infty} r(1/n, 1) = (0, 0)$.

Lemma 4.1.25 *If (X, A) is a relative CW complex and $\varphi \colon A \to Y$ is a map, then $(X \cup_\varphi Y, Y)$ is a relative CW complex whose relative cells are in bijection with the relative cells of (X, A), given by composition with $X \to X \cup_\varphi Y$. If Y is a CW complex, then $X \cup_\varphi Y$ is as well with cells being the cells of Y and the relative cells of (X, A).*

Proof This is just an application of the pasting lemma at each step in the skeleton. Indeed write $X^{-1} = A$, $Z = X \cup_\varphi Y$ with $Z^{-1} = Y$. Then for each $n \geq 0$ we have inductively

$$\begin{array}{ccccc} \bigsqcup_\alpha \partial D^n_\alpha & \longrightarrow & X^{n-1} & \longrightarrow & Z^{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \bigsqcup_\alpha D^n_\alpha & \longrightarrow & X^n & \longrightarrow & Z^n \end{array}$$

where when $n = 0$ the top right map is φ. Taking the filtered colimit along the right-hand diagrams, we obtain a cocartesian square

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & \varinjlim_n Y^n \end{array}$$

showing that $\varinjlim_n Z^n = Z$. The rest is similar. □

Proposition 4.1.26 *Let X be a CW complex. Then for each $x_0 \in X$, there is a CW structure on X with x_0 as a 0-cell. In particular, X is well-pointed.*

Proof Assume that X is n-dimensional, and x_0 is contained in a unique cell e^n_α of dimension n. Then $Y = X - e^n_\alpha$ is a subcomplex, and we have a pushout diagram:

$$\begin{array}{ccc} \partial D^n_\alpha & \xrightarrow{\varphi_\alpha} & Y \\ \downarrow & & \downarrow \\ D^n_\alpha & \xrightarrow{\Phi_\alpha} & X \end{array}$$

Using an automorphism of the pair $(D^n, \partial D^n)$ sending $\Phi^{-1}(x_0) \in D^n - \partial D^n$ to 0, we may assume $\Phi(0) = x_0$. Now $(D^n, \partial D^n) \cong (CS^{n-1}, S^{n-1})$, and the cone of a CW complex has a 0-cell at the apex, i.e. at $0 \in D^n$. Hence $(D^n, \partial D^n)$ is a relative CW complex with a relative 0-cell at 0, so by the lemma and the above square, X is a CW complex with a zero cell given by $0 \to D^n \to X$, i.e. the inclusion of x_0. To reduce to the n-dimensional case for $n = \dim(e^n_\alpha)$, simply replace X by its n-skeleton X^n. □

Now let us illustrate the main thing we can do with well-pointed pairs. Fix a space X. For each well-pointed pair (Z, z_0), we are going to define a functor

$$\langle (Z, z_0), X \rangle : \Pi_1(X) \to \mathbf{Set}$$

sending $x_0 \mapsto \langle (Z, z_0), (X, x_0) \rangle$, i.e. pointed homotopy classes of maps. Given a path $\gamma \colon z_0 \to z_1$, we define

$$\gamma_* \colon \langle Z, (X, x_0) \rangle \to \langle Z, (X, x_1) \rangle$$

as follows: a based map $f \colon Z \to (X, x_0)$, provides an initial lift of the homotopy $\gamma \colon \{z_0\} \times I \to X$, hence we obtain a homotopy $F \colon Z \times I \to X$ such that $F_t(z_0) = \gamma(t)$ and $F_0 = f$. In particular, $F_1(z_0) = \gamma(0) = x_0$, so we can define $[\gamma]^*([f]) = [F_1]$.

Lemma 4.1.27 *With this definition, $\langle Z, X \rangle$ is a well-defined functor for any well-pointed space $Z = (Z, z_0)$.*

Proof Recall from Lemma 4.1.16 that

$$z_0 \times I \underset{z_0 \times \partial I}{\cup} Z \times \partial I \to Z \times I \qquad (*)$$

is a cofibration. Suppose that $F, G \colon Z \times I \to X$ are homotopies for which:

1. $F_0(z_0) = G_0(z_0) = x_0$, and $h \colon F_0 \simeq G_0$ is a homotopy based at z_0.
2. $F(z_0, -)$ and $G(z_0, -)$ are both paths from x_0 to x_1, and $g \colon F(z_0, -) \simeq G(z_0, -)$ is a homotopy of paths rel ∂I, so that $g(s, 0) = F(z_0, s)$, $g(s, 1) = G(z_0, s)$, and $g(0, t) = x_0$, $g(1, t) = x_1$.

For each t, $(F_t, G_t) \colon Z \times \partial I \to X$ whose restriction to $z_0 \times \partial I$ is $(F_t(z_0), G_t(z_0))$. Similarly for each t, $g(t, -) \colon z_0 \times I \to X$ is a map whose restriction to $z_0 \times \partial I$ is $(F_t(z_0), G_t(z_0))$.

Thus these glue to a homotopy of two maps from the domain of the cofibration $(*)$ to X. This homotopy starts at $g(0, -) = x_0$ and (F_0, G_0). Thus,

the homotopy h provides an initial lift, because $h(z_0, -) = x_0$ and $h(-, 0) = F_0, h(-, 1) = G_0$. Hence, h is homotopic to a homotopy $h' \colon Z \times I \to X$ satisfying:

$$h'(z_0, -) = g(1, -) = z_0, \quad h'(-, 0) = F_1, \quad h'(-, 1) = G_1.$$

In other words, h' exhibits $F_1 = G_1$ in $\langle Z, (X, x_0)\rangle$ so this shows that the map γ_* above is well-defined, and depends only on the path homotopy class. Hence $\langle Z, X\rangle$ provides a well-defined assignment on $\Pi_1(X)$

Now functoriality: if γ is the constant path at x_0, then for each based map $f \colon Z \to (X, x_0)$ we can choose the constant lift $(x, t) \mapsto f(x)$ at f, so $\gamma_*([f]) = [f]$. Now let $\gamma \colon x_0 \to x_1$, $\eta \colon x_1 \to x_2$, and $f \colon Z \to (X, x_0)$. Choose a homotopy F such that $F_0 = f$, $F_t(z_0) = \gamma(t)$. Do the same for F_1, i.e. find G with $G_0 = F_1$ and $G_t(z_0) = \eta(t)$. Then $F * G$ provides a homotopy which at z_0 is the composite path $\gamma * \eta$, and $(F * G)_1 = G_1 = \eta_*(\gamma_*([f]))$. □

Definition 4.1.28 Let (Z, z_0) be a well-pointed space. The functor

$$\langle (Z, z_0), X\rangle : \Pi_1(X) \to \mathbf{Set}$$

is defined on objects by $x_0 \mapsto \langle (Z, z_0), (X, x_0)\rangle$, and uniquely specified on morphisms as follows: given classes $\alpha \colon x_0 \to x_1$ and $\beta \in \langle Z, (X, x_0)\rangle$, choose any $\gamma \in \alpha$, $f_0 \in \beta$, and homotopy $f \colon Z \times I \to X$ starting at f_0 with $f(z_0, -) = \gamma$. Then $\big(\langle (Z, z_0), X\rangle\, \alpha\big)(\beta) = [f_1]$.

Thus in particular, for each $x_0 \in X$ we obtain a canonical map

$$\mathrm{Aut}_{\Pi_1(X)}(x_0) = \pi_1(X, x_0)^{\mathrm{op}} \to \mathrm{Aut}(\langle (Z, z_0), (X, x_0)\rangle),$$

i.e. a right action of $\pi_1(X, x_0)$ on $\langle Z, X\rangle$ (suppressing the chosen basepoints). Note also that there is the canonical map $\langle Z, (X, x_0)\rangle \to [Z, X]$, and this is a natural transformation from $\langle Z, X\rangle$ to the constant functor at $[Z, X]$. Indeed, by definition a based homotopy class and its image under a path class are freely homotopic.

Proposition 4.1.29 *Let (Z, z_0) be well-pointed and (X, x_0) be pointed. Then the canonical map*

$$\langle (Z, z_0), (X, x_0)\rangle \to [Z, X]$$

induces an injection

$$\langle (Z, z_0), (X, x_0)\rangle \,/\, \pi_1(X, x_0) \to [Z, X]$$

on the orbit set. If X is path connected, then this is a bijection.

Proof The aforementioned naturality implies that the map factors through the quotient by the automorphism group. Naturality also gives surjectivity in the path connected case, in which case the fundamental groupoid is connected so the images for all basepoints x_0 are the same. For injectivity, if based maps f_0, f_1 are freely homotopic by $f \colon Z \times I \to X$, then $\gamma(t) = f_t(z_0)$ is loop at x_0 with $[f_0] \cdot [\gamma] = [f_1]$. □

Let $h\colon X \to Y$ be a map with $h(x_0) = y_0$. For based maps $f_0, f_1\colon (W, w_0) \to (X, x_0)$, and a based homotopy $f\colon f_0 \sim f_1$ between them, $hf\colon hf_0 \simeq hf_1$ is also a based homotopy. Hence we get induced maps and a commutative diagram

$$\begin{array}{ccc} \langle (W,w_0),(X,x_0)\rangle & \longrightarrow & [W,X] \\ {\scriptstyle h_*}\downarrow & & {\scriptstyle h_*}\downarrow \\ \langle (W,w_0),(Y,y_0)\rangle & \longrightarrow & [W,Y] \end{array}$$

as so.

Proposition 4.1.30 *Let $h\colon X \to Y$ be a homotopy equivalence. Then if $Z = (Z, z_0)$ is well-pointed, the natural map*

$$h_*\colon \langle Z, (X, x_0)\rangle \to \langle Z, (Y, h(x_0))\rangle$$

is a bijection for any $x_0 \in X$.

Proof Let $h^{-1}\colon Y \to X$ be a homotopy inverse to h, and let $F\colon 1_X \simeq h^{-1}h$. Then let $\gamma := F(x_0, -)$ be the corresponding path from x_0 to $h^{-1}h(x_0)$, and let us compute the induced bijection

$$[\gamma]_*\colon \langle Z, (X, x_0)\rangle \to \langle Z, (X, h^{-1}h(x_0))\rangle\,.$$

Given $f\colon Z \to (X, x_0)$, the homotopy F induces a homotopy $f \simeq h^{-1}hf$ given by $F \circ (f \times I)$, and $(F \circ (f \times I))(z_0, t) = F(x_0, t) = \gamma(t)$, hence $[\gamma]_*([f]) = [h^{-1}hf]$. Hence the above is the same map as that induced by $h^{-1}h\colon X \to X$, so we have a commutative diagram

$$\begin{array}{ccc} \langle Z,(X,x_0)\rangle & & \\ {\scriptstyle h_*}\downarrow & \overset{[\gamma]_*}{\searrow} & \cong \\ \langle Y,(X,hx_0)\rangle & \xrightarrow[h_*^{-1}]{} & \langle Z,(X,h^{-1}hx_0)\rangle \end{array}$$

Hence the left-hand h_* is a split monomorphism. But since h was arbitrary, the same argument applies to $h^{-1}\colon (Y, h(x_0)) \to (X, h^{-1}h(x_0))$, hence the lower h_*^{-1} is also a split monomorphism. Hence both maps are bijections. □

Let us briefly discuss how one computes the functors $\langle Z, X\rangle$ in practice. First of all, when $(Z, z_0) = (I^n/\partial I^n, *)$ one can explicitly construct the homotopy lifts of maps along paths using smaller and smaller concentric squares. This recovers the standard action of π_1 on π_n. Second, if X is simply connected, then Prop. 4.1.29 shows that $\langle Z, X\rangle$ is canonically isomorphic to the constant functor at $[Z, X]$.

Lastly, we can use covering spaces. Let $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ be a universal cover of a locally path-connected space X. Then there are a few basic observations we can make using standard covering space theory. Fix a well-pointed

space $Z = (Z, z_0)$; first observe that the diagram

$$\begin{array}{ccc} \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle & \xrightarrow{p_*} & \langle Z, X\rangle \\ \cong\downarrow & & \downarrow \\ [Z, \widetilde{X}] & \xrightarrow{p_*} & [Z, X] \end{array}$$

commutes, where the left map is bijective since $\widetilde{X}$ is simply connected, and the bottom p_* is injective by the homotopy lifting property. Hence the top p_* is injective too. Now suppose $[\gamma] \in \pi_1(X, x_0)$, and let $\widetilde{\gamma}$ be the lifted path going from $\widetilde{x}_0$ to $\widetilde{x}_1$. Then we claim the diagram

$$\begin{array}{ccc} \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle & \xrightarrow{\widetilde{\gamma}_*} & \langle Z, (\widetilde{X}, \widetilde{x}_1)\rangle \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} \\ \langle Z, X\rangle & \xrightarrow{\gamma_*} & \langle Z, X\rangle \end{array}$$

commutes. Indeed, given $f\colon Z \to (\widetilde{X}, \widetilde{x}_0)$, choose a homotopy F such that $F_1 = pf$ and $F(z_0, -) = \gamma$, so that $\gamma_* p_*([f]) = [F_1]$. By the homotopy lifting property, there's a lift of this homotopy $\widetilde{F}$ with $\widetilde{F}_0 = f$, and thus $\widetilde{F}(z_0, -)$ is a lift of γ starting at $\widetilde{x}_0$, so $\widetilde{F}(z_0, -) = \widetilde{\gamma}$. Therefore $\widetilde{\gamma}_*([f]) = [\widetilde{F}_1]$, and $p_*[\widetilde{F}_1] = [F_1]$, as desired.

Now, recall that $\pi_1(X, x_0)$ acts on $\widetilde{X}$ by deck transformations. These are unbased homeomorphisms, hence they give a left action of $\pi_1(X, x_0)$ on $[Z, \widetilde{X}]$ by post-composition, or a right action on $[Z, \widetilde{X}]$ where $[\gamma]$ acts by post composition by the inverse deck transformation $D_{[\gamma]}^{-1}$. The canonical bijection

$$\langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle \xrightarrow{\sim} [Z, \widetilde{X}]$$

allows us to pullback the right action to $\langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle$. We can alternatively describe this action through the following commutative diagram:

$$\begin{array}{ccccc} \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle & \xrightarrow{\widetilde{\gamma}_*} & \langle Z, (\widetilde{X}, \widetilde{x}_1)\rangle & \xrightarrow{D_{[\gamma],*}^{-1}} & \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ [Z, \widetilde{X}] & = & [Z, \widetilde{X}] & \xrightarrow{D_{[\gamma],*}^{-1}} & [Z, \widetilde{X}] \end{array}$$

This shows that a class $[\gamma] \in \pi_1(X, x_0)$ acts by $D_\gamma^{-1} \circ \widetilde{\gamma}_*$.

Lemma 4.1.31 *Keeping the above notation, the map*

$$p_*\colon \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle \to \langle Z, X\rangle$$

is an equivariant monomorphism of right $\pi_1(X, x_0)$-sets. If in addition Z is path connected, locally path connected, and simply connected, then the map

is an isomorphism. This applies in particular for a 1-connected CW complex with any basepoint.

Proof We have already seen that it is injective. In the case with additional hypotheses on Z, surjectivity is just an application of the standard lifting criterion for covering spaces. To see the equivariance, observe that the diagram

$$\begin{array}{ccccc} \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle & \xrightarrow{\widetilde{\gamma}_*} & \langle Z, (\widetilde{X}, \widetilde{x}_1)\rangle & \xrightarrow{D^{-1}_{\gamma,*}} & \langle Z, (\widetilde{X}, \widetilde{x}_0)\rangle \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & \swarrow{\scriptstyle p_*} & \\ \langle Z, X\rangle & \xrightarrow{\gamma_*} & \langle Z, X\rangle & & \end{array}$$

commutes. The left square we showed earlier, and the right-hand triangle is by definition of a deck transformation. But the upper composite is the bijection by which γ acts, as is the bottom. □

Note that the extra hypotheses for surjectivity are clearly necessary, because e.g. if $Z = S^1$, then the domain is trivial while the codomain may not be. In general, the lifting criterion for covering spaces says that a map $f\colon (Z, z_0) \to (X, x_0)$ lifts to the universal cover iff the induced map f_* on fundamental groups is zero.

Example 4.1.32 Consider $\mathbb{R}P^n$ for $n \geq 2$ with its universal double cover $p\colon S^n \to \mathbb{R}P^n$. The deck grouip is $\mathbb{Z}_2 = \{\pm 1\}$, where -1 acts on S^n by $x \mapsto -x$ (the antipodal map), and the covering exhibits $\mathbb{R}P^n = S^n/\mathbb{Z}_2$. The action of π_1 on itself is trivial because $\pi_1 = \mathbb{Z}_2$ is abelian. Of course $\pi_k(\mathbb{R}P^n) = 0$ for $2 \leq k < n$, and then

$$p_*\colon \pi_n(S^n) \to \pi_n(\mathbb{R}P^n)$$

gives an isomorphism from $\mathbb{Z}$, generated by the quotient map p. Let us consider the free homotopy set $[S^n, \mathbb{R}P^n]$, for which we must find the action of $\pi_1(\mathbb{R}P^n) = \mathbb{Z}_2$ on $\pi_n(S^n)$. Observe that

$$\pi_n(S^n) \xrightarrow{\sim} [S^n, S^n] \xrightarrow{\deg} \mathbb{Z}$$

are isomorphisms, and $-1 \in \mathbb{Z}_2$ sends $\mathrm{Id} \in [S^n, S^n]$ to the antipodal map, which has degree $(-1)^{n+1}$, -1 sends $[\mathrm{Id}] \in \pi_1(S^n)$ to $(-1)^{n+1}[\mathrm{Id}]$. In particular

$$\mathbb{Z}\langle [p]\rangle \cong \pi_n(\mathbb{R}P^n) \xrightarrow{\sim} [S^n, \mathbb{R}P^n]$$

when n is odd, since the action is trivial. When n is even however,

$$[S^n, \mathbb{R}P^n] \cong \{0, [p], 2[p], \dots\}.$$

In other words, for n even, the maps $x \mapsto p(x)$ and $x \mapsto p(-x)$ are not based homotopic but they are freely homotopic.

References

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