The Gelfand-Fuchs Cohomology of Vector Fields on S^1

Vihaan Dheer

1 Introduction

1.1 Background

This is an exposition and survey of certain aspects of Lie algebra cohomology, with the aim of computing the cohomology of a specific infinite-dimensional Lie algebra: The space of vector fields on circle S^1. More precisely, we study the *Gelfand-Fuchs cohomology* of this Lie algebra, which was a slight variant of standard Lie algebra cohomology defined by Gelfand and Fuchs in the late 1960s specifically for Lie algebras of smooth vector fields.

Lie algebras of vector fields on a smooth manifold are infinite-dimensional, and their standard cohomology rings are generally not finitely-generated, and usually do not contain uesful information. The Gelfand-Fuchs cohomology, as we will discuss, is finitely generated for vector fields of smooth manifolds. This cohomology theory has concrete practical applications, and most important is the cohomology of degree 2, which is crucial for considering loop algebras such as the Virasoro algebra. This is explained in Sec. 1.2

To compute the cohomology of vector fields on S^1, we use Gelfand and Fuchs' machinery developed in their paper on the subject [GF69], which is via 'diagonal cohomology'. This theory is the subject of Sec. 2. In 1975, Haefliger used the diagonal cohomology of Gelfand and Fuchs to compute the full cohomology ring of vectors field over any closed orientable manifold [Hae75]. It is also worth noting that Gelfand and Fuchs computed the cohomology in the specific case of S^1 before developing this general machinery [GF68].

In the remainder of this subsection, we briefly explain the layout of the proof of the result presented, and then the structure of this paper itself. We denote the topological Lie algebra of smooth vector fields S^1 by $\mathfrak{X}(S^1)$. Throughout the course of the discussion, we will define two distinguished cohomology classes

$$\alpha \in H^2(\mathbb{C}\mathfrak{X}(S^1); \mathbb{C}) \qquad \beta \in H^3(\mathbb{C}\mathfrak{X}(S^1)).$$

Here $\mathbb{C}\mathfrak{X}(S^1)$ denotes the complexification, but the result holds just as well if we consider $H^*(\mathfrak{X}(S^1); \mathbb{R})$, so we will not differentiate between the two notions. Our goal will be following result:

Theorem *The Gelfand-Fuchs cohomology ring of $\mathfrak{X}(S^1)$ has the presentation*

$$H^*(\mathfrak{X}(S^1)) = \mathbb{R}[\alpha, \beta]/(\beta^2) = \mathbb{R}[\alpha] \otimes_{\mathbb{R}} \Lambda_{\mathbb{R}}[\beta] \qquad \deg \alpha = 2, \deg \beta = 3.$$

Namely, we have for all $k \geq 0$

$$H^{2k}(\mathfrak{X}(S^1)) = \langle \alpha^k \rangle \cong \mathbb{R} \qquad H^{2k+1}(\mathfrak{X}(S^1)) = \langle \alpha^{k-1}\beta \rangle \cong \mathbb{R}.$$

The analagous result by replacing $\mathbb{R}$ with $\mathbb{C}$ holds if we replace $\mathfrak{X}(S^1)$ with its complexification $\mathbb{C}\mathfrak{X}(S^1)$ as a topological Lie algebra.

Rather than giving in-depth proofs, we survey the ideas used along the way; our presentation generally follows that of Fuchs'book [Fuk86]. First, in the rest of Sec. 1, we briefly review Gelfand-Fuchs cohomology and discuss the various applications of Lie algebra cohomology. We also state some preliminary results on the cohomology of formal vector fields, which significantly different from the present case, and comes up in the proof.

Next, in Sec. 2, we explain the primary technique to compute Gelfand-Fuchs cohomology, namely diagonal cohomology. The focus is on the *diagonal filtration* of the Gelfand-Fuchs cochain complex, which we might loosely interpret as breaking up the problem into cochains vanishing on vector fields with sufficiently small support. In our case of $\mathfrak{X}(S^1)$, the associated spectral sequence becomes trivial, reducing the problem to computing the cohomology of suffcessive diagonal filtration quotients.

This will be done by defining a new filtration on each term of the diagonal sequence, based on the vanishing order of certain zeroes of vector fields. We can explicitly compute the E_2-page of the associated spectral sequence, and will be able to deduce that the Gelfand-Fuchs cohomology of smooth vector fields on any smooth manifold is finite dimensional.

All of the main work will be done in that section, so in Sec. 3 we will be able to deduce from our previous work enough information for an explicit computation in the case of $\mathfrak{X}(S^1)$, proving the aforemnetioned theorem of Gelfand and Fuchs. We will conclude with a short discussion of further work on this topic.

1.2 Lie Algebra Cohomology

In this section we develop the basic cohomology theory of topological Lie algebras with coefficients in a topological module, and define the ring structure on the cohomology of a Lie algebra with coefficients in the trivial representation. We observe that the somewhat non-standard theory of topological Lie algebras is what differentiates standard Lie algebra cohomology from Gelfand-Fuchs cohomology. Additional continuity conditions on cochains are what makes the resulting cohomology useful in the case of infinite-dimensional algebras.

In what follows, fix a field k not of characteristic 2.

Definition 1.2.1 A *topological Lie algebra* over k is a Lie algebra $\mathfrak{g}$ over k which is also a topological space, such that the commutator and addition

$$[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g} \quad \text{and} \quad (-)+(-)\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

are continuous maps. A *topological module over* $\mathfrak{g}$ is a left $U(\mathfrak{g})$-module V - equivalently a k-vector space V equipped with a Lie algebra homomorphism $\phi\colon \mathfrak{g}\to\mathrm{End}(V)$ - which is also a topological space, and the action and addition

$$\mathfrak{g}\times V\to V \quad \text{and} \quad V\times V\to V$$

are continuous maps.

All of the standard constructions on Lie algebras and their representations import to the topological case. For example, a *morphism* of $\mathfrak{g}$-modules $V\to W$ is just a continuous map $V\to W$ which is also a homomorphism when viewed as regular modules over the Lie algebra $\mathfrak{g}$. Note that even if k is $\mathbb{R}$ or $\mathbb{C}$, we don't require $\mathfrak{g}$ or V to be topological vector spaces; so by giving $\mathfrak{g}$ the discrete topology we get a valid topological Lie algebra, and moreover in this case the obvious functor

$$\mathrm{discrete}\colon \mathrm{Rep}(\mathfrak{g})\to\mathrm{TopRep}(\mathfrak{g})$$

sending a $\mathfrak{g}$-module to the same one with the discrete topology is fully faithful. In particular, the standard theory of Lie algebras and their representations is a special case of the topological version.

As mentioned, the basic constructions of $\mathfrak{g}$-modules carry over to the topological case. First, any topological space V which is also a k-vector space is a $\mathfrak{g}$-module with the trivial action, i.e. where V is annihilated by any element of $\mathfrak{g}$. This is continuous because the preimage of any open subset of V is either empty or all of $\mathfrak{g}\times V$. If V,W are $\mathfrak{g}$-modules, then $V\otimes W$ is one by taking the final topology with respect to the canonical map $V\times W\to V\otimes W$, and we can use this to define S^rV and Λ^rV, the symmetric and alternating powers respectively, as quotient spaces of a tensor power.

Now we move to cohomology. Let $\mathfrak{g}$ be a topological Lie algebra and V a (topological) $\mathfrak{g}$-module. We define

$$C^q(\mathfrak{g};V) \coloneqq \mathrm{Hom}_{\mathfrak{g}}(\Lambda^q\mathfrak{g},V),$$

i.e. the k-vector space of continuous multilinear skew-symmetric maps

$$\mathfrak{g}\times\cdots\times\mathfrak{q}\to V.$$

We define differentials $d\colon C^q(\mathfrak{g};V)\to C^{q+1}(\mathfrak{g};V)$ by the formula

$$\begin{aligned}(dc)(x_1,\dots,x_{q+1}) = &\sum_s (-1)^s x_s\cdot c(x_1,\dots,\widehat{x_s},\dots,x_{q+1})\\ &-\sum_{s\le t}(-1)^{s+t}([x_s,x_t],\dots,\widehat{x_s},\dots,\widehat{x_t},\dots).\end{aligned}$$

It is obvious that dc is continuous and a multilinear skew-symmetric map, and one checks $d^2 = 0$ so that we get a cochain complex:

$$0 \to C^0(\mathfrak{g}; V) \xrightarrow{d} C^1(\mathfrak{g}; V) \xrightarrow{d} C^2(\mathfrak{g}; V) \xrightarrow{d} C^3(\mathfrak{g}; V) \cdots$$

We in this case call it the Gelfand-Fuchs complex because of the requirement of continuity, and we call its cohomology $H^q(\mathfrak{g}; V)$ the *Gelfand-Fuchs cohomology with coefficients in* V.

Much of this cohomology theory resembles cohomology in topology. For example, let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra with the subspace topology. We define the relative cocycles, forming a subspace $C^q(\mathfrak{g}, \mathfrak{h}; V)$, to be those cochains c for which both c and dc vanish on any tuple of $\mathfrak{g}$-elements which contains an element of $\mathfrak{h}$. It is easy to check that there is a short exact sequence

$$0 \to C^*(\mathfrak{g}, \mathfrak{h}; V) \to C^*(\mathfrak{g}; V) \to C^*(\mathfrak{h}; V) \to 0$$

of complexes, where V is viewed as an $\mathfrak{h}$-module by restriction of the action, and hence we get a long exact sequence in cohomology. For the same reason, given a filtration

$$\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \cdots$$

by subalgebras there is a canonical spectral sequence which, under reasonable hypotheses, converges to the cohomology of $\mathfrak{g}$. Similarly, given an exact sequence of $\mathfrak{g}$-modules induces another short exact coefficient sequence of complexes, so we get another long exact sequence. This can be used to upgrade results from trivial coefficients to nontrivial coefficients, but we will not do that here.

Whenever we take $V = k$ to be the trivial module, which has either the discrete topology or the natural one if $k = \mathbb{R}, \mathbb{C}$, we just write $C^*(\mathfrak{g}) = C^*(\mathfrak{g}; k)$, and likewise for cohomology. In this case there is a natural graded ring structure on $C*(\mathfrak{g})$, coming directly from the graded ring structure on the exterior algebra $\Lambda\mathfrak{g}$ with wedge product. This respects the differential, giving a cohomology ring $H^*(\mathfrak{g})$. As expected, this is graded-commutative.

We conclude with some elementary observations regarding the applications of cohomology, and then a brief explanation of its practical uses in the theory of finite-dimensional discrete semisimple Lie algebras. First, we note that

$$H^0(\mathfrak{g}; V) = V^{\mathfrak{g}},$$

i.e. the set of $\mathfrak{g}$-invariants $\{v \in V : xv = 0, x \in \mathfrak{g}\}$. In particular $H^0(\mathfrak{g}) = k$, the ground field. It is easy to see that $H^1(\mathfrak{g}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ from the definition. An elementary computation shows that

$$H^1(\mathfrak{g}; \mathrm{Hom}(V, W)) =: \mathrm{Ext}^1_{\mathfrak{g}}(V, W)$$

classifies extensions of W by V, so that in particular all exact sequences

$$0 \to V \to ? \to W \to 0$$

split exactly when $H^1(\mathfrak{g}; \mathrm{Hom}(V, W)) = 0$. In the theory of finite-dimensional Lie algebras, there is the following theorem:

Theorem 1.2.2 (Whitehead's Lemma) *Let k be a field of characteristic 0, $\mathfrak{g}$ a finite-dimensional semisimple Lie algebra and V a finite-dimensional $\mathfrak{g}$-module. Then:*

1. $H^1(\mathfrak{g}; V) = 0$.
2. $H^2(\mathfrak{g}; V) = 0$.
3. *If further V is simple and nontrivial, then $H^q(V) = 0$ for all $q \geq 0$.*

We mention that the first statement is not too difficult to prove from the existence of the Casimir element in the universal enveloping algebra, yet it has the following important consequence:

Corollary 1.2.3 (Weyl) *Every finite-dimensional module over a finite-dimensional semsimple Lie algebra in characteristic 0 is a direct sum of simple modules.*

Proof By induction on the dimension of module X, if it is not already simple then take a proper submodule V and write $W = V/X$ so that the sequence

$$0 \to V \to X \to W \to 0$$

is exact and both V and W are direct sums of simple modules by induction. As $H^1(\mathfrak{g}; \mathrm{Hom}(V, W)) = 0$ by Whitehead's lemma, the sequence splits. □

This begins to demonstrate the power of cohomology even in the finite-dimensional setting. Moving onto H^2, it is another elementary property that property that $H^2(\mathfrak{g}; V)$ classifies central extension of $\mathfrak{g}$ by V, namely V is equipped with the compatible trivial bracket, and we have an exact sequence

$$0 \to V \to \hat{\mathfrak{g}} \to \mathfrak{g} \to 0$$

of Lie algebras so that $\hat{g}$ is the central extension. Note that by Whitehead's lemma, if $\mathfrak{g}$ is semismple then there are no central extensions by finite-dimensional modules except for $\mathfrak{g} \oplus V$.

The infinite-dimensional case is different from this, and this classification will be our main motivation for studying $H^*(\mathfrak{X}(S^1))$. In particular, the result that $H^2(\mathbb{C}\mathfrak{X}(S^1)) = \mathbb{C}$ will show that up to scaling there is a unique central extension of $\mathbb{C}\mathfrak{X}(S^1)$; this central extension is the *Virasoro algebra* which has wide applications and provides a fundamental example of an infinite-dimensional Lie algebra.

Remark 1.2.4 We must mention that the Virasoro is usually defined as the unique central extension of the complexification of the *polynomial* vector fields on S^1, not all of $\mathbb{C}\mathfrak{X}(S^1)$. But, in light of the use of Gelfand-Fuchs cohomology,

there is a topological basis for $\mathfrak{X}(S^1)$ whose commutator matches that of the polynomial vector field case, so the result here will be sufficient to establish that cohomology as well. See [Fuk86, 1.2].

1.3 Basic Preliminaries

Having motivated and defined the Gelfand-Fuchs cohomology of Lie algebras, we end Sec. 1 by formally defining the topological Lie algebras whose cohomology we aim to compute, as well as the definition and some properties of Lie algebras of formal vector fields. First we review the C^∞-topology on a space of smooth maps.

If M, N are smooth manifolds, we recall that the k-jet space $J^k_x(M,N)$ at some $x \in M$ is the quotient of the set $C^\infty_x(M,N)$ of germs of smooth maps at x by the following equivalence relation: $f \sim g$ if there is a neighborhood centered at x on which, for each $i \leq k$, the ith partial derivatives of f at x agree with those of g. Then by setting

$$J^k(M,N) \coloneqq \bigsqcup_{x \in M} J^k_x(M,N)$$

equipped with the projection $J^k(M,N) \to M$ sending $J^k_x(M,N)$ to x, one can show that there is a canonical manifold structure on $J^k(M,N)$ making $J^k(M,N) \to M$ a smooth submersion. Under this structure, the projection map becomes a fiber bundle for which the fiber over a point is the k-jet space at that point. This is the *jet bundle* between M and N.

For each k and $f \in C^\infty(M,N)$, we have a canonical section of the jet bundle

$$j^k f \colon M \to J^k(M,N)$$

sending $x \in M$ to the image of f under the composite

$$C^\infty(M,N) \to C^\infty_x(M,N) \to J^k_x(M,N) \hookrightarrow J^k(M,N).$$

The image $\operatorname{im} j^k f$ of this section in the jet bundle might be thought of as a smooth representative of the kth order Taylor polynomial of f. For an open set $U \subset J^k(M,N)$ of the jet bundle, we define

$$N^k(U) \coloneqq \{f \in C^\infty(M,N) : \operatorname{im}(j^k f) \subset U\} \subset C^\infty(M,N).$$

The collection $\{N^k(U) : k \geq 0, U \subset J^k(M,N) \text{ open}\}$ form a basis for the topology on $C^\infty(M,N)$ that they generate; this topology on $C^\infty(M,N)$ is called the *Whitney C^∞-topology*. See [Mic80, §1] for more details of this construction.

Now let M be a smooth manifold. We denote by $\mathfrak{X}(M)$ the $C^\infty(M)$-module of smooth sections of $TM \to M$ i.e. vector fields, but usually we will regard it as vector space. Recall that the commutator of first order differential operators remains a first order operator, so it makes $\mathfrak{X}(M)$ into a real Lie algebra. Using

the inclusion

$$\mathfrak{X}(M) \subset C^\infty(M, TM),$$

we give $\mathfrak{X}(M)$ the subspace topology with respect to the C^∞-topology. It is easy to verify from the topology on each jet bundle $J^k(M, TM)$ that addition and commutation of sections are continuous (as it suffices to do the case $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$), so $\mathfrak{X}(M)$ is a topological Lie algebra.

