Derived Categories in Algebraic Geometry

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CHAPTER 1

Background

1.1. Homological algebra

At least at the beginning, we will take the following very standard inputs from algebra:

1. The notion of an abelian category, which will usually be denoted by letters like $\mathcal{A}, \mathcal{B}$, etc. Moreover, if we make element-wise arguments then we may be implicitly using the fact that arguments can be extended via Mitchell embedding.
2. Elementary homological algebra facts like the five-lemma, the four lemmas, the snake lemma, and the fact that a short exact sequence of chain complexes gives rise to a long exact sequence in cohomology.
3. The spectral sequence associated to a filtration, which is described below.
4. The basic theory of cones, which is described below.

Let $\mathcal{A}$ be an abelian category. We denote by $\mathrm{Ch}(\mathcal{A})$ the category of $\mathbb{Z}$-graded chain complexes with values in $\mathcal{A}$, with cohomological indexing convention, meaning its objects are sequences $C^\bullet$ of objects $\{C^i : i \in \mathbb{Z}\}$ in $\mathcal{A}$ with differentials $d\colon C^i \to C^{i+1}$ satisfying $d^2 = 0$. Its cohomology is denoted $H^i(C^\bullet)$ as usual. The homotopy category $\mathrm{K}(\mathcal{A})$ is the category with the same objects as $\mathcal{A}$, with morphisms as homotopy classes of maps.

1.1.1. Standard spectral sequences. Let $C^\bullet$ be an object of $\mathrm{Ch}(\mathcal{A})$. A *(descending) filtration* of this complex is a sequence $\{F^pC^\bullet : p \in \mathbb{Z}\}$ of subcomplexes of $C^\bullet$ such that:

(a) $F^{p+1}C^\bullet F^pC^\bullet$ for all $p \in \mathbb{Z}$; and
(b) $\bigcup_p F^pC^\bullet = C^\bullet$.

The filtration is *separated* if $\bigcap_p F^pC^\bullet = 0$. Given a filtration, we can form its *associated graded* $\mathrm{gr}_F C^\bullet = \bigoplus_{p\in\mathbb{Z}} \mathrm{gr}_F^p C^\bullet$, where $\mathrm{gr}_F^p C^\bullet = F^pC^\bullet/F^{p+1}C^\bullet$. For each $p, q \in \mathbb{Z}$, write $E_0^{p,q} := (\mathrm{gr}_F^p C^\bullet)^{p+q} = \mathrm{gr}_F^p C^{p+q}$. Then the differential on $\mathrm{gr}_F^p C$ goes between

$$d_0\colon E_0^{p,q} = \mathrm{gr}_F^p C^{p+q} \to \mathrm{gr}_F^p C^{p+q+1} = E_0^{p,q+1},$$

i.e. it is a vertical differential. Now take cohomology: so,

$$E_1^{p,q} := H^{p+q}(\mathrm{gr}_F^p C) = H^{p+q}(F^pC/F^{p+1}C).$$

Now there is a natural horizontal differential $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$ which works as in the following diagram:

$$\begin{array}{cc} F^pC^{p+q+1}/F^{p+1}C^{p+q+1} & F^{p+1}C^{p+q+2}/F^{p+2}C^{p+q+2} \\ {\scriptstyle d_0}\uparrow & {\scriptstyle d_0}\uparrow \\ F^pC^{p+q}/F^{p+1}C^{p+q} & F^{p+1}C^{p+q+1}/F^{p+2}C^{p+q+1} \\ {\scriptstyle d_0}\uparrow & {\scriptstyle d_0}\uparrow \\ F^pC^{p+q-1}/F^{p+1}C^{p+q-1} & F^{p+1}C^{p+q}/F^{p+2}C^{p+q} \end{array}$$

This is part of the E_0 page, and to get to E_1 we take cohomology at each step. Suppose we have a cocycle in $E_1^{p,q}$ represented by an element $[x]$ with $x \in F^pC^{p+q}$. Since $d[x] = 0$, $dx \in F^{p+1}C^{p+q+1}$, which we claim represents a well-defined element of $E_1^{p+1,q}$. Here's what we have to check:

i. $[dx] \in F^{p+1}C^{p+q+1}/F^{p+2}C^{p+q+1}$ is a cocycle: $d[dx] = [d^2x] = 0$.
ii. If $x \in F^{p+1}C^{p+q}$, then dx is by definition in the image of the differential $E_0^{p+1,q-1} \to E_0^{p+1,q}$, so we obtain a well-defined map $Z_1^{p,q} \to E_1^{p+1,q}$.
iii. Finally, if $x = dy$ for $y \in F^pC^{p+q-1}$, then $dx = 0$, so this descends to a well-defined map $d_1 \colon E_1^{p,q} \to E_1^{p+1,q}$.

Lemma 1.1.1. *Let $C^\bullet$ be an object of $\operatorname{Ch}\mathcal{A}$ and $F^*C^\bullet$ be a descending filtration. There is a canonical spectral sequence $(E_r^{\bullet,\bullet}, d_r)$ with initial pages*

$$E_0^{p,q} = \operatorname{gr}_F^p C^{p+q} \qquad E_1^{p,q} = H^{p+q}(\operatorname{gr}_F^p C^\bullet),$$

and with d_0 and d_1 as described above. In general, the differential has signature

$$d_r \colon E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

This is called the spectral sequence associated to the filtration F of $C^\bullet$.

Now we state the standard conditions for convergence. First of all, the filtration $F^*C^\bullet$, because it respects the differential, induces a descending filtration $F^*H^\bullet(C)$ on the cohomology of C, which is induced from the chain map $F^pC^\bullet \to C^\bullet$ as follows:

$$F^pH^n(C) \coloneqq \operatorname{im}\big(H^n(F^pC) \to H^n(C)\big).$$

We say that the filtration is *finite in cohomological degree* if for each $n \in \mathbb{Z}$, there are integers $a_n \le b_n$ such that $F^pC^n = C^n$ for $p \le a_n$, and $F^pC^n = 0$ for $p \ge b_n$. In other words, for every n, the set

$$\{p \in \mathbb{Z} : \operatorname{gr}_F^p C^n \neq 0\}$$

is finite; equivalently, along each diagonal $\{(p,q) : p+q = n\}$ of $\mathbb{Z}^2$, $E_0^{p,q}$ is nonzero for only finitely many terms. So, this includes cases like when E_0 is contained in the first or third quadrant, or e.g. if it is contained in an infinitely long horizontal strip but bounded above and below.

Now in general, at a point (p,q), the differentials leaving this point for all r are of the form $(p,q) \to (p+r, q-r+1)$, which always adds to $p+q+1$. In particular, they all lie on a fixed diagonal, and likewise for those differentials

entering (p, q). Thus under this condition, the point (p, q) will stop changing once $r \gg 0$ (depending on p and q). So in this case, the E_∞ page is well-defined without having to take any limits.

Lemma 1.1.2. *Let $C^\bullet$ be an object of $\operatorname{Ch}\mathcal{A}$, let F be a descending filtration of $C^\bullet$, and let (E_r, d_r) be the associated spectral sequence. If F is finite in cohomological degree, then there is a canonical isomorphism*

$$E_\infty^{p,q} \cong F^p H^{p+q}(C)/F^{p+1}H^{p+q}(C) =: \operatorname{gr}_F^p H^{p+q}(C).$$

In other words, the spectral sequence converges

$$E_r^{p,q} \implies H^{p+q}(C^\bullet)$$

with the associated filtration F of the cohomology.

Recall how this convergence is used: to get information about the nth cohomology of C, we look along the diagonal $E_\infty^{p,q}$ for $p + q = n$, which will necessarily be contained in the finite horizontal strip $a_n \le p \le b_n$. If we take the direct sum along this diagonal, we get the graded object $\operatorname{gr}_F H^n(C)$ in p.

There are a few immediate deductions: if there's at most one object $E_\infty^{p,q}$ in the n-diagonal which is nonzero, then $H^n(C) \cong E_\infty^{p,q}$, because the filtration must be of the form

$$H^n = F^{a_n}H^n \supset \cdots \supset F^pH^n \supset \cdots F^{b_n}H^n = 0,$$

where all terms to the right (and excluding) F^pH^n are zero, and those to the left are $E_\infty^{p,q}$. In general, we have to solve some extension problems to conclude: namely, for each p we have an exact sequence

$$0 \to F^{p+1}H^n \to F^pH^n \to E_\infty^{p,n-p} \to 0,$$

and starting with $F^{b_n}H^n = 0$, so $F^{b_n-1}H^n = E_\infty^{b_n-1,n+1-b_n}$, we can try to solve downwards for F^pH^n until we've gotten to $F^{a_n}H^n = H^n$. In practice, it is worth mentioning that when we are working with finite-dimensional vector spaces, we have $\dim \operatorname{gr}_F H^n(C) = \dim H^n(C)$, so it is especially easy to conclude dimensions from the spectral sequence.

In these notes, we won't care too much about this spectral sequence in this level of generality; instead we will use much more the *spectral sequence associated to a bicomplex or double complex*. A double complex is a bigraded collection of objects $\{C^{i,j} : i, j \in \mathbb{Z}\}$ of an abelian category $\mathcal{A}$, equipped with a vertical differential d_v and a horizontal differential d_h satisfying the following properties:

(a) For each (i, j), the vertical differentials

$$d_v \colon C^{i,j} \to C^{i,j+1} \qquad j \in \mathbb{Z}$$

make $(C^{i,\bullet}, d_v)$ a chain complex for all i.

(b) Likewise, the horizontal differentials $d_h \colon C^{i,j} \to C^{i+1,j}$ for all i, j make $(C^{\bullet,j}, d_h)$ a chain complex for all j.

(c) The differentials satisfy $d_v d_h + d_h d_v = 0$ for all i, j.

If $C^{\bullet,\bullet}$ is a bicomplex with values in $\mathcal{A}$, then there is an associated chain complex called the *totalization*, given by

$$(\operatorname{Tot} C)^n \coloneqq \bigoplus_{p+q=n} C^{p,q},$$

where the differential $(\operatorname{Tot} C)^n \to (\operatorname{Tot} C)^{n+1}$ is defined with (p,q)th component $C^{p,q} \to \bigoplus_{r+s=n+1} C^{r,s}$ given by

$$C^{p,q} \xrightarrow{d_v+d_h} C^{p+1,q} \oplus C^{p,q+1} \hookrightarrow (\operatorname{Tot} C)^{n+1}.$$

Then $(d_v + d_h)^2 = d_v^2 + d_v d_h + d_h d_v + d_h^2 = 0$ by definition.

We will have legitimate interest in complexes which are totalizations of bicomplexes.

Example 1.1.3. Suppose $\mathcal{A}$ has a symmetric monoidal structure compatible with the abelian structure, and let A, B be chain complexes. There is a natural bicomplex given on objects by $C^{p,q} \coloneqq A^p \otimes B^q$, where the horizontal differential is

$$d_h = d_A \otimes B^q \colon A^p \otimes B^q \to A^{p+1} \otimes B^q,$$

and the vertical differential is

$$d_v = (-1)^p A^p \otimes d_B \colon A^p \otimes B^q \to A^p \otimes B^{q+1}.$$

Without the sign change on d_v, the two differentials would have commuted, but as defined they anticommute. Thus $C^{\bullet,\bullet}$ forms a bicomplex, and we define the *tensor product* $A \otimes B \coloneqq \operatorname{Tot} C$. Explicitly,

$$(A \otimes B)^n = \bigoplus_{p+q=n} A^p \otimes B^q$$

with differential $d = d_A \otimes 1 + (-1)^p 1 \otimes d_B$. It would be useful to know the relation between the cohomology of the tensor product complex and the cohomologies of the individual complexes.