We aim to learn about its Gelfand-Fuchs cohomology, and compute it fully in the case $M = S^1$. To do this, we need to simultaneously know about the Lie algebra of formal vector fields. The Lie algebra W_n *of formal vector fields* is informally written

$$W_n = \mathbb{R}[[x_1, \ldots, x_n]] \left(\frac{\partial}{\partial x_i} \right)_{i=1,\ldots,n},$$

so we interpret elements $f \in W_n$ as formal power series in $x_1, \ldots, x_n$ with commutator

$$[\sum_i f_i \partial_i, \sum_i g \partial_i] = \sum_i \sum_j \left(f_j \partial_j g_i - g_j \partial_j f_i \right) \partial_j.$$

W_n is given the projective limit topology: For example, in the case of $n = 1$ this means we use the isomorphism

$$A[[x]] \xrightarrow{\sim} \varprojlim_{j \geq 1} A[x]/(x^j)$$

and take the limit in the category of topological space treating each quotient $A[x]/(x^j)$ as discrete. It can alternatively be described as a subset of the product topology $\prod_{j \geq 1} A[x]/(x^j)$. Addition and the commutator are continuous in this topology, where the former is obvious and the latter is because taking partial derivatives only reduces the degree. Consequently W_n is a real topological Lie algebra of infinite dimension.

W_n is closely related to the construction $\mathfrak{X}(M)$ defined before. Firstly, we observe that the commutator formula on W_n is exatcly that of $\mathfrak{X}(M)$ in local coordinates of $\mathbb{R}^n$, for $\dim M = n$. Next, considering the case $\mathfrak{X}(\mathbb{R}^n)$, we get a canonical map

$$\mathfrak{X}(\mathbb{R}^n) \to W_n$$

sending a section ξ to its Taylor power series, which is injective modulo non-analytic smooth sections. For precisely, a vector field maps to its ∞-jet of smooth vector fields on $\mathbb{R}^n$ in the direct limit of the jet bundles. Loosely, we know that the module of vector fields is a sheaf, so $\mathfrak{X}(M)$ is locally $\mathfrak{X}(\mathbb{R}^n)$, and moreover given our definition of the $\mathbb{C}^\infty$-topology on sections via arbitrarily high degree jets, the topologies agree too. Hence, we might think of W_n as the completion to a topological Lie algebra of a local approximation to $\mathfrak{X}(M)$.

There is a remarkable result of, unsurprisingly, Gelfand and Fuchs, which computes fully the cohomology ring of W_n - the solution of this problem in light of the above remarks might be seen as a local solution to the cohomology of $\mathfrak{X}(M)$. We state without much discussion the theorem here.

Let $U(n)$ be the nth unitary group and consider the universal principal $U(n)$-bundle

$$U(n) \to EU(n) \xrightarrow{p} BU(n).$$

Recall from topology the explicit construction as $BU(n) = G_n(\mathbb{C}^\infty)$, the infinite Grassmanian of n-planes, and $EU(n) = V_n(\mathbb{C}^\infty)$, the Stiefel manifold of orthonormal n-frames. The standard Schubert cell structure on $G_n(\mathbb{C}^\infty)$ has cells in even dimensions, and we set

$$X_n \coloneqq p^{-1}(\mathrm{sk}_{2n} G_n(\mathbb{C}^\infty)) \subset V_n(\mathbb{C}^\infty)$$

with the subspace topology. The theorem is then:

Theorem 1.3.1 (Gelfand-Fuchs) *There is an isomorphism of graded rings*

$$H^*(W_n) \cong H^*(X_n; \mathbb{R})$$

between the Gelfand-Fuchs cohomology of W_n and the singular cohomology of X_n in $\mathbb{R}$-coefficients. Moreover, the multiplication in both sides is trivial in positive degrees.

A somewhat complicated exposition is given in [Fuk86, §2.2.2]. We note that the cohomology of X_n *is* computable in practice in light of the explicit description of Schubert cells of dimension less than $2n$ via Young diagrams; hence X_n is simply the subspace of orthonormal bases of all the n-planes for these cells. $H^q(X_n; \mathbb{R})$ is zero for q outside of the interval $2n + 1 \leq q \leq n(n + 2)$, and inside the interval it can be given in terms of the partition number. The case we will need is $n = 1$, and it can be done by hand easily so we will prove that case in Sec. 3.

Remark 1.3.2 It is worth mentioning from where the seemingly random topological space X_n comes from in Thm. 1.3.1. The point is that the proof relies on the Serre-Hochschild spectral sequence for W_n with respect to its Lie subalgebra $\mathfrak{gl}_n(\mathbb{R})$ (see [Fuk86, 1.5] for its construction in general), and this is shown to have the same E_2-page and differentials as the Serre spectral sequence for the fibration

$$U(n) \to X_n \to \mathrm{sk}_{2n} G_n(\mathbb{C}^\infty),$$

which eventually gives rise to the theorem.

2 Diagonal Cohomology for Vector Fields

As explained in Sec. 1, we will now explore the main tool of Gelfand-Fuchs cohomology, which is the diagonal filtration of the cochain complex for the Lie algebra $\mathfrak{X}(M)$. Throughout this section, M is a fixed closed oriented smooth manifold of dimension n. We will leave the special case of $M = S^1$, our main goal, to the next section.

2.1 Diagonal Filtration & Generalized Sections

We will need some basic results from distribution theory for some our notions here, particularly for that of a 'generalized section'. If $E \to M$ is a vector bundle on M, then $\Gamma(E)$ denotes the $C^\infty(M)$-module of smooth sections of E, and we recall that $\Gamma(E)$ is topologized with the C^∞-topology as explained in Sec. 1.3. Viewing $\Gamma(E)$ as simply an $\mathbb{R}$-vector space, we write $\Gamma(E)_c^*$ for the *continuous dual space*, namely the continuous functions $\Gamma(E) \to \mathbb{R}$ which are also $\mathbb{R}$-linear maps. A *generalized section of* E is an element of $\Gamma(E)_c^*$.

Proposition 2.1.1 *Let $E_i \to M_i$ be vector bundles for $i = 1, 2$ and write π_1, π_2 for the projection maps from $M_1 \times M_2$.*

1. *The natural map*
$$\Gamma(E_1) \otimes_{\mathbb{R}} \Gamma(E_2) \to \Gamma(\pi_1^* E_1 \otimes \pi_2^* E_2)$$
is a topological embedding, where the source is given the final topology with respect to the canonical bilinear map $\Gamma(E_1) \times \Gamma(E_2) \to \Gamma(E_1) \otimes \Gamma(E_2)$.
2. *Every continuous linear functional on $\Gamma(E_1) \otimes \Gamma(E_2)$ extends uniquely to a continuous linear functional on $\Gamma(\pi_1^* E_1 \otimes \pi_2^* E_2)$. In other words, the map*
$$\Gamma(\pi_1^* E_1 \otimes \pi_2^* E_2)_c^* \to \big(\Gamma(E_1) \otimes_{\mathbb{R}} \Gamma(E_2)\big)_c^*$$
induced by the embedding in (1) is an isomorphism of vector spaces.

Proof (1) is just unwinding the definitions, and (2) relies on the Schwartz kernel theorem and a partition of unity. See [GF69, 2.3] for details. □

By the definitions, a continuous linear functional on $\Gamma(E_1) \otimes \Gamma(E_2)$ is the same data as a bilinear map $\Gamma(E_1) \times \Gamma(E_2) \to \mathbb{R}$ which is continuous with respect to the product topology. Therefore, Prop. 2.1.1 shows that such a continuous bilinear functional can be regarded as a generalized section of a tensor product bundle on the product manifold.

Definition 2.1.2 Let $E \to M$ be a vector bundle, and $A \subset M$ be a closed subspace. Let $F \in \Gamma(E)_c^*$ be a generalized section of E.

1. F is *concentrated on* A if for any section s such that $s \equiv 0$ on a neighborhood of A, we have $F(s) = 0$.
2. If F is concentrated on A, then F has *order* $\leq k$ if $F(s) = 0$ for every section s such that the corresponding section
$$j^k s \colon M \to J^k(M, E)$$
of the jet bundle vanishes on A.

By linearity, if F is concentrated on A, then $F(s) = F(t)$ if $s \equiv t$ on a neighborhood of A. So, F may be regarded as a linear functional on sections over the generalized stalk on A, that is $\varinjlim_{U \supset A} \Gamma(U)$. By the compactness of

M, it is a fact that there always exists some k such that F has order $\leq k$, so we can say that the order of F is the smallest such k. For example, if F has order zero, then it is determined completely by sections over A. If F has order one, it is determined by the linear approximations on A of sections defined on a neighborhood of A.

This langauge will help us study the diagonal filtration we now define. A collection $\{V_1, \dots, V_q\}$ has the property (Δ_k) for some $k \geq 0$ if for any subset $A \subset M$ of k points, there is some i and a neighborhood U of A such that $V_i \equiv 0$ on A.

Any collection vacuously has (Δ_0), and it has (Δ_1) iff $\bigcap_i \mathrm{Supp} V_i = \varnothing$. When $k \geq q$ the, the fields have (Δ_k) iff one of them is identically zero, because if not then there are k distinct points on which none of the fields vanish.

Definition 2.1.3 1. A cochain $\alpha \in C^q(\mathfrak{X}(M))$ *has diagonal filtration* $\leq k$ if α vanishes on any collection of vector fields with (Δ_k).

2. The subspace $\Delta_k C^q(\mathfrak{X}(M)) \subset C^q(\mathfrak{X}(M))$ consists of cochains of diagonal filtration $\leq k$. For the case $k = 1$, we write $C^q_\Delta(\mathfrak{X}(M))$.

Proposition 2.1.4 $\Delta_* C^*(\mathfrak{X}(M))$ *defines an increasing multiplicative filtration of* $C^*(\mathfrak{X}(M))$ *bounded in each degree by the cochain dimension.*

Proof First, we show $d(\Delta_k C^q(\mathfrak{X}(M))) \subset \Delta_k C^{q+1}(\mathfrak{X}(M))$, for which we may prove that if $V_1, \dots, V_q$ have (Δ_q) then the collection

$$[V_i, V_j], V_1, \dots, \widehat{V_i}, \dots, \widehat{V_j}, \dots, V_q$$

also has Δ_k. This is obvious because if e.g. V_i vanishes on an open set U, then we have $[V_i, V_j]|_U = [V_i|_U, V_j|_U] = 0$. Next, it is also clear that

$$\Delta_k C^q(\mathfrak{X}(M)) \subset \Delta_{k+1} C^q(\mathfrak{X}(M))$$

as any collection of vector fields with (Δ_{k+1}) must also have (Δ_k). The multiplication rule

$$\Delta_k C^q(\mathfrak{X}(M)) \cdot \Delta_l C^r(\mathfrak{X}(M)) \subset \Delta_{k+l} C^{q+r}(\mathfrak{X}(M))$$

is also evident from the definition of the product. The boundedness is clear because we already said all collections have (Δ_0), so the corresponding cochains are zero, and a collection has (Δ_q) exactly when one of the fields are zero, hence all cochains vanish on it by multilinearity. □

Therefore we can take diagonal cohomology, and it will inherit a natural grading as well. Of course, we can also take the associated spectral sequence but we won't do this until Sec. 3. Now we interpret this with generalized sections. The remark after Prop. 2.1.1 shows that continuous multilinear functions of vector fields

$$\mathfrak{X}(M)^q = \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to \mathbb{R}$$

are identified with generalized sections of the tensor product bundle $TM^{\otimes q} := \bigotimes_i \pi_i^* TM$ over the manifold $M^q = M \times \cdots \times M$. Therefore, a q-dimensional cochain is such a generalized section which is also *skew-symemtric* with respect to the factors. More precisely, this means that for any $\sigma \in S_q$, which induces a diffeomorphism $\sigma \colon M^q \to M^q$ permuting the factors, we have

$$F(M^q \xrightarrow{\sigma} M^q \xrightarrow{s} TM^{\otimes q}) = \operatorname{sgn}(\sigma) F(s) \qquad s \in \Gamma(TM^{\otimes q})$$

where F is the generalized section in question.

Continuing this identification, let us define the subset

$$M_k^q = \left\{(x_1, \dots, x_q) \in M^q : |\{x_1, \dots, x_q\}| \leq k\right\}$$

whose points have at most k distinct coordinates. This is a closed submanifold of M^q because it is a union of the images of the finitely-many maps $M^k \to M^q$ that correspond on the factors to the order-preserving maps $\{1, \dots, k\} \to \{1, \dots, q\}$. By definition, a q-dimensional cochain is in $\Delta_k C^q(\mathfrak{X}(M))$ if and only if it is concentrated on $M_k^q \subset M^q$. So, $C_\Delta^q(\mathfrak{X}(M))$ consists of those skew-symmetric generalized sections concentrated on the diagonal.

In what follows we will mostly focus on the chain complex $C_\Delta^*(\mathfrak{X}(M))$, with its cohomology $H_\Delta^*(\mathfrak{X}(M))$. To compute it, we will define a new filtration $F^* C_\Delta^*(\mathfrak{X}(M))$ on $C_\Delta^*(\mathfrak{X}(M))$ incorporating the order. Namely, write $F^m C_\Delta^q(\mathfrak{X}(M))$ for the subspace of $C_\Delta^q(\mathfrak{X}(M))$ consisting of cochains whose corresponding generalized sections have order $\leq q - m$ on the diagonal $\Delta \subset M^q$. If $m > q$, we define this to be 0.

Having a trivial $q-m$-jet on Δ implies having a trivial $q-m+1 = q-(m-1)$-jet, so we have a decreasing filtration

$$0 = F^{q+1} C_\Delta^q(\mathfrak{X}(M)) \subset F^q C_\Delta^q(\mathfrak{X}(M)) \subset F^{q-1} C_\Delta^q(\mathfrak{X}(M)) \subset \cdots,$$

but we observe that q need not stop at 0, and instead goes to $-\infty$. However, since every cochain in $C_\Delta^q(\mathfrak{X}(M))$ is concentrated on the diagonal Δ, then as explained compactness implies that it has an order, and hence $\bigcup_{m=-\infty}^q F^m C_\Delta^q(\mathfrak{X}(M)) = C_\Delta^q(\mathfrak{X}(M))$. Moreover:

Proposition 2.1.5 *$F^* C_\Delta^*(\mathfrak{X}(M))$ is a decreasing filtration of $C_\Delta^*(\mathfrak{X}(M))$.*

Proof All that remains is to prove that the differential d preserves the grading F^*. By linearity, this reduces to showing that the generalized section

$$F(V_1, \dots, V_{q+1}) = G([V_i, V_j], \dots, \widehat{V_i}, \dots, \widehat{V_j}, \dots)$$

has a trivial $(q-m)$-jet on the diagonal, assuming that $G(V_1, \dots, V_q)$ does. Working in local coordinates, this is induction, a simple application of the multivariable chain rule, and the fact that $[V, V] = 0$ for all V. □

Denote by $(E_*^{p,q}, d_*)$ the corresponding cohomological spectral sequence associated to this filtration. So, E_2 gives us the associated graded of $F^* H_\Delta^*(\mathfrak{X}(M))$.

2.2 Results on Diagonal Cohomology

Now we come to the computation of the E_2-page of the spectral sequence defined in the previous section. The two major theorems in this section, Thm. 2.2.2 and Thm. 2.2.4, will be the main tools in computing the cohomology of $\mathfrak{X}(M)$. We will give a brief overview of Gelfand and Fuchs' proof of these results without going into too much detail.

Throughout this subsection we will abbreviate $a \coloneqq p+q$. As $\Delta \subset M^a$ is a closed submanifold, the immersion $\Delta \hookrightarrow M^a$ canonically identifies

$$T\Delta \subset TM^a = TM \oplus \cdots \oplus =: (TM)^{\oplus a}.$$

as a subbundle. Recall that the quotient bundle pulled back (i.e. restricted) to Δ is the *normal bundle* written $N\Delta$, a bundle on Δ. We will frequently identify $M = \Delta$.

Consider the following inclusion of vector bundles over Δ:

$$\mathcal{E}_0^{p,q} \coloneqq \mathrm{Alt}(S^q(N\Delta), TM^{\otimes a}\big|_\Delta) \subset \widehat{\mathcal{E}}_0^{p,q} \coloneqq \mathrm{Hom}(S^q(N\Delta), TM^{\otimes a}\big|_\Delta).$$

On the right, we consider the Hom-bundle of multilinear symmetric maps

$$N\Delta \times \cdots \times N\Delta \to TM^{\otimes a}\big|_\Delta.$$

For $x \in M$, the fiber of $\widehat{\mathcal{E}}_0^{p,q}$ over x consists of maps

$$(T_xM)^{\oplus a} \times \cdots \times (T_xM)^{\oplus a} \to (T_xM)^{\otimes a}$$

which are symmetric in the $\times$-factors on the left, and which send the diagonal subspace of each factor to zero, i.e.

$$\big((v_1, \ldots, v_1), \ldots, (v_q, \ldots, v_q)\big) \mapsto 0$$

for any tangent vectors $v_1, v_2, \ldots, v_q \in T_xM$. The notation Alt in the definition of $\mathcal{E}_0^{p,q}$ means to only take those homomorphisms which develop the sign $\mathrm{sgn}(\sigma)$ when the factors of M^a are permuted by $\sigma \in S_a$, where we note that the permutation takes place in *every* $\times$-factor, and the $\otimes$-factors.

Lemma 2.2.1 *We have a natural identification* $E_0^{p,q} = \Gamma(\mathcal{E}_0^{p,q})_c^*$.

Proof The E_0-page of the spectral sequence associated to a filtration has the form

$$E_0^{p,q} = F^pC_\Delta^a(\mathfrak{X}(M))/F^{p+1}C_\Delta^a(\mathfrak{X}(M)),$$

hence we take the quotient of the cochains with trivial $a-p=q$-jet by the chains with trivial $(q-1)$-jet on $\Delta \subset M^a$. Therefore, we work up to cochains vanishing exactly up to order q. As explained before, since all the cochains are concentrated on the diagonal, the data is determined by a generalized section on a bundle over Δ, defined up to choice of a q-jet.

On a local neighborhood, the q-jet is given by qth symmetric power because it is just a choice of qth Taylor polynomial: the description of the diagonal factored out is because the sections are concentrated on the diagonal. Finally, we take

Alt because our cochains are not only multilinear, but are skew-symmetric as before. □

The computation of the E_1-page is significantly more difficult because it involves a filtration of the algebra of formal vector fields W_n defined in Sec. 1.3; so we state here the result without proof.

Theorem 2.2.2 *There is a canonical isomorphism*

$$E_1^{p,q} \cong \Gamma(\Lambda^{-p} TM')_c^* \otimes H^q(W_n)$$

for all p, q.

Corollary 2.2.3 ([Fuk86, Thm. 2.4.1a]) *For all* p, q *such that* $p + q > 0$,

$$E_2^{p,q} \cong H_{-p}(M; \mathbb{R}) \otimes H^q(W_n),$$

where the first tensor factor is the integral homology of M.

Proof The first tensor factor in the isomorphism of Thm. 2.2.2 can be identified as the de Rham cohomology of M with shifted differential, i.e. $\Omega^{-p}(M) \otimes H^q(W_n)$. A corollary to that proof shows that the differential on the E_1-page is just the de Rham differential

$$\Omega^{-p}(M)' \to \Omega^{-p-1}(M)'$$

tensored with the identity map $H^q(W_n) \to H^q(W_n)$. Therefore

$$E_2^{p,q} = H^{-p}(M)' \otimes H^q(W_n) = H_{-p}(M) \otimes H^q(W_n)$$

which is what we wanted. □

As explained, in most cases this lets us give a complete description of the 1-diagonal cohomology $H^*_\Delta(\mathfrak{X}(M))$. The same proof with many more symbols applies to the complexes $\Delta_k C^*(\mathfrak{X}(M))$, which are also filtered by order bound of generalized sections as in the case of $k = 1$. The associated spectral sequence is written ${}^{(k)}E_r^{p,q}$, and the following result is the generalization to $k > 1$:

Theorem 2.2.4 *Let* $k > 1$. *Then* ${}^{(k)}E_2^{p,q}$ *is the alternating subspace of*

$$H_{-p}(M^k, M^k_{k-1}) \otimes \bigoplus_{q_1 + \cdots + q_k = q} (H^{q_1}(W_n) \otimes \cdots \otimes H^{q_k}(W_n)),$$

that is the subspace of those elements multiplied by $\operatorname{sgn}(\sigma) \in \{\pm 1\}$ *for some* $\sigma \in S_{p+q}$ *whenever the automorphism on* M^{p+q} *induced by* σ *is applied to the tensor factors and in* M^k *itself.*

Once again, this is practically computable; one simply uses the long exact sequence of a pair in homology, the Kunneth formula, and cellular cohomology on the space X_n whose cohomology computes $H^*(W_n)$ as discussed in Sec. 1.3.

Therefore, the problem has been completely converted into an elementary use of ordinary topology. Let us mention one more important result here.

Corollary 2.2.5 *We have* $\dim H^q(\mathfrak{X}(M)) < \infty$ *for all* q.

Proof The spectral sequences of Cor. 2.2.3 and Thm. 2.2.4, as well as substituting Thm. 1.3.1 (which is the cohomology of a finite CW complex with $\mathbb{R}$-coefficients), together show that $H^*(\Delta_k C^*/\Delta_{k-1}C^*(\mathfrak{X}(M))) < \infty$. The spectral sequence associated to the diagonal filtration shows that the associated graded of $\Delta_* H^*(\mathfrak{X}(M))$ is finite-dimensional, and this is bounded below so we are done by induction. □

3 Cohomology of the Loop Algebra

Here we will finally discuss the Gelfand-Fuchs cohomology of $\mathfrak{X}(S^1)$, the topological Lie algebra of vector fields on a circle. In this particular case, the spectral sequences from the previous section give us much more information and the one associated to the diagonal filtration degenerates immediately. Moreover, we can find explicit cocycles which generate the cohomology ring.

First, to apply either Cor. 2.2.3 or Thm. 2.2.4, we need to know the cohomology of W_1 because $\dim(S^1) = 1$.

Proposition 3.0.1 *There is an isomorphism of rings* $H^*(W_1) \cong \mathbb{R}[g]/g^2$, *where* $\deg(g) = 3$.

Proof Recall from Thm. 1.3.1 that

$$H^*(W_1) \cong H^*(X_1),$$

where X_n is the preimage of the $2n$-skeleton under the fibration

$$V_n(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty).$$

When $n = 1$, this reduces to the standard fiber bundle $S^\infty \to \mathbb{C}P^\infty$, which is the direct limit of the finite Hopf bundles. In particular, the diagram

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^{2k+1} & \longrightarrow & \mathbb{C}P^k \\ \| & & \downarrow & & \downarrow \\ S^1 & \longrightarrow & S^\infty & \longrightarrow & \mathbb{C}P^\infty \end{array}$$

commutes for all k. But the cell structure of $\mathbb{C}P^\infty$ is particularly simple, with $2k$-skeleton equal to $\mathbb{C}P^k$; hence, taking $k = 1$ shows that the preimage of $\mathbb{C}P^1$ under the fibration is just S^3, so $X_1 = S^3$. This implies the result. □

Knowing $H_1(S^1)$, this shows by Cor. 2.2.3 that $E_2^{p,q}$ is $\mathbb{R}$ if

$$(p, q) = (0, 0), (-1, 3), (0, 3),$$

and 0 otherwise. So, for degree reasons $H^q_\Delta(\mathfrak{X}(M))$ is $\mathbb{R}$ if $q = 2, 3$ and 0 otherwise. We can do a similar computation for the higher Δ_k's; the key to that computation is the fact that there is a homotopy equivalence

$$M^k/M^k_{k-1} \simeq \Sigma M^{k-1}$$

when $M = S^1$ with the reduced suspension, so by the Mayer-Vietoris sequence, the mth homology is equal to $\widetilde{H}_{m-1}(M^{k-1})$, which lets us apply the $k = 1$ case and use induction. In general, we find that

$$H^q(\Delta_k C^*/\Delta_{k-1}C^*(\mathfrak{X}(M)))$$

is $\mathbb{R}$ when $q = 2k, 2k+1$ and zero otherwise. We will derive our main result by explicitly finding cochains to represent the generators in the $2k$ and $2k+1$ case. Since S^1 is parallelizable, we can view vector fields as certain smooth real-valued functions $S^1 \to \mathbb{R}$. Define a cocycle α in $C^2(\mathfrak{X}(S^1))$ by

$$\alpha(f, g) = \int_{S^1} (f'g'' - f''g').$$

Observe that

$$(d\alpha)(f, g, h) = \alpha([f, g], h) - \alpha([f, h], g) + \alpha([g, h], f) = 0$$

because the integrands all cancel out. In addition, this is a 2x2 determinant of linear functionals in f and g, so it is skew-symmetric. Therefore α defines a cocycle. We do the same thing by defining $\beta \in C^3(\mathfrak{X}(S^1))$, except take a 3x3 determinant, using the 0th, first, and second derivatives across the rows in the integrand. A similar verification shows that β is a cocycle.

Theorem 3.0.2 *We have an identification of rings*

$$H^*(\mathfrak{X}(S^1)) = \mathbb{R}[\alpha, \beta]/(\beta^2) \qquad \deg \alpha = 2, \deg \beta = 3$$

in cohomology.

Proof Identify α^k and $\alpha^{k-1}\beta$ with their images inside the cohomologies

$$H^{2k}(\Delta_k C^*/\Delta_{k-1}C^*(\mathfrak{X}(M))) \qquad H^{2k+1}(\Delta_k C^*/\Delta_{k-1}C^*(\mathfrak{X}(M))),$$

the only nontrivial cohomology groups. We know these spaces are isomorphic to $\mathbb{R}$, a field, so to show α^k and $\alpha^{k-1}\beta$ are generators it suffices to show they aren't boundaries. Note that all of the cohomology is one-dimensional and again $\mathbb{R}$ is a field, so we may assume $k = 1$ from the definition of the multiplicative structure (they commute and $\beta^2 = 0$ for degree reasons).

It is easy to that $[\alpha] \neq 0$ by directly inspecting the formula for the first boundary map, whereas α can produce other nontrivial values. For β, a similar thing can be done but it is not as obvious. Instead, we observe that β is cohomologous to the cocycle where we remove the integral and just evaluate at some arbitrary $t \in S^1$ (the difference is clearly a 3-coboundary). This gives what we want, and in light of these generators it follows that the diagonal spectral sequence (for the diagonal filtration) collapses immediately. □

References

[Fuk86] D. B. Fuks, *Cohomology of infinite-dimensional Lie algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986, Translated from the Russian by A. B. Sosinskiĭ. MR 874337

[GF68] I. M. Gel'fand and D. B. Fuks, *Cohomologies of the Lie algebra of vector fields on the circle*, Funkcional. Anal. i Priložen. **2** (1968), no. 4, 92–93. MR 245035

[GF69] ———, *Cohomologies of Lie algebra of tangential vector fields of a smooth manifold*, Functional Analysis and Its Applications **3** (1969), no. 3, 194–210.

[Hae75] A. Haefliger, *Sur la cohomologie de Gelfand-Fuchs*, Differential Topology and Geometry (Berlin, Heidelberg) (Gérard P. Joubert, Robert P. Moussu, and Robert H. Roussarie, eds.), Springer Berlin Heidelberg, 1975, pp. 121–152.

[Mic80] Peter W. Michor, *Manifolds of differentiable mappings*, Shiva Mathematics Series, vol. 3, Shiva Publishing Ltd., Nantwich, 1980. MR 583436