Now, let C be a bicomplex. There are two natural filtrations of the associated totalization $\operatorname{Tot} C$, induced by two filtrations on the bicomplex as follows. Arrange $C^{p,q}$ in the plane $\mathbb{Z}^2$: the horizontal filtration $F_h C$ is defined by setting $F_h^p C$ to be the same bicomplex except we set everything to the left of the line $\{(p,q) : q \in \mathbb{Z}\}$ to zero. Likewise, $F_v^q C$ is defined by setting everything below $\{(p,q) : p \in \mathbb{Z}\}$ to zero. This induces filtrations on the totalizations:

$$F_h^p \operatorname{Tot} C \coloneqq \operatorname{Tot} F_h^p C \qquad F_v^q \operatorname{Tot} C \coloneqq \operatorname{Tot} F_v^q C,$$

which explicitly on nth cohomological components are

$$F_h^r \operatorname{Tot} C^n = \bigoplus_{p+q=n, p \geq r} C^{p,q} \qquad F_v^s \operatorname{Tot} C^n = \bigoplus_{p+q=n, q \geq s} C^{p,q}.$$

These filtrations, by virtue of Lemma 1.1.1, are going to induce two spectral sequences associated to the totalization. Our goal is to compute the E_0, E_1, and

E_2 pages of these two sequences. From the above description, clearly:

$$F_h^{p+1}(\operatorname{Tot} C)^n = C^{p,n-p} \oplus F_h^p(\operatorname{Tot} C)^n$$
$$F_h^{q+1}(\operatorname{Tot} C)^n = C^{n-q,q} \oplus F_v^q(\operatorname{Tot} C)^n$$

So, if the associated spectral sequences are $E_{h,r}$ and $E_{v,r}$, then we will have $E_{h,0}^{p,q} = \operatorname{gr}_{F_h}(\operatorname{Tot} C)^{p+q} = C^{p,q}$ and likewise $E_{v,0}^{q,p} = C^{p,q}$. Thus, it is clear that the differentials on the first page are just the vertical and horizontal differentials respectively:

$$E_{h,0}^{p,q} = C^{p,q} \xrightarrow{d_v} E_{h,0}^{p,q+1} = C^{p,q+1} \qquad E_{v,0}^{q,p} = C^{p,q} \xrightarrow{d_h} E_{v,0}^{q,p+1} = C^{p+1,q}$$

This is because we took the totalization differential $d_v + d_h$ but quotiented the target of the d_h differential (and likewise for the E_v case). Thus we take vertical (resp. horizontal) cohomology to get to E_1:

$$E_{h,1}^{p,q} = H^q(C^{p,\bullet}, d_v) =: H_v^q(C^{p,\bullet}) \qquad E_{v,1}^{q,p} = H^p(C^{\bullet,q}, d_h) = H_h^p(C^{\bullet,q}).$$

Now return to the definition of the E_1-page differential: we just used the cocycle condition and took the total differential of the objects before taking the filtration quotients. This corresponds to simply viewing the horizontal (resp. vertical) differentials as chain maps and looking at the induced map on cohomology. Thus our E_1 differentials are

$$d_h \colon H_v^q(C^{p,\bullet}) \to H_v^q(C^{p+1,\bullet}) \qquad d_v \colon H_h^p(C^{\bullet,q}) \to H_h^p(C^{\bullet,q+1}).$$

In particular, we have shown the following:

THEOREM 1.1.4. *Let $C^{\bullet,\bullet}$ be a bicomplex with values in $\mathcal{A}$. There are two spectral sequences $(E_{h,r}, d_r)$ and $(E_{v,r}, d_r)$ as described above, which in particular have E_2-pages given by*

$$E_{h,2}^{p,q} = H_h^p(H_v^q(C)) \qquad E_{v,2}^{q,p} = H_v^q(H_h^p(C)).$$

If for each $n \in \mathbb{Z}$, $\{(p,q) : p+q = n, C^{p,q} \neq 0\}$ is finite, then both spectral sequences converge to $H^{p+q}(\operatorname{Tot} C)$, where in the first case the associated filtration is induced by F_h, and in the second it is induced by F_v.

Remark 1.1.5. This data comes with some natural 'edge maps': for simplicity, suppose we are in the circumstance where $C^{p,q}$ is a first-quadrant bicomplex. Let us just consider the first spectral sequence. There are two edge maps

$$H^n(\operatorname{Tot} C) \to H_h^0(H_v^n(C)) \qquad H_h^n H_v^0(C) \to H^n(\operatorname{Tot} C)$$

which either project a total cocycle to the $(0,n)$ component, or include an $(n,0)$ cocycle in the total cohomology. These natural maps factor through E_∞ and the filtration on cohomology. So, if some of these edge components survive till E_∞, then we already know how they are subquotients of the total cohomology.

Example 1.1.6. Suppose we're in the situation of Example 1.1.3. An object X of $\mathcal{A}$ is *flat* if $X \otimes (-)$ is an exact functor. So, if $B^\bullet$ is a complex which has no cohomology (i.e. it is an exact sequence), then $X \otimes B^\bullet$ also has no cohomology.

Now suppose $A^\bullet$ is a complex of flat objects; does the tensor product $A \otimes B$ also have no cohomology?

Let's answer this under the additional assumption that $A^\bullet$ and $B^\bullet$ are bounded above complexes. Form the bicomplex $C^{i,j} = A^i \otimes B^j$ as in Example 1.1.3, and observe that our hypothesis ensures that the bicomplex is cut off at some point along the x-axis and the y-axis, so this looks like a shifted third quadrant spectral sequence. Thus by Thm. 1.1.4, we have a spectral sequence

$$E_2^{p,q} = H_h^p(H_v^q(C)) \implies H^{p+q}(A \otimes B).$$

Note that $H_v^q(C)^p = H^q(A^p \otimes B^\bullet, d_B) = 0$ because each A^p is flat, so $E_2^{p,q} = 0$, and thus $H^n(A \otimes B)$ is zero for all $n \in \mathbb{Z}$. Note that this simple fact would be somewhat difficult to prove just using the definitions, but has a trivial proof via spectral sequences.

Remark 1.1.7. Let us note that Lemma 1.1.2 and Thm. 1.1.4 actually hold in a slightly more general setting: namely, e.g. for the former, we don't need the filtration to be finite in cohomological degree, only bounded towards the right. In other words, we should have the condition that for fixed n, $F^pC^n = 0$ for $p \gg 0$. In our bicomplex situation, this means that we assume $C^{p,n-p} = 0$ for $p \gg 0$ for the first sequence, or $C^{n-q,q} = 0$ for $q \gg 0$ in the second one. However, in this case, to get the convergence result, we need to assume that countable direct sums are exact in the abelian category. In the examples we're interested in this will always hold, because filtered colimits will be exact.

The subtlety is that at a given point $(p, q) \in \mathbb{Z}^2$, the sequence may not stabilize anymore. However, the differentials leaving $E_r^{p,q}$ are all going further down and to the right along the $p + q + 1$ diagonal, so eventually they are zero. Thus, we do have a well-defined cocycle space $Z_\infty^{p,q}$. To take the image of the incoming differential, for large enough r we use the image $B_r^{p,q}$ in $Z_\infty^{p,q}$, assuming $Z_r^{p,q} = Z_\infty^{p,q}$. Then $B_\infty^{p,q} = \bigcup_r B_r^{p,q}$, and $E_{p,q}^\infty = Z_\infty^{p,q}/B_\infty^{p,q}$.

In practice, we usually don't need to worry about this because we hope that the whole sequence degenerates quickly. For example, using this we can now extend Example 1.1.6 to the case where $A^\bullet$ is still a bounded above complex of flat objects, but $B^\bullet$ is an arbitrary acyclic complex.

Remark 1.1.8. Again let $C^{\bullet,\bullet}$ be a bicomplex. Sometimes we care instead about the *product totalization*, denoted $\mathrm{Tot}^\Pi(C)$,

$$\mathrm{Tot}^\Pi(C)^n \coloneqq \prod_{p+q=n} C^{p,q}$$

which is defined if $\mathcal{A}$ has countable products. The differential component mapping to $C^{p,q}$ for $p + q = n + 1$, is given by

$$(dx)_{p,q} = d_h(x_{p-1,q}) + d_v(x_{p,q-1}),$$

which we can still write as $d_h + d_v$. Then again $d^2 = 0$ and this forms a complex. We still want a spectral sequence in this case, so we can use the following trick:

observe that $D^{p,q} \coloneqq C^{-p,-q}$ is a bicomplex in the opposite category $\mathcal{A}^{\mathrm{op}}$, and

$$\left(\operatorname{Tot}_{\mathcal{A}^{\mathrm{op}}} D^{\bullet,\bullet}\right)^n = \bigoplus_{p+q=n}^{\mathcal{A}^{\mathrm{op}}} C^{-p,-q} = \prod_{-p-q=-n} C^{-p,-q},$$

so $\left(\operatorname{Tot}_{\mathcal{A}^{\mathrm{op}}} D^{\bullet,\bullet}\right)^n = \operatorname{Tot}^{\prod}_{\mathcal{A}}(C^{\bullet,\bullet})^{-n}$, and the differentials coincide with what we just wrote down. So, we have two spectral sequences with reversed indices as before, and let us focus on the first one. The boundedness assumption from the previous remark was that along all diagonals, we must eventually get zero if we go far enough to the right. Thus the condition flips, and we obtain the statement

$$E_2^{p,q} = H_h^p(H_v^q(C)) \implies H^{p+q}(\operatorname{Tot} C)$$

assuming that for each n, $C^{p,n-p} = 0$ for $p \ll 0$. If we do the second spectral sequence, then the condition should be that $C^{n-q,q} = 0$ for $q \ll 0$. We will see an example of this in use when defining the Hom complex. Since we used the opposite category, in this case we have to assume that countable products are exact. This will *not* always be true for the categories we care about, so we should be careful.

1.1.2. Cones and homotopy. Let's review some standard facts about cones. Again, fix $\mathcal{A}$ an abelian category, and we generally work in either the category of cochain complexes $\mathrm{Ch}(\mathcal{A})$, or its homotopy category $\mathrm{K}(\mathcal{A})$.

Let A be a chain complex. For $i \in \mathbb{Z}$, we define $A[i]$ to be $(A[i])^n = A^{i+n}$, with $d_{A[i]} = (-1)^i d_A$. Note that e.g. $A[1]$ is the shift of A to the *left*, not the right. This defines a functor $\mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(srcA)$. It is clear that $H^n(A[i]) = H^{i+n}(A)$, and in particular $H^0(A[i]) = H^n(A)$.

Let $f\colon A \to B$ be a chain map. Then $\operatorname{Cone}(f)$ is the chain complex with

$$\operatorname{Cone}(f)^i \coloneqq A^{i+1} \oplus B^i \qquad d_{\operatorname{Cone}(f)} = \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} : A^{i+1} \oplus B^i \to A^{i+2} \oplus B^{i+1}.$$

There is a canonical exact sequence

$$0 \to B \to \operatorname{Cone}(f) \to A[1] \to 0 \tag{1.1.1}$$

where the maps on the object level are the obvious ones: $B^i \to A^{i+1} \oplus B^i$ and $A^{i+1} \oplus B^i \to A^{i+1}$, so this sequence is 'termwise-split', but not split globally because of the extra f component in the differential of the cone. Considering the corresponding long exact sequence, we obtain

$$\cdots \to H^i(B) \to H^i(\operatorname{Cone}(f)) \to H^{i+1}(A) \xrightarrow{\delta} H^{i+1}(B) \to \cdots,$$

where δ is the connecting map; it is easy to check in this case that $\delta = f_*$ is the induced map, and thus we see that f is a quasi-isomorphism if and only if $\operatorname{Cone}(f)$ is acyclic.

Many properties of maps are easily translated into properties of cones. The following properties are important to define the derived category as a well-behaved localization of the homotopy category of chain complexes. They will also be used later to prove that the latter is a triangulated category.

Lemma 1.1.9. *Let $f\colon A \to B$ be a chain map. Then there is a natural homotopy equivalence*

$$\mathrm{Cone}(B \xrightarrow{\tau} \mathrm{Cone}(f)) \simeq A[1],$$

where in fact $A[1]$ is a deformation retract of $\mathrm{Cone}(f)$.

Proof (Sketch). Let us describe what the deformation retract is:

$$A^{i+1} \xrightarrow{(-f \quad 1 \quad 0)^T} B^{i+1} \oplus (A^{i+1} \oplus B^i) \xrightarrow{(0 \quad 1 \quad 0)} A^{i+1}.$$

Then the composite written above is the identity on $A[1]$, and the other composite is a 3x3 matrix which is homotopy equivalent to the identity. □

As in the lemma, let $\tau\colon B \to \mathrm{Cone}(f)$ be the canonical map. Then we have a commutative diagram:

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & A[1] & \xrightarrow{-f} & B[1] \\ & & \| & & \| & & \downarrow{\scriptstyle he} & & \| \\ & & B & \longrightarrow & \mathrm{Cone}(f) & \longrightarrow & \mathrm{Cone}(\tau) & \longrightarrow & B[1] \end{array}$$

This diagram will tell us that we can shift distinguished triangles in the homotopy category. More precisely, we will want to study triangle sequences of the form $A \to B \to \mathrm{Cone}(f) \to A[1]$, and if we 'rotate' such a triangle we get

$$B \to \mathrm{Cone}(f) \to A[1] \xrightarrow{-f} B[1].$$

The diagram above and the lemma show that this rotated triangle is isomorphic to a triangle of the first form.

Lemma 1.1.10. *Let $f\colon A \to B$ be a chain map, and consider the natural map $\eta\colon \mathrm{Cone}(f)[-1] \to A$. Then $\mathrm{Cone}(\eta)$ is canonically homotopy equivalent to $B[1]$.*

Proof. Follows from Lemma 1.1.9 and the homotopy invariance of cones under homotopy equivalences, which is standard. □

Next, we will prove a lemma which is explained by the next section:

Lemma 1.1.11. *Suppose we have morphisms $X \xrightarrow{f} Y \xleftarrow{g} Z$, where f is a quasi-isomorphism. Then we can extend this data to a commutative diagram*

$$\begin{array}{ccc} T & \overset{f'}{\dashrightarrow} & Z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathrm{K}(\mathcal{A})$ where f' is a quasi-isomorphism.

PROOF (SKETCH). Let $\tau\colon Y \to \mathrm{Cone}(f)$ be the canonical map. By our previous results, we have a diagram:

$$\begin{array}{ccccccc}
\mathrm{Cone}(\tau g)[-1] & \longrightarrow & Z & \xrightarrow{\tau g} & \mathrm{Cone}(f) & \longrightarrow & \mathrm{Cone}(\tau g) \\
\downarrow{\scriptstyle \overline{g}'[-1]} & & \downarrow{\scriptstyle g} & & \| & & \downarrow{\scriptstyle \overline{g}} \\
\mathrm{Cone}(\tau)[-1] & \longrightarrow & Y & \xrightarrow[\tau]{} & \mathrm{Cone}(f) & \longrightarrow & \mathrm{Cone}(\tau) \\
\simeq\downarrow{\scriptstyle he} & & \| & & \| & & \simeq\downarrow{\scriptstyle he} \\
X & \xrightarrow{f} & Y & \xrightarrow{\tau} & \mathrm{Cone}(f) & \longrightarrow & X[1]
\end{array}$$

Importantly, the two squares which involve a term that says $he, \simeq$ only commute up to homotopy, because in fact the truly commutative square is the one where we replace them by their homotopy inverses. Also, $\mathrm{Cone}(\tau g) \to \mathrm{Cone}(\tau)$ is the natural map, induced by commutativity of the middle upper square, and this map makes the upper left square commute because we could have added a downward arrow $Z[1] \to Y[1]$ on the right side of the diagram.

To conclude, we want to check that $\mathrm{Cone}(\tau g)[-1] \to Z$ is a qis. Since f is a qis, $\mathrm{Cone}(f)$ is acyclic. Then Lemma 1.1.10 tells us that the cone of $\mathrm{Cone}(\tau g)[-1] \to Z$ is homotopy equivalent to $\mathrm{Cone}(f)$, hence acyclic. □

Lemma 1.1.12. *Let $f, g\colon A \to B$ be chain maps. Then there exists a quasi-isomorphism q for which $qf \sim qg$ if and only if there exists a quasi-isomorphism q' for which $fq' \sim gq'$.*

PROOF. We may assume $g = 0$. The point is that both conditions are equivalent to f factoring through an acyclic complex. □

Lemma 1.1.13. *Let f, g, h be composable morphisms for which fg and gh are quasi-isomorphisms. Then g is a quasi-isomorphism.*

PROOF. To check this we have to check it after applying H^n for all n, after which fg and gh become isomorphisms. But this property is true for isomorphisms in a category. □

1.1.3. Localizing classes. In this section we define localizations of categories in the case of the existence of a localizing class. Let $\mathcal{C}$ be a category and S a subclass of morphisms. By analogy with rings, S is *multiplicative* if it includes all identity morphisms, and is closed under composition.

Recall the definition of the localization $\gamma\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$, which is a category $\mathcal{C}[S^{-1}]$ equipped with a functor γ such that for every category $\mathcal{D}$, restriction along γ induces an equivalence of categories

$$\gamma^*\colon \mathrm{Fun}(\mathcal{C}[S^{-1}], D) \xrightarrow{\sim} \mathrm{Fun}^S(\mathcal{C}, \mathcal{D}),$$

where Fun^S denotes the full subcategory of functors which send elements of S to isomorphisms. Recall moreover that if we allow locally large categories, then we can always construct this category with the same objects as $\mathcal{C}$, so that γ is an identity-on-objects functor, and γ^* above is an isomorphism of categories, not just an equivalence.

The following setup is one way to make localizations behave well (another is to go directly to triangulated categories, and a third is to use model categories).

Definition 1.1.14. Let $\mathcal{C}$ be a category and $S \subset \mathrm{Mor}(\mathcal{C})$ a multiplicative subclass of morphisms. S is a *localizing class* if the following additional conditions hold:

(a) For every $s \in S$ and map $f\colon X \to Y$, we can extend this data to a commutative diagram

$$\begin{array}{ccc} T & \dashrightarrow & Z \\ {\scriptstyle s'}\downarrow & & \downarrow{\scriptstyle s} \\ X & \xrightarrow{f} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s'} \\ Z & \dashrightarrow & T \end{array}$$

so that in both cases, $s' \in S$ as well.

(b) Given two parallel maps $f, g\colon X \to Y$, there exists $s \in S$ for which $sf = sg$ iff there exists $s' \in S$ for which $fs' = gs'$.

The point of this definition is that we have a very explicit description of the localization. Let S be a multiplicative subclass of morphisms; a *roof diagram* from A to B is a pair of arrows of the form $A \xleftarrow{s} X \xrightarrow{f} B$ for which $s \in S$. We temporarily denote this set $\mathrm{Roof}_S(A, B)$, then there is a natural equivalence relation on this set, where

$$(A \xleftarrow{s} X \to B) \sim (A \xleftarrow{s'} X' \to B)$$

iff there exists a commutative diagram of the form

$$\begin{array}{ccccc} A & \xleftarrow{s} & X & \xleftarrow{t} & Y \\ & \nwarrow^{s'} & & & \| \\ B & \leftarrow & X' & \xleftarrow{t'} & Y \end{array}$$

for which $t, t' \in S$.

Now suppose in addition that S is localizing. We are going to construct a category $\mathcal{C}[S^{-1}]$ with objects equal to those objects of $\mathcal{C}$, and morphisms

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(A, B) \coloneqq \mathrm{Roof}_S(A, B)/\sim$$

being equivalence classes of roof diagrams with respect to S. The identity maps $A \to A$ will be represented by the roof diagrams $A \xleftarrow{1} A \xrightarrow{1} A$. The composition of two roof diagrams is defined by the following diagram:

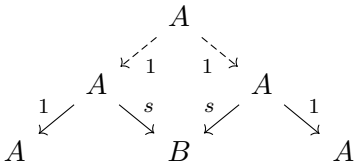
$$\begin{array}{ccccc} & & Z & & \\ & {\scriptstyle s_3}\swarrow & & \searrow & \\ & X & & Y & \\ {\scriptstyle s_1}\swarrow & & \searrow & {\scriptstyle s_2}\swarrow & \searrow \\ A & & B & & C \end{array}$$

In particular, we use the first axiom of a localizing class to choose an element $s_3 \in S$ and $Z \to Y$ as in the diagram, and then our composite is the outer perimeter of the triangle.

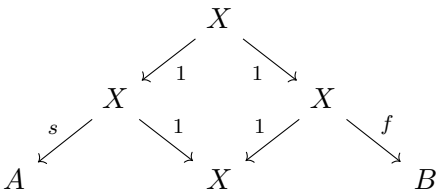
Lemma 1.1.15. *Let S be a localizing class of morphisms in $\mathcal{C}$. The composition law defined above is well-defined on equivalence classes of roofs and independent of any choices. With this composition, $\mathcal{C}[S^{-1}]$ is a well-defined category, possibly locally large.*

Remark 1.1.16. In general, note that if $\mathcal{C}$ was small then the localization is small. Thus we will ignore all set-theoretic technicalities by subtly referring to Grothendieck universes.

Given this construction, there is a natural map $\gamma\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$, which is the identity on the objects, and on morphisms we send $A \to B$ to the roof diagram $A \xleftarrow{1} A \to B$. Under this map, note that everything in S becomes invertible. If $s \in S$, the corresponding map $A \xleftarrow{1} A \xrightarrow{s} B$, and we claim the inverse is $B \xleftarrow{s} A \xrightarrow{1} A$. Indeed, we can take as our composite roof:



The same thing works in the other direction. More generally, if we are given any roof of the form $A \xleftarrow{s} X \xrightarrow{f} B$, we can always write it as the following composite:



In particular, this is equal to $\gamma(f) \circ \gamma(s)^{-1}$ as we just showed.

Lemma 1.1.17. *Keep the notation above. Then:*

- *(a) Every morphism of $\mathcal{C}[S^{-1}]$ is of the form $\gamma(f) \circ \gamma(s)^{-1}$ for some morphism f and $s \in S$.*
- *(b) γ exhibits $\mathcal{C}[S^{-1}]$ as the categorical localization of $\mathcal{C}$ with respect to the class S.*

Proof. We just proved (a). We also showed that γ sends elements of S to isomorphisms. To verify the universal property, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which sends S to isomorphisms. An extension along γ is clearly unique by (a), and moreover, defining $F(\gamma(f) \circ \gamma(s)^{-1}) = F(f) \circ F(s)^{-1}$ is (we claim) well-defined. Checking this comes down to using the equivalence relation on roof diagrams, and this gives the result. □

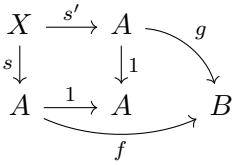
Now let us study the basic properties of the roofs more closely. Once again, we will fix γ to be our localization functor. First, if a map $g = \gamma(f)\gamma(s)^{-1}$ is

represented by the roof consisting of s and f, then by the first axiom of localizing classes we can write $s'f = f's$ for some $s' \in S$, which implies that

$$g = \gamma(f)\gamma(s)^{-1} = \gamma(s')^{-1}\gamma(f').$$

So, any element can also be represented as a fraction with the denominator coming first as well.

Second, suppose f, g are parallel arrows in $\mathcal{C}$, we want to know when $\gamma(f) = \gamma(g)$ holds. According to the roof diagrams, this happens iff there is a commutative diagram



with $s, s' \in S$. Thus $s = s'$, and so this happens iff $gs = fs$ for some element s, which happens iff $s'g = s'f$ for some element $s' \in S$, by Axiom 2 of a localizing class.

Third, given morphisms f, g in $\mathcal{C}[S^{-1}]$, we can always write them as a roof with the same denominator, i.e. we can write them in the form

$$f = \gamma(a)\gamma(s)^{-1} \qquad g = \gamma(b)\gamma(s)^{-1}$$

for a single s (and similarly the other way too). To see this, first write $f = \gamma(a)\gamma(s_1)^{-1}$ and $g = \gamma(b)\gamma(s_2)^{-1}$, and then by axiom 1 we may write $s_1s_3 = s_2r$ for some s_3, r. Let $s \coloneqq s_1s_3$. Then

$$f = \gamma(a)\gamma(s_3)\gamma(s_3)^{-1}\gamma(s_1)^{-1} = \gamma(as_3)\gamma(s)^{-1},$$

and likewise $g = \gamma(br)\gamma(s)^{-1}$. More generally, we can do this simultaneously for n elements: in the induction step, we'll have

$$f_i = \gamma(a_i)\gamma(s)^{-1} \qquad f_n = \gamma(b)\gamma(t)^{-1}$$

for $i \leq n-1$, then we do the same trick as in the $n = 2$ case to obtain the desired representation.

Lemma 1.1.18. *Keep the same notation as above. The following are equivalent:*

i. S satisfies the 2-out-of-6 property, namely if f, g, h are composable in $\mathcal{C}$, then

$$fg, gh \in S \implies g \in S.$$

ii. If g is in $\mathcal{C}$ and $\gamma(g)$ is an isomorphism, then $f \in S$.

If any of these conditions hold, we say S is saturated. In this case, the isomorphisms in $\mathcal{C}[S^{-1}]$ are exactly the elements of the form $\gamma(s)\gamma(s')^{-1}$.

Proof. If (i) holds, then it also satisfies 2-out-of-3, because we can take f or h to be the identity. If g has an inverse of the form $\gamma(h)\gamma(s)^{-1}$, then $\gamma(fh) = \gamma(s)$, so $fhs' = ss' \in S$, hence by 2-of-3 we have $gh \in S$. Likewise, it has a left inverse of the form $\gamma(s_2)^{-1}\gamma(f)$, so $fg \in S$ as well. Thus $g \in S$.

If (ii) holds, then $\gamma(f)\gamma(g)$ and $\gamma(g)\gamma(h)$ are isomorphisms, so $\gamma(g)$ is both a split epi and split mono, thus an isomorphism. Hence $g \in S$.

Lastly, if e.g. the second holds, then consider a morphism $f = \gamma(g)\gamma(s)^{-1}$ which is an isomorphism. Thus $\gamma(g)$ is an isomorphism, so $g \in S$. □

Lemma 1.1.19. *Let $\mathcal{C}$ be an additive category and S a localizing class. Then $\mathcal{C}[S^{-1}]$ is an additive category, and the localization functor $\gamma\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ is an additive functor. Addition of morphisms is uniquely specified by the rule*

$$\gamma(f)\gamma(s)^{-1} + \gamma(g)\gamma(s)^{-1} = \gamma(f+g)\gamma(s)^{-1}$$

for parallel morphisms f and g in $\mathcal{C}$.

Proof. We already know that the specification is unique, because any parallel arrows in $\mathcal{C}[S^{-1}]$ can be written in the form on the LHS. The only thing remaining to check, in light of the above remarks, is that this is well-defined. This follows (e.g. one step at a time) from the equivalence relations on roofs. □

We will very quickly state some final results about localizing subcategories.

Remark 1.1.20. Via opposites (and the symmetry in the definition), it is clear that everything said here is equivalent if we use 'right roofs' instead of left roofs. So, e.g., morphisms in $\mathcal{C}[S^{-1}]$ can be represented as equivalence classes of diagrams

$$A \xrightarrow{f} X \xleftarrow{s} B$$

where $s \in S$, where we define things similarly as in the previous case.

Proposition 1.1.21. *Let S be a localizing class in $\mathcal{C}$ and suppose $\mathcal{B} \subset \mathcal{A}$ is a full subcategory. Suppose $S \cap \mathcal{B}$ is a localizing class in $\mathcal{B}$, and that one of the following conditions holds:*

(a) For any $s\colon X' \to X$ with $s \in S$ and X in $\mathcal{B}$, there exists $f\colon X'' \to X'$ with X'' in $\mathcal{B}$ such that $sf \in S$; or

(b) The same as (a) but with arrows reversed.

Then, writing $S_{\mathcal{B}} \coloneqq S \cap \mathcal{B}$, the canonical map $\mathcal{B}[S_{\mathcal{B}}^{-1}] \to \mathcal{C}[S^{-1}]$ is a fully faithful embedding.

1.2. The Derived Category

Let $\mathcal{A}$ be an abelian category, and as usual we denote by $\mathrm{K}(\mathcal{A})$ the homotopy category of chain complexes of $\mathcal{A}$. We will from now also write

$$\mathrm{K}^+(\mathcal{A}) \qquad \mathrm{K}^-(\mathcal{A}) \qquad \mathrm{K}^b(\mathcal{A})$$

for the full (additive) subcategories of bounded below, bounded above, and bounded complexes, respectively. Note that for chain maps inside one of these categories, taking the cone lands us back inside the same category. Therefore, Lemma 1.1.11 and Lemma 1.1.12 tell us that the class of quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$ forms a localizing class of morphisms, and from the above remark it is also clear that the same is true for the three categories written above.

Definition 1.2.1. The additive categories $D(\mathcal{A}), D^+(\mathcal{A}), D^-(\mathcal{A})$, and $D^b(\mathcal{A})$ are all defined as the localizations of the respective homotopy categories with respect to the quasi-isomorphisms. Any of these is called the *derived category* of $\mathcal{A}$. We always consider these categories with their natural functor from $\mathrm{K}(\mathcal{A})$ or $\mathrm{Ch}(\mathcal{A})$, or a subcategory of one of these.

Using Prop. 1.1.21, we identify the latter three categories as full subcategories of $D(\mathcal{A})$. Lemma 1.1.13 with Lemma 1.1.18 shows that the quasi-isomorphisms form a saturated class in all four of the categories. Thus a map is a quasi-isomorphism iff it becomes an isomorphism in the derived category.

For the sake of time, we will very briefly state some important results regarding the homotopy categories and derived categories (to be filled in later).

1.2.1. The triangulated structure. In any of the categories $D^*(\mathcal{A})$ or $\mathrm{K}^*(\mathcal{A})$, a *triangle* will be a sequence of maps

$$A \to B \to C \to A[1].$$

A morphism of triangles is a commutative diagram

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

A *standard* triangle is one of the form

$$A \xrightarrow{f} B \to \mathrm{Cone}(f) \to A[1],$$

and a *distinguished* (or exact) triangle is one isomorphic to a standard triangle.

Lemma 1.2.2. *A short exact sequence* $0 \to A \to B \to C \to 0$ *of complexes induces a distinguished triangle in* $D^*(\mathcal{A})$*. More precisely, there is a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \longrightarrow & \mathrm{Cone}(i) \\ \| & & \| & & \downarrow \\ A & \xrightarrow{i} & B & \longrightarrow & C \end{array}$$

where $\mathrm{Cone}(i) \to C$ *is a quasi-isomorphism.*

Proof. The map $\mathrm{Cone}(i)^n = A^{n+1} \oplus B^n \to C^n$ is given by projecting to the B factor and applying $B \to C$. This is surjective, and the kernel is termwise of the form $A^{n+1} \oplus A^n$, so we get an exact sequence

$$0 \to \mathrm{Cone}(1_A) \to \mathrm{Cone}(i) \to C \to 0.$$

Since $\mathrm{Cone}(1_A)$ is acyclic the map is a quasi-isomorphism. □

Lemma 1.2.3. *Let* $A \xrightarrow{f} B \to \mathrm{Cone}(f) \to A[1]$ *be a standard triangle in* $\mathrm{K}^*(\mathcal{A})$*. Then the rotations*

$$B \to \mathrm{Cone}(f) \to A[1] \xrightarrow{-f} B[1]$$

and

$$\mathrm{Cone}(f)[-1] \to A \to B \to \mathrm{Cone}(f)$$

are both distinguished in $\mathrm{K}^*(\mathcal{A})$*. In particular, the same is true in* $D^*(\mathcal{A})$*.*

Proof. We have already discussed this via a deformation retract. □

Lemma 1.2.4. *Every distinguished triangle in* $D^*(\mathcal{A})$ *comes from an exact sequence in the sense of Lemma 1.2.2.*

Proof. Every distinguished triangle is isomorphic to a standard triangle, but it's also isomorphic to the rotation of some standard triangle. Indeed, if $A \to B \to C \to A[1]$ is distinguished, then by the above lemma so is $C[-1] \to A \to B \to C$, and thus this is isomorphic to a standard triangle.

In particular, they are all isomorphic to a triangle of the form $B \to \mathrm{Cone}(f) \to A[1] \xrightarrow{-f} B[1]$, and we know that this comes from the short exact sequence

$$0 \to B \to \mathrm{Cone}(f) \to A[1] \to 0$$

in $\mathrm{Ch}(\mathcal{A})$. □

As a special case of this, we see that every distinguished triangle in $D^*(\mathcal{A})$ induces a long exact sequence in cohomology. We will now partially axiomatize the above results and revisit this more carefully.

Let $\mathcal{C}$ be a category equipped with an autoequivalence $T\colon \mathcal{C} \to \mathcal{C}$. For simplicity, we will assume T is actually an automorphism. Just as above, we can define triangles in $\mathcal{C}$ to be sequences of the form $A \to B \to C \to T(A)$, and likewise morphisms.

Definition 1.2.5. A *triangulated category* is an additive category $\mathcal{C}$ equipped with an additive automorphism T and a class of triangles called *distinguished* or *exact triangles* satisfying the following properties:

(TR1) (a) For A in $\mathcal{C}$, $A \xrightarrow{1_A} A \to 0 \to T(A)$ is distinguished.
(b) Any triangle isomorphic to a distinguished triangle is distinguished.
(c) Any morphism $f\colon A \to B$ can be completed to a distinguished triangle $A \to B \to C \to T[1]$.

(TR2) A triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} T(A)$ is distinguished iff its *rotation*

$$B \xrightarrow{v} C \xrightarrow{w} T(A) \xrightarrow{-T(u)} T(B)$$

is distinguished.

(TR3) Given two distinguished triangles as in the diagram below

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & T(A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle T(f)} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A') \end{array}$$

with f and g given, we can find an h making the diagram commute.

(TR4) The octahedral axiom, which we will skip.

One often calls a pre-triangulated category a category satisfying all axioms except (TR4), and in fact we will mostly just be dealing with pre-triangulated categories when we discuss basic properties.

Proposition 1.2.6. *Let $\mathcal{C}$ be a triangulated category.*

(a) The composite of any two morphisms in a distinguished triangle is zero.

(b) For any distinguished triangle $A \to B \to C \to T(A)$, we have long exact sequences

$$\cdots \to \mathcal{C}(X, T^i(A)) \to \mathcal{C}(X, T^i(B)) \to \mathcal{C}(X, T^i(C)) \to \mathcal{C}(X, T^{i+1}(A)) \to$$

and likewise in the contravariant direction for any object X in $\mathcal{C}$.

(c) Given a diagram as in (TR3), if f and g are isomorphisms, then h is an isomorphism.

(d) The distinguished triangle of (TR1) is unique up to a non-canonical isomorphism.

(e) If in a morphism of triangles, any two of the first three maps are isomorphisms, the third is an isomorphism.

Proof. (a) By rotation, consider $A \to B \to C \to T(A)$ and we show that $A \to B \to C$ is zero. The axioms imply that we must be able to complete the diagram below,

$$\begin{array}{ccccccc} A & \xrightarrow{1} & A & \longrightarrow & 0 & \longrightarrow & T(A) \\ \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle f} & & \vdots & & \downarrow{\scriptstyle T(f)} \\ A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & T(A) \end{array}$$

but then $A \to B \to C$ factors through 0.

(b) By rotation, we just have to check that if $A \to B \to C \to T(A)$ is distinguished, then

$$\mathrm{Hom}(X, A) \to \mathrm{Hom}(X, B) \to \mathrm{Hom}(X, C)$$

is exact in the middle. By (a) the composite is zero. If $a\colon X \to B$ satisfies $X \to B \to C = 0$, then consider the diagram of rotated triangles

$$\begin{array}{ccccccc} X & \longrightarrow & 0 & \longrightarrow & T(X) & \longrightarrow & T(X) \\ {\scriptstyle a}\downarrow & & \downarrow & & \vdots{\scriptstyle h} & & \downarrow \\ B & \longrightarrow & C & \longrightarrow & T(A) & \longrightarrow & T(B) \end{array}$$

where our assumption means that the first square commutes. Then we win by using $\pm h[-1]$.

(c) By functoriality of T and the LES of (a), we obtain a diagram of LES's and the 5-lemma shows that for every X, $\mathrm{Hom}(X, h)$ is an isomorphism. Thus h is an isomorphism.

(d) It's unique because if we had two such C or C', we can use (TR3) to get a map between them with maps on A and B being identities, and thus we're done by (c).

(e) Consider a morphism of triangles:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\ \downarrow f & & \downarrow g & & \downarrow h & & \downarrow Tf \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA' \end{array}$$

We have just explained the case when f and g are isomorphisms. If f and h are isos or g and h are isos, then rotate once to the right or once to the left respectively to get into that case. □

Let $\mathcal{C}$ be triangulated and $\mathcal{A}$ be abelian. An additive functor $F\colon \mathcal{C} \to \mathcal{A}$ is *cohomological* if for any distinguished triangle $A \to B \to C \to T(A)$ the induced sequence

$$F(A) \to F(B) \to F(C)$$

is exact in $\mathcal{A}$. Given a distinguished triangle, we can always extend it to a sequence

$$\cdots \to T^i(A) \to T^i(B) \to T^i(C) \to T^{i+1}(A) \to T^{i+1}(B) \to \cdots,$$

where any subset of four terms is a distinguished triangle. Thus, a cohomological functor induces a long exact sequence in $\mathcal{A}$. For example, we have shown that $\mathrm{Hom}(X, -)$ into abelian groups is cohomological for any X, and so is $\mathrm{Hom}(-, X)$ if we view it as a functor $\mathcal{C} \to \mathbf{Ab}^{\mathrm{op}}$.

Here are two related notions:

Definition 1.2.7. Let $\mathcal{C}, \mathcal{C}'$ be triangulated categories, and $\mathcal{A}$ be an abelian category.

(a) An *exact* or *triangulated* functor $F\colon \mathcal{C} \to \mathcal{C}'$ is an additive functor F together with a natural isomorphism

$$\mu\colon F \circ T \xrightarrow{\sim} T' \circ F$$

such that whenever $A \to B \to C \to T(A)$ is distinguished, the induced triangle

$$F(A) \to F(B) \to F(C) \to (F(T(A)) \xrightarrow{\mu} T'(F(A)))$$

is distinguished.

(b) A morphism of triangulated functors is a natural transformation which commutes with the natural isomorphisms μ above.

(c) A δ-functor $G\colon \mathcal{A} \to \mathcal{C}$ is an additive functor which assigns to each short exact sequence

$$0 \to A \to B \to C \to 0$$

a map $\delta\colon G(C) \to G(A)[1]$ satisfying the following properties:

i. $GA \to GB \to GC \xrightarrow{\delta} GA[1]$ is a distinguished triangle.

ii. δ is natural with respect to morphisms of exact sequences.

Lemma 1.2.8. *Let $F\colon \mathcal{C} \to \mathcal{C}'$ be a triangulated functor which is also an equivalence of categories. Then F is a triangulated equivalence.*

Proof. After unwinding the definitions, and even making the simplifying assumption that F commutes with the translation functor, the main thing to prove is that if

$$FA \xrightarrow{Ff} FB \to FC \to T'(FA) = F(T(A))$$

is distinguished in $\mathcal{C}'$, then $A \to B \to C \to TA$ was already distinguished. Indeed, to see this choose a distinguished triangle $A \xrightarrow{f} B \to D \to TA$, and apply F to it; we are going to get a diagram:

$$\begin{array}{ccccccc} FA & \xrightarrow{Ff} & FB & \longrightarrow & FC & \longrightarrow & F(TA) \\ \| & & \| & & \downarrow & & \| \\ FA & \xrightarrow{Ff} & FB & \longrightarrow & FD & \longrightarrow & T'F(A) \end{array}$$

Since both rows are distinguished we obtain a map $FC \to FD$, which by the previous proposition is an isomorphism. By full faithfulness, this is of the form $Fi\colon FC \to FD$ for an isomorphism $i\colon C \xrightarrow{\sim} D$, so $A \to B \to C \to TA$ was isomorphic to a distinguished triangle already. □

The definition of a δ-functor is made so that for every cohomological functor $\mathcal{C} \to \mathcal{D}$, the composite $\mathcal{A} \to \mathcal{C} \to \mathcal{D}$ is a δ-functor of abelian categories. We will discuss some general properties of such functors later.

Our main examples of triangulated categories are the homotopy categories $\mathrm{K}^*(\mathcal{A})$ and their localizations $D^*(\mathcal{A})$. Once we know that they are triangulated, it will be obvious that the localization map $\mathrm{K}^*(\mathcal{A}) \to D^*(\mathcal{A})$ is a triangulated functor.

It will also be obvious that $H^0\colon D^*(\mathcal{A}) \to \mathcal{A}$ is a cohomological functor: we may show the property on the middle portion of the sequence $A \to B \to \mathrm{Cone}(f) \to A[1]$, and then we are done because we know that $H^0(B) \to H^0(\mathrm{Cone}(f)) \to H^0(A[1])$ is part of a long exact sequence in cohomology. This gives another proof of the fact that any distinguished triangle gives a long exact sequence in cohomology.

Third, it will be clear that $\mathrm{Ch}^*(\mathcal{A}) \to D^*(\mathcal{A})$ is a δ-functor. Indeed, we have proven in Lemma 1.2.2 that for each SES $0 \to A \to B \to C \to 0$, we have a corresponding exact triangle

$$A \xrightarrow{i} B \to \mathrm{Cone}(i) \to A[1]$$

induced because of the quasi-isomorphism $q_i\colon \mathrm{Cone}(i) \to C$. Therefore, $A \to B \to C \to A[1]$ is an exact triangle in the derived category, and the map δ is the natural map corresponding to $\mathrm{Cone}(i) \to A[1]$. This is called *the universal δ-functor* (not to be confused with universal δ-functors between abelian categories).

Theorem 1.2.9. *Let $\mathcal{A}$ be an abelian category. The homotopy categories $\mathrm{K}^*(\mathcal{A})$ are all triangulated categories with the shift functor* [1] *and the distinguished triangles as defined.*

PROOF. (TR1) The cone $C(1_A)$ is homotopy equivalent to 0, so the standard triangle $A \to A \to C(1_A) \to A[1]$ is isomorphic to the triangle $A \to A \to 0 \to A[1]$. (b) is obvious, and we have (c) by taking $C = \mathrm{Cone}(A \to B)$.

(TR2) We already proved that both rotations of cones are cones up to homotopy equivalence.

(TR3) By replacing with a homotopy equivalent diagram, we may assume that $C = \mathrm{Cone}(A \to B)$, etc. Then $h = f[1] \oplus g$ and we are done.

(TR4) As usual, we skip the octahedral axiom. □

Let us now state a general means for constructing localizations in the presence of a triangulated structure on a category. We will say a localizing class on a triangulated category is *compatible with the triangulation* if $s \in S$ iff $T(s) \in S$ for all s, and if in TR3, when both $f, g \in S$, we can take $h \in S$ as well.

Proposition 1.2.10. *Let $\mathcal{C}$ be a triangulated category, and S a localizing class of morphisms which is compatible with the triangulation. The localization $\mathcal{C}[S^{-1}]$ admits a natural triangulated structure making*

$$\mathcal{C} \to \mathcal{C}[S^{-1}]$$

a triangulated functor. A triangle in $\mathcal{C}[S^{-1}]$ is distinguished if and only if it is isomorphic to the image of a distinguished triangle in $\mathcal{C}$.

PROOF. Everything is easy to check using roofs with the exception of (TR4), which we skip as usual. □

Corollary 1.2.11. *Let $\mathcal{A}$ be an abelian category. The derived categories $D^*(\mathcal{A})$ are triangulated categories with the shift functor* [1] *and exact triangles as defined previously. Moreover, the localization functors*

$$\mathrm{K}^*(\mathcal{A}) \to D^*(\mathcal{A})$$

are all triangulated functors. The functors

$$\mathrm{Ch}^*(\mathcal{A}) \to D^*(\mathcal{A})$$

are δ-functors, and a triangle in $D^*(\mathcal{A})$ is exact iff it is isomorphic to the image of an exact sequence in $\mathrm{Ch}^*(\mathcal{A})$.

PROOF. We just have to check the hypotheses of the previous theorem for quasi-isomorphisms. Clearly, a map is a qis iff its shifts are qis's, and for the (TR3) property, we have said that we can take $f[1] \oplus g$ to be the third map. If f and g are qis, then this map is also a qis. We have already explained the second statement. □

Any general lemmas about the basic structure of the derived category will be added here.

Lemma 1.2.12. *Let $\mathcal{A}$ be an abelian category and $f\colon X \to Y$ a morphism in $D^*(\mathcal{A})$. Let $Q\colon \mathrm{K}^*(\mathcal{A}) \to D^*(\mathcal{A})$ denote the localization functor. The following are equivalent:*

(a) f is an isomorphism.

(b) $f = Q(s) \circ Q(t)^{-1}$ *for* s *and* t *quasi-isomorphisms.*
(c) $f = Q(s)^{-1} \circ Q(t)$ *for* s *and* t *quasi-isomorphisms.*
(d) $H^i(f)$ *is an isomorphism in* $\mathcal{A}$ *for all* $i \in \mathbb{Z}$.

Proof. We already know that $(a) = (b)$ because of isomorphisms in a saturated localization, Lemma 1.1.18. If (c) holds then clearly it is an isomorphism, and conversely if (b) holds, we can choose a diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & Y \\ {\scriptstyle t}\downarrow & & \vdots{\scriptstyle q} \\ X & \overset{r}{\dashrightarrow} & B \end{array}$$

by Axiom 1 of a localizing class, with q a quasi-isomorphism. Then by 2-out-of-3 r is a quasi-isomorphism, and we get (c). Finally, if (d) holds, then write f in the form $f = Q(b) \circ Q(s)^{-1}$, so that $f \circ Q(s) = Q(b)$. Applying H^i, this implies b is a quasi-isomorphism, so we're in the case of (b). □

1.2.2. Derived functors. We are now going to define derived functors on the derived categories of an abelian category $\mathcal{A}$. We will at first not do this using injective/projective resolutions, and instead define adapted classes directly. In the next section, we will explain what assuming the existence of enough injectives (or projectives) buys us.

Let $\mathcal{A}$ be an abelian category, and let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. What we say now will depend on a specific parity, so by convention our chosen 'parity' is left exact functor, right derived functor, etc., and the other case is delegated to parentheses.

Definition 1.2.13. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left (resp. right) exact functor of abelian categories. A subclass $\mathcal{R} \subset \operatorname{Ob}\mathcal{A}$ of objects is *adapted to* F if the following properties hold:

(a) $\mathcal{R}$ is closed under finite direct sums (meaning it includes 0).
(b) For each acyclic complex $C^\bullet$ in $\mathrm{Ch}^+(\mathcal{R})$ (resp. $\mathrm{Ch}^-(\mathcal{R})$), the complex $F(C^\bullet)$ in $\mathrm{Ch}^+(\mathcal{B})$ (resp. $\mathrm{Ch}^-(\mathcal{B})$) is acyclic.
(c) $\mathcal{C}$ has *enough objects in* $\mathcal{R}$, or $\mathcal{R}$ is large enough, meaning every object of $\mathcal{C}$ admits a monomorphism into (resp. an epimorphism from) an object of $\mathcal{R}$.

If $\mathcal{R}$ only satisfies conditions (a) and (c), then it is called *preadapted*, independent of the functor F.

The main theorem which will allow us to define derived functors is the following:

Theorem 1.2.14. *Let* $\mathcal{R}$ *be a preadapted class of objects of* $\mathcal{A}$. *Then* $S_\mathcal{R}$, *the class of quasi-isomorphisms between complexes with terms in* $\mathcal{R}$, *is a localizing class, and the canonical functor*

$$\mathrm{K}^+(\mathcal{R})[S_\mathcal{R}^{-1}] \to D^+(\mathcal{A})$$

is an equivalence of categories. Moreover, $\mathrm{K}^+(\mathcal{R})$ *is a triangulated subcategory of* $\mathrm{K}^+(\mathcal{A})$, $S_\mathcal{R}$ *is compatible with the triangulation, and the equivalence above is an equivalence of triangulated categories.*

PROOF. We do the case that everything has a monomorphism into $\mathcal{R}$. $S_{\mathcal{R}}$ is a localizing class, because to prove that qis's were a localizing class in the homotopy category we constructed it out of shifts and cones of some maps given. If those maps lie in $\mathrm{K}^+(\mathcal{R})$, then since $\mathcal{R}$ is closed under finite direct sums the cones (and shifts) are also in the same category. Thus $S_{\mathcal{R}}$ is a localizing class in $\mathrm{K}^+(\mathcal{R})$.

Claim For any complex A in $\mathrm{K}^+(\mathcal{A})$ there exists a qis $A \to R$ where R is in $\mathrm{K}^+(\mathcal{R})$.

Once we have shown the claim, to check that this canonical functor is fully faithful, we apply Prop. 1.1.21(b), so we need to show that if $s\colon X \to X'$ is a qis with X in $\mathcal{R}$, then we can find $f\colon X' \to X''$ such that fs is a qis in $\mathcal{R}$. Indeed, just take f to be a quasi-isomorphism in the claim where R is in $\mathcal{R}$.

Next, to check that this is essentially surjective, the claim says that every object in $D^+(\mathcal{A})$ is isomorphic to an object in $\mathrm{K}^+(\mathcal{R})$. This means it's an equivalence. For the statements about the triangulated structure, note that we have said $\mathrm{K}^+(\mathcal{R})$ is closed under shifts, cones (and contains 0 by definition). This is all we needed to establish the triangulated structure on the homotopy category. To prove that $S_{\mathcal{R}}$ is compatible, we already said it is closed under shifts, and the other condition is one on maps, not on objects.

It is clear from this that the natural map is an exact functor of triangulated categories, and hence it is a triangulated equivalence. Thus we have reduced to proving the claim. This will be deduced from the following lemma. □

Lemma 1.2.15. *Let $\mathcal{A}$ be an abelian category, and $\mathcal{R}$ a class of objects of $\mathcal{A}$ such that every object of $\mathcal{A}$ admits a monomorphism into an object of $\mathcal{R}$. Then for each $A^\bullet$ in $\mathrm{Ch}^+(\mathcal{A})$, there exists $R^\bullet$ in $\mathrm{Ch}^+(\mathcal{R})$ and a quasi-isomorphism $A^\bullet \to R^\bullet$, which we may take to be a monomorphism. Moreover, if $A^i = 0$ for $i < k$, then we may assume that $R^i = 0$ for $i < k$.*

PROOF. Assume $A^i = 0$ for $i < 0$, choose a mono $f^0\colon A^0 \to R^0$, and take a pushout as in the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & A^0 & \xrightarrow{d_A^1} & A^1 & = & A^1 \\
 & & \downarrow & & {\scriptstyle f^0}\downarrow & & \downarrow & & \downarrow{\scriptstyle f^1} \\
0 & \longrightarrow & 0 & \longrightarrow & R^0 & \dashrightarrow & P & \hookrightarrow & R^1
\end{array}$$

The bottom composite $R^0 \to R^1$ is what we define as d_R^0. Since pushouts of monics are monic in an abelian category, $A^1 \to P$ is monic, and thus so are f^0 and f^1. Now if we've constructed the ith step of R already, we do a similar thing:

$$\begin{array}{ccccccc}
A^{i-1} & \xrightarrow{d_A} & A^i & \xrightarrow{q} & \operatorname{cok} d_A^{i-1} & \xrightarrow{\overline{d_A^i}} & A^{i+1} & = & A^{i+1} \\
\downarrow{\scriptstyle f^{i-1}} & & \downarrow{\scriptstyle f^i} & & {\scriptstyle \overline{f^i}}\downarrow & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle f^{i+1}} \\
R^{i-1} & \xrightarrow{d_R} & R^i & \xrightarrow{p} & \operatorname{cok}(d_R^{i-1}) & \dashrightarrow & P & \xhookrightarrow{c} & R^{i+1}
\end{array}$$

Namely, construct the same pushout, except first pass to the cokernels. This ensures $d^2 = 0$ because we factor through the cokernel of d^{i-1}. Moreover, we have

the relation $\overline{d_A^i}q = d_A^i$. Then define $d_R^i = cbp$. This construction defines a map $f\colon A^\bullet \to R^\bullet$. Observe also that if $\overline{f^i}$ is monic, then a is monic, so $f^{i+1} = ca$ is monic.

We have to show that f is a quasi-isomorphism and that all the f^i are monos. To do this, in the diagram above we are going to show that $H^i(f)$ is an epimorphism, and then that the map $\operatorname{cok} d_A^i \to \operatorname{cok} d_R^i$ induced by f^{i+1} is monic, which will imply that its restriction to the subobjects $H^i(A) \to H^i(R)$ is monic as well. As explained in the construction this will keep each f^i monic.

For the former, let $[x] \in H^i(R)$ so $cbp(x) = 0$, so $bp(x) = 0$, meaning that the image of $(p(x), 0)$ in P is 0. Thus we can find $\overline{y}$ such that $(p(x), 0) = (\overline{f^i}, \overline{d_A^i})\overline{y}$, where we may assume $\overline{y} = q(y)$. Thus $(p(x), 0) = (pf^i(y), d_A^i(y))$, so $[y] \in H^i(A)$, and $x - f^i(y) \in \ker p = \operatorname{im} d_R^{i-1}$, i.e. $[x] = [f^i(y)]$.

To show that the induced map $\overline{f^{i+1}}$ on cokernels is monic, we want to show that if $x \in A^{i+1}$ maps to the image of d_R^i, then it is in the image of d_A^i. So suppose $ca(x) = cbp(x)$, i.e. $a(x) = bp(x)$, then the element $(p(y), -x)$ maps to zero in P, so is of the form $(\overline{f}^i, -\overline{d_A^i})\overline{z}$, where we may assume $\overline{z} = q(z)$. Thus $-d_A^i(z) = -x$, as desired. □

Here's another statement about preadapted classes which is sometimes useful.

Lemma 1.2.16. *Let $\mathcal{R}$ be a (right) preadapted class in $\mathcal{A}$. Let $X \in \mathrm{K}(\mathcal{A})$ satisfy the property that*

$$\operatorname{Hom}_{\mathrm{K}(A)}(X, R) = 0$$

for every acyclic R in $\mathrm{K}^+(\mathcal{R})$. Then the natural map

$$\operatorname{Hom}_{\mathrm{K}(A)}(X, B) \to \operatorname{Hom}_{D(A)}(X, B)$$

is an isomorphism for all $B \in \mathrm{K}^+(\mathcal{R})$.

Proof. Let $q\colon B \to B'$ be a quasi-isomorphism in $\mathrm{K}^+(\mathcal{R})$. Then $B \to B' \to \operatorname{Cone}(q) \to B[1]$ is an exact triangle in $\mathrm{K}(\mathcal{A})$, so applying the cohomological functor $\operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(X, -)$ and noting that $\operatorname{Cone}(q)$ is acyclic, we deduce that

$$q_*\colon \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(X, B) \to \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(X, B')$$

is an isomorphism. Now consider the natural map in the lemma statement: represent a map $X \to B$ by a right roof $X \to A \leftarrow B$ with $B \to A$ a quasi-isomorphism. By considering $A \to \tau_{\geq m}A$ for $m \gg 0$, we may assume A is bounded below. Choosing a quasi-isomorphism $A \to B'$ with B' in $\mathrm{K}^+(\mathcal{R})$, we may assume we have a roof

$$X \to B' \xleftarrow{q} B$$

for q a quasi-isomorphism in $\mathrm{K}^+(\mathcal{R})$. By the above remark, there's a unique map $X \to B$ in $\mathrm{K}(\mathcal{A})$ making the diagram commute.

Conversely, if a map $X \to B$ in $\mathrm{K}(\mathcal{A})$ is sent to zero in $D(\mathcal{A})$, we can repeat the argument above to get a qis $q\colon B \to B'$ in $\mathrm{K}^+(\mathcal{R})$ such that $X \to B \to B' = 0$. By the above remark, $X \to B$ is zero in the homotopy category. □

Now we can come to the definition and construction of derived functors in the presence of an adapted class. As usual, we state the left exact case, where the other one follows just as well.

Definition 1.2.17. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor of abelian categories, and denote the standard localization functors by

$$Q_{\mathcal{A}}\colon \mathrm{K}^+(\mathcal{A}) \to D^+(\mathcal{A}) \qquad Q_{\mathcal{B}}\colon \mathrm{K}^+(\mathcal{B}) \to D^+(\mathcal{B}).$$

A *(total) right derived functor of* F is a triangulated left Kan extension of $Q_{\mathcal{B}} \circ \mathrm{K}^+(F)$ along $Q_{\mathcal{A}}$. More precisely, it is a triangulated functor

$$RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$$

equipped with a natural transformation $\varepsilon\colon Q_{\mathcal{B}} \circ \mathrm{K}^+(F) \to RF \circ Q_{\mathcal{A}}$ with the universal property that for every triangulated functor $G\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and map $\pi\colon Q_\beta \circ \mathrm{K}^+(F) \to G \circ Q_{\mathcal{A}}$, there is a unique map $\eta\colon RF \to G$ such that

$$\begin{array}{ccc} Q_{\mathcal{B}} \circ \mathrm{K}^+(F) & & \\ \downarrow{\scriptstyle \varepsilon} & \searrow^{\pi} & \\ RF \circ Q_{\mathcal{A}} & \xrightarrow{\eta \circ Q_{\mathcal{A}}} & G \circ Q_{\mathcal{A}} \end{array}$$

commutes.

In particular, the derived functor, if it exists, is unique up to canonical isomorphism commuting with the natural transformation.

Example 1.2.18. The universal property is made exactly so that natural transformations induce natural maps of our derived functors. More specifically, suppose $F, G\colon \mathcal{A} \to \mathcal{B}$ are functors for which (RF, ε_F) and (RG, ε_G) are derived functors. Suppose we have a map $u\colon F \to G$. Then this induces a natural map $u\colon \mathrm{K}^+(F) \to \mathrm{K}^+(G)$, so by the universal property there is a unique natural map $\eta\colon RF \to RG$ such that

$$\begin{array}{ccc} Q_{\mathcal{B}} \circ \mathrm{K}^+(F) & \xrightarrow{u} & Q_{\mathcal{B}} \circ \mathrm{K}^+(G) \\ \downarrow{\scriptstyle \varepsilon_F} & & \downarrow{\scriptstyle \varepsilon_G} \\ RF \circ Q_{\mathcal{A}} & \xrightarrow{\eta} & RG \circ Q_{\mathcal{A}} \end{array}$$

commutes. More concretely, for every $A^\bullet$ in $\mathrm{K}^+(\mathcal{A})$, the diagram

$$\begin{array}{ccc} F(A^\bullet) & \xrightarrow{u} & G(A^\bullet) \\ \downarrow & & \downarrow \\ RF(A^\bullet) & \xrightarrow{\eta} & RG(A^\bullet) \end{array}$$

commutes. Usually, this is the best way to define maps out of/into derived functors, and in particular out of/into their cohomology.

We now show the main theorem of this section, that the derived functor exists (and how to construct it) in the presence of an adapted class.

THEOREM 1.2.19. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, and $\mathcal{R}$ an F-adapted class of objects of $\mathcal{A}$. Then the derived functor RF exists with the following construction: For any choice Φ of inverse to the canonical equivalence of Thm. 1.2.14, define RF by the diagram:*

$$\begin{array}{ccccc} \mathrm{K}^+(\mathcal{R}) & \longrightarrow & \mathrm{K}^+(\mathcal{R})[S_{\mathcal{R}}^{-1}] & \xleftarrow{\Phi} & D^+(\mathcal{A}) \\ & \searrow^{\mathrm{K}^+(F)} & \downarrow & & \vdots\, RF \\ & & \mathrm{K}^+(\mathcal{B}) & \xrightarrow{Q_{\mathcal{B}}} & D^+(\mathcal{B}) \end{array}$$

Then there is a canonical construction of ε making (RF, ε) the right derived functor of F, which satisfies the property that for each $R^\bullet$ in $\mathrm{K}^+(\mathcal{R})$, the component

$$\varepsilon_R\colon F(R^\bullet) \xrightarrow{\sim} RF(R^\bullet)$$

is an isomorphism in $D^+(\mathcal{B})$.

To prove this theorem, let us make a few observations. Firstly, it is clear from the definition that for any additive functor F, we have $F(\mathrm{Cone}(f)) = \mathrm{Cone}(Ff)$ (where we are implicitly using F for the functor on $\mathrm{Ch}^*(\mathcal{A})$ and $\mathrm{K}^*(\mathcal{A})$ as well). Therefore, since F preserves acyclics between objects of $\mathcal{R}$, it preserves quasi-isomorphisms between objects of $\mathcal{R}$. Hence it extends along the localization $\mathrm{K}^+(\mathcal{R}) \to \mathrm{K}^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$, justifying the left triangle in the diagram above. Note that this extension is triangulated because F preserves cones, so if $A \to B \to \mathrm{Cone}(f) \to A[1]$ is distinguished then

$$FA \to FB \to F(\mathrm{Cone}\, f) = \mathrm{Cone}(Ff) \to FA[1]$$

is distinguished. Then Thm. 1.2.14 shows that $\Psi\colon D^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{R})[S_{\mathcal{R}}^{-1}]$, so we can choose a homotopy inverse Φ. With this choice, the diagram above indeed makes sense and we get a well-defined functor RF between the derived categories. By the above, RF is triangulated because Ψ is a triangulated equivalence (so Φ is one too).

Now we construct the natural map ε. By definition, there is a natural isomorphism

$$\beta\colon \mathrm{Id}_{D^+(\mathcal{A})} \xrightarrow{\sim} \Psi \circ \Phi,$$

so for any A in $\mathrm{K}^+(\mathcal{A})$, viewed as an object of $D^+(\mathcal{A})$, we have an isomorphism $\beta_A\colon A \xrightarrow{\sim} \Psi(\Phi(A))$, where $\Phi(A) = Y$ is some object of $\mathrm{K}^+(\mathcal{R})$. On objects, $\Psi(Y) = Y$. Thus we may write $\beta_A = Q_{\mathcal{A}}(s)^{-1}Q_{\mathcal{A}}(t)$ for some quasi-isomorphisms s, t, in a commutative diagram

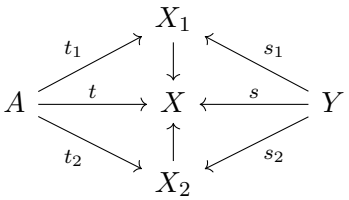
$$\begin{array}{ccc} & A & \\ {\scriptstyle Q_{\mathcal{A}}(t)}\swarrow & & \searrow^{\beta_A} \\ X & \xrightarrow[Q_{\mathcal{A}}(s)^{-1}]{} & Y \end{array}$$

By taking a quasi-isomorphism $X \to R$ for some R in $\mathrm{K}^+(\mathcal{R})$ and composing, we may assume that X is in $\mathrm{K}^+(\mathcal{R})$. Now define ε_A to be

$$F(A) \xrightarrow{Q_{\mathcal{B}}(F(t))} F(X) \xrightarrow{Q_{\mathcal{B}}(s)^{-1}} F(Y) =: RF(A),$$

viewed inside $D^+(\mathcal{B})$, where the second map is defined because s was a qis between objects of $\mathrm{K}^+(\mathcal{R})$, so $F(s)$ is a qis. First of all, note immediately that if A was in $\mathrm{K}^+(\mathcal{R})$, the left map would also be a quasi-isomorphism so $Q_{\mathcal{B}}F(A) \xrightarrow{\sim} RFQ_{\mathcal{A}}(A)$ would be an isomorphism in $D^+(\mathcal{B})$. Second, we claim that this map is independent of the choice of middle object X in $\mathrm{K}^+(\mathcal{R})$.

This is clear, because for any two X_1 and X_2 middle objects over $\mathcal{R}$ we can complete this to a commutative diagram of the form:



Here X may not be over $\mathcal{R}$, but once we apply F, s_1 and s_2 become invertible, so commutativity of the diagram gives the desired statement. It is not difficult from this to check naturality of ε. Finally, checking the universal property of the left Kan extension is also a standard roof argument, where at each step we can assume our right-roofs have middle term in $\mathcal{R}$. This completes the sketch of Thm. 1.2.19.

Example 1.2.20. Suppose we are in the situation of Thm. 1.2.19, and A is a complex for which $A^i = 0$ for $i < k$. Then $H^i(RF(A)) = 0$ for $i < k$. Indeed, by Lemma 1.2.15, we can choose our resolution $R^\bullet$ to have no terms below degree k, and then the cohomology of $F(R^\bullet)$ will have the same property.

Example 1.2.21. Suppose A and B are complexes. Since $\mathrm{Ch}^*(\mathcal{A}) \to D^*((A))$ is additive, $A \oplus B$ can refer unambiguously to the termwise direct sum complex. Since H^i is an additive functor (e.g. on complexes or the derived category), we have $H^i(A) \oplus H^i(B) = H^i(A \oplus B)$. Since the derived functors are (in particular) additive, $RF(A \oplus B) = RF(A) \oplus RF(B)$.

In the rest of this section, we discuss what happens when we want to compute a composition of two derived functors. This is the first place where the derived category makes something significantly easier than classical derived functors.

THEOREM 1.2.22. *Consider two left exact functors between abelian categories*

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}.$$

Let $\mathcal{R}_{\mathcal{A}}$ be a class adapted to F, and $\mathcal{R}_{\mathcal{B}}$ a class adapted to G. If $F(\mathcal{R}_{\mathcal{A}}) \subset \mathcal{R}_{\mathcal{B}}$, then the three derived functors RF, RG, and $R(G \circ F)$ all exist on the (bounded below) derived categories, and the natural map

$$R(G \circ F) \xrightarrow{\sim} RG \circ RF$$

is a natural isomorphism.

PROOF. RF and RG exist by Thm. 1.2.19. Note that $\mathcal{R}_{\mathcal{A}}$ is adapted to $G \circ F$, since if $A \in \mathrm{K}^+(\mathcal{R}_{\mathcal{A}})$ is acyclic, then $F(A)$ is acyclic and in $\mathrm{K}^+(\mathcal{R}_{\mathcal{B}})$, so $GF(A)$ is acyclic. Thus $R(G \circ F)$ exists; note $RG \circ RF$ is exact and the natural maps

$\varepsilon_G(B)\colon G(B) \to RG(B)$ and $\varepsilon_F(A)\colon F(A) \to RF(A)$ in the derived category give a composite

$$\pi(A) = GF(A) \xrightarrow{\varepsilon_G(FA)} RG(FA) \xrightarrow{RG(\varepsilon_F(A))} RG(RF(A)).$$

Hence there is a map $R(G \circ F) \to RG \circ RF$ such that

$$\begin{array}{ccc} (G \circ F)A & \longrightarrow & RG(FA) \\ \downarrow & & \downarrow \\ R(G \circ F)A & \text{-----} & RG(RF(A)) \end{array}$$

commutes in $D^+(\mathcal{C})$ for each $A^\bullet$ in $\mathrm{K}^+(\mathcal{A})$. If we take A to be in $\mathrm{K}^+(\mathcal{R}_A)$, then the left map is an isomorphism, and $FA \to RF(A)$ is an isomorphism so the right one is too, and FA is in $\mathrm{K}^+(\mathcal{R}_\mathcal{B})$ so the top map is an isomorphism too. Now we are done since every object is isomorphic to one of this form. □

So, in practice, we now have a good idea of how to compute derived functors in these situations: pick a quasi-isomorphism $A^\bullet \to R^\bullet$ where $R^\bullet$ has terms in our adapted class, and then $RF(A) \xrightarrow{\sim} RF(R) = F(R)$. Finally, let us discuss the trivial case of the above.

Lemma 1.2.23. *Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is an exact functor of abelian categories. The class of all objects of $\mathcal{A}$ is adapted to F, and the right/left derived functors of F both exist and coincide with the restrictions of the natural induced functor*

$$F\colon D(\mathcal{A}) \to D(\mathcal{B}).$$

Proof. This natural functor exists because F sends acyclic complexes to acyclic complexes, and hence preserves quasi-isomorphisms. Thus we clearly get a triangulated functor as above on the full derived category. Now if we restrict e.g. to $D^+(\mathcal{A})$, it is clear that ε is an isomorphism of F and RF because it is an isomorphism on all objects, since indeed the class of all objects is adapted by definition. □

1.2.3. Basic objects of the derived category. Here are some useful things to work with the derived category $D^*(\mathcal{A})$ of an abelian category $\mathcal{A}$. First, we have some standard filtrations associated to any complex $A^\bullet$: The *stupid filtrations* are $\sigma_{\geq n}A$ and $\sigma_{\leq n}A$, which are given by truncating the complex.

$$(\sigma_{\geq n}A)^i = \begin{cases} 0 & i < n \\ A^i & i \geq n \end{cases} \qquad (\sigma_{\leq n}A)^i = \begin{cases} 0 & i < n \\ A^i & i \leq n \end{cases},$$

and the differentials are the obvious ones. There are some obvious properties, namely there is an inclusion $\sigma_{\geq n}C \hookrightarrow C$, and

$$H^i(\sigma_{\geq n}A) = \begin{cases} 0 & i < n \\ \ker d^n & i = n \\ H^i(A) & i > n \end{cases}.$$

For the upper filtration we likewise have a quotient map $C \to \sigma_{\leq n} C$ and a similar statement as above. Finally, the main property is that we have an exact sequence

$$0 \to \sigma_{\geq n+1} C \to \sigma_{\geq n} C \to C^n[-n] \to 0$$

of complexes.

The *smart* or *canonical filtration* is $\tau_{\geq n} C$, defined as follows:

$$\cdots \to 0 \to 0 \to \operatorname{cok}(d^{n-1}) \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} C^{n+2} \to \cdots$$

The differential is well-defined because of $d^2 = 0$ in the original complex, and now the nth cohomology is

$$\ker d^n / \operatorname{cok}(d^{n-1}) = \ker d^n / \operatorname{im}(d^{n-1}) = H^n(C),$$

with the natural map $C \to \tau_{\geq n} C$ identifying cohomology $\geq n$. There is a natural map $\tau_{\geq n} C \to \tau_{\geq n+1} C$ because the diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & \operatorname{cok} d^{n-1} & \xrightarrow{\overline{d^n}} & C^{n+1} \longrightarrow C^{n+2} \\ & & \downarrow & & \downarrow \quad \| \\ & & 0 & \longrightarrow & \operatorname{cok} d^n \xrightarrow{\overline{d^{n+1}}} C^{n+2} \end{array}$$

commutes. This map is surjective, and the kernel is clearly

$$0[n-1] \to \operatorname{cok} d^{n-1}[n] \to \operatorname{im} d^n[n+1] \to 0.$$

The nontrivial differential is surjective, so the H^{n+1} of this complex is zero, and $H^n \cong H^n(C)$. Thus the kernel is quasi-isomorphic to $H^n(C)[-n]$, and hence there is a distinguished triangle

$$H^n(C)[-n] \to \tau_{\geq n} C \to \tau_{\geq n+1} C \to H^n(C)[-n+1]$$

in the derived category. Likewise, the upper filtration $\tau_{\leq n} C$ is given by

$$\cdots \to C^{n-2} \to C^{n-1} \to \ker d^n \to 0 \to 0 \to \cdots,$$

so we have a natural inclusion $\tau_{\leq n} C \to C$ preserving cohomology in degrees $\leq n$, and a similar distinguished triangle

$$\tau_{\leq n-1} C \to \tau_{\leq n} C \to H^n(C)[-n] \to \tau_{\leq n-1} C[1]$$

in $D^*(\mathcal{A})$.

Example 1.2.24. If $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ is a right derived functor in the presence of an adapted class, Example 1.2.20 showed that if A has no terms below degree k, then $RF(A)$ has no cohomology below degree k. Now we can weaken the assumption to A having no *cohomology* in degree less than k. Indeed in this case, $A \to \tau_{\geq k} A$ is a quasi-isomorphism, and Example 1.2.20 applies to this latter complex.

We next would like to discuss complexes concentrated at 0 (called 0-complexes), and complexes whose cohomology is concentrated in degree 0, called H^0-complexes. Obviously, $\mathcal{A} \to \mathrm{Ch}^*(\mathcal{A})$ sending $A \mapsto A[0]$ is a fully faithful embedding. Composing this with $\mathrm{Ch}^*(\mathcal{A}) \to \mathrm{K}^*(\mathcal{A})$ remains a fully faithful embedding, because a homotopy between 0-complexes must be 0.

On the other hand, suppose $A^\bullet$ is an H^0-complex. Then $A \to \tau_{\geq 0}C$ is a quasi-isomorphism, and then $\tau_{\leq 0}\tau_{\geq 0}C \to \tau_{\geq 0}C$ is a quasi-isomorphism, so A is isomorphic in the derived category to a complex concentrated in degree zero, namely $H^0(A)[0]$.

Lemma 1.2.25. *The natural map $\mathcal{A} \to D^*(\mathcal{A})$ defined by $A \mapsto A[0]$ is a fully faithful embedding, giving an equivalence to the full subcategory of H^0-complexes of $D^*(\mathcal{A})$.*

Proof. Consider the cohomology functor $H^0\colon D^*(\mathcal{A}) \to \mathcal{A}$, and restrict to the H^0-complexes. The composite $\mathcal{A} \to D^*(\mathcal{A}) \to \mathcal{A}$ is the identity, so our functor is at least faithful. It is also essentially surjective because of the remark made above.

Now let $\alpha\colon A[0] \to B[0]$ be a map in the derived category, and write it as a roof $A[0] \xleftarrow{s} X \xrightarrow{u} B[0]$ in the homotopy category. Then X is an H^0-complex, so by precomposing our roof with the natural inclusion $\tau_{\leq 0}X \to X$ (which is a quasi-isomorphism here), we may assume X has no positive degree terms.

The quasi-isomorphism $X \to \tau_{\geq 0}X = H^0(X)[0]$ sends X^i to 0 for $i \neq 0$, and $X^0 \to \operatorname{cok} d^{-1} = H^0(X)$. On the other hand, since $B[0]$ is concentrated in 0, the map $X \to B[0]$ satisfies $u^0 d^{-1} = 0$, namely u^0 factors through the cokernel $H^0(X)$. Thus u factors through $X \to H^0(X)[0]$ and likewise for s, so we have an equivalent roof

$$A[0] \xleftarrow{s'} H^0(X)[0] \xrightarrow{u'} B[0].$$

Since the left map is a quasi-isomorphism, it must be an isomorphism of complexes, so we have found the representative $u' \circ s'^{-1}$ as a chain map for $A[0] \to B[0]$. This shows full faithfulness. □

Here is an application. If A and B are objects of $\mathcal{A}$, we define

$$\operatorname{Ext}^i_{\mathcal{A}}(A, B) \coloneqq \operatorname{Hom}_{D^*(\mathcal{A})}(A[0], B[i]) \qquad i \in \mathbb{Z}$$

(as an abelian group), where this is independent of $*$ because the respective embeddings are fully faithful. Since the shift functors $[k]$ are additive equivalences of categories, it is also isomorphic to look at $\operatorname{Hom}_{D^*(\mathcal{A})}(A[k], B[i+k])$ instead of 0.

Lemma 1.2.26. *Let $\mathcal{A}$ be an abelian category and A, B be objects of $\mathcal{A}$. Then*

(a) $\operatorname{Ext}^i_{\mathcal{A}}(A, B) = 0$ *for all* $i < 0$.

(b) $\operatorname{Ext}^0_{\mathcal{A}}(A, B) = \operatorname{Hom}_{\mathcal{A}}(A, B) \equiv \mathcal{A}(A, B)$.

(c) Given a short exact sequence $0 \to B' \to B \to B'' \to 0$ *in* $\mathcal{A}$, *there is a long exact sequence*

$$0 \to \mathcal{A}(A, B') \to \mathcal{A}(A, B) \to \mathcal{A}(A, B'') \to \operatorname{Ext}^1_{\mathcal{A}}(A, B') \to \cdots$$

of abelian groups.

(d) Given a short exact sequence $0 \to A' \to A \to A'' \to 0$ *in* $\mathcal{A}$, *there is a long exact sequence*

$$\cdots \leftarrow \operatorname{Ext}^1_{\mathcal{A}}(A'', B) \leftarrow \mathcal{A}(A', B) \leftarrow \mathcal{A}(A, B) \leftarrow \mathcal{A}(A'', B) \leftarrow 0$$

of abelian groups.

PROOF. We already proved (b) above, and (c) and (d) follow from (a) and the fact that Homs in either variable are a cohomological functor to abelian groups in any triangulated category. For (a), represent a map $A[0] \to B[-i]$ by a roof

$$A[0] \xleftarrow{s} X \xrightarrow{f} B[-i]$$

with s a quasi-isomorphism. Since $\tau_{\leq 0}X \to X$ is a quasi-isomorphism, we may assume X has no positive terms. But $B[-i]$ is concentrated in degree $i > 0$, hence $f = 0$, so $A[0] \to B[-i]$ is zero in $D^*(\mathcal{A})$. □

From the lemma, it is obvious that if $\mathrm{Ext}^1_{\mathcal{A}}(P, B) = 0$ for every B in $\mathcal{A}$, then $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact, so P is projective. Likewise, if $\mathrm{Ext}^1_{\mathcal{A}}(A, I) = 0$ for all A, then I is injective. We will in greater depth explore the converse in the next section. These are especially nice functors on the derived category, because they exist even independently of injective or projective resolutions.

Here are some elementary observations we can make in the case of Ext. The proofs will also illustrate general techniques for working with roofs. A main idea will be the following remark: Suppose we have an exact sequence

$$0 \to P^{-1} \to P^0 \to A \to 0$$

and a map $P^{-1} \to B$. Then we can extend this to a diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \vdots & & \| & & \\ 0 & \longrightarrow & B & \dashrightarrow & B' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

by simply taking B' to be the pushout (since the pushout of monos in an abelian category is a mono), where the bottom row is still exact by the pasting lemma (paste the square with 0, so $B'/B = P^0/P^{-1} = A$). Thus, if we have an exact sequence $0 \to P^{-i} \to \cdots \to P^0 \to A \to 0$ and a map $P^{-i} \to B$, we can do the same thing: First split this up into short exact sequences

$$0 \to P^{-i} \to P^{-i+1} \to C^{-i+1} \to 0 \qquad 0 \to C^k \to P^{k+1} \to C^{k+1} \to 0$$

ending with $0 \to C^{-1} \to P^0 \to A \to 0$. Then fill in each sequence, compose the squares together, and recombine the exact sequence to obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P^{-i} & \longrightarrow & \cdots & \longrightarrow & P^0 & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & \cdots & \longrightarrow & B' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

with exact rows. Now the result:

Lemma 1.2.27. *Let A, B be objects of an abelian category $\mathcal{A}$. Morphisms in the derived category from $A[0]$ to $B[i]$, i.e. elements of $\mathrm{Ext}^i_{\mathcal{A}}(A, B)$, are equivalence classes of roofs of the form*

$$A[0] \xleftarrow{q} P \to B[i]$$

where P is concentrated in the interval $[-i, 0]$ and q is a quasi-isomorphism, with $P^{-i} = A$ and the map $P^{-i} \to A$ is the identity.

PROOF. If a map is represented by a roof $A[0] \leftarrow Q \rightarrow B[i]$, then we may immediately precompose the roof with the map $\tau_{\geq 0}Q \rightarrow Q$ (which is a qis since Q is an H^0 complex), so we may assume Q is concentrated in $(-\infty, 0]$. Now we want to factor this roof through the quasi-isomorphism $Q \rightarrow \tau_{\geq -i}Q$; the map $Q \rightarrow A[0]$ obviously factors through $\tau_{\geq -i}Q$ because A is zero in degree < 0.

For $B[i]$, we are considering a diagram:

$$\begin{array}{ccccc} Q^{-i-1} & \xrightarrow{d^{-i-1}} & Q^{-i} & \xrightarrow{d^{-i}} & Q^{-i+1} \\ \downarrow & & \downarrow f & & \downarrow \\ 0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

The fact that $fd^{-i-1} = 0$ means exactly that f factors through the cokernel of this differential, which is how we defined $\tau_{\geq -i}Q$. This gives an equivalent roof where the middle term is concentrated in $[-i, 0]$.

Equivalently, we may view this as an exact sequence $0 \rightarrow P^{-i} \rightarrow \cdots P^0 \rightarrow A$ equipped with a map $P^{-i} \rightarrow B$, and now by the remark above we can choose another complex Q concentrated in $[-i, 0]$ with $Q \rightarrow A[0]$ a quasi-isomorphism, $Q^{-i} = B$, and a quasi-isomorphism $P \rightarrow Q$ so that the diagram

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & \searrow & \\ A[0] & \longleftarrow & Q & \longrightarrow & B[i] \end{array}$$

commutes. This shows what we wanted. □

This lemma shows that elements of $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ are roof equivalence classes of exact sequences,

$$0 \rightarrow B = P^{-i} \rightarrow P^{-i+1} \rightarrow \cdots \rightarrow P^0 \rightarrow A \rightarrow 0.$$

Let us examine what the equivalence relation is: Two roofs $A[0] \leftarrow P \rightarrow B[-i]$ and $A[0] \leftarrow Q \rightarrow B[-i]$ of this form are equivalent iff there are quasi-isomorphisms $q\colon R \rightarrow P$ and $r\colon R \rightarrow Q$ making the diagrams

$$\begin{array}{ccc} R^0 & \xrightarrow{r^0} & Q^0 \\ {\scriptstyle q^0}\downarrow & & \downarrow \\ P^0 & \twoheadrightarrow & A \end{array} \qquad \begin{array}{ccc} R^{-i} & \xrightarrow{r^{-i}} & Q^{-i} \\ {\scriptstyle q^{-i}}\downarrow & & \| \\ P^{-i} & = & B \end{array}$$

commute. By the standard trick we may assume R is concentrated in negative degrees, which as before implies $R^0 \rightarrow A$ is an epimorphism. This looks like a big commutative diagram below, the middle resolution row R is exact, as are the other two rows. Moreover, the two maps $R^{-i} \rightarrow B$ on either side are the same.

$$\begin{array}{ccccccccccc}
 & & 0 & \longrightarrow & B & \longrightarrow & Q^{-i+1} & \longrightarrow \cdots \longrightarrow & Q^0 & \longrightarrow & A \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| \\
\cdots & \longrightarrow & R^{-i-1} & \longrightarrow & R^{-i} & \longrightarrow & R^{-i+1} & \longrightarrow \cdots \longrightarrow & R^0 & \longrightarrow & A \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| \\
 & & 0 & \longrightarrow & B & \longrightarrow & P^{-i+1} & \longrightarrow \cdots \longrightarrow & P^0 & \longrightarrow & A \longrightarrow 0
\end{array}$$

Given such a diagram, commutativity means that we may replace R^{-i} with $\operatorname{cok} R^{-i-1} \to R^{-i}$ and the lower R terms with zero without changing the diagram. Thus we can also assume $R^k = 0$ for $k < -i$. Finally, we can use the same trick from earlier: pushout the map $R^{-i} \to B$ along the sequence $0 \to R^{-i} \to R^{-i+1} \to C \to 0$. This will give a diagram like this:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R^{-i} & \longrightarrow & R^{-i+1} & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \vdots & & \| & & \\
0 & \longrightarrow & B & \dashrightarrow & X & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

Now replace $R^{-i} \to R^{-i+1}$ with $B \to X$, which leaves the middle row exact and the maps still factor through as desired.

Corollary 1.2.28. *Let $\mathcal{A}$ be an abelian category with objects A, B. Then elements of $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$ correspond canonically to equivalence classes of acyclic complexes $P \in \operatorname{Ch}^{[-i,1]}(\mathcal{A})$ such that $P^{-i} = B$ and $P^1 = A$, as follows:*

(a) P, Q are equivalent iff there are morphisms $P \leftarrow R \to Q$ with R also in $\operatorname{Ext}^i_{\mathcal{A}}(A, B)$.

(b) Given such an acyclic complex P, and considering the associated quasi-isomorphism $\sigma_{\leq 0} P \to A$, its Ext *class is the one corresponding under*

$$\operatorname{Ext}^0(A, A) \xleftarrow{\cong} \operatorname{Hom}_{D(\mathcal{A})}(A, \sigma_{\leq 0} P) \to \operatorname{Ext}^i(A, B),$$

to the identity map, where the second map is $\operatorname{Hom}(A, -)$ applied to the projection $\sigma_{\leq 0} P \to B[i]$.

Proof. We already proved (a). Note that the arrows in (b) are obtained by applying $\operatorname{Hom}_{D(\mathcal{A})}(A, -)$ to $A[0] \leftarrow \sigma_{\leq 0} P \to B[i]$, which is exactly the roof that corresponds to the sequence P. Applying Hom and taking the element corresponding to 1 recovers the map associated to this roof. □

Notice that this puts a natural abelian group structure on the equivalence classes of exact sequences from B to A. To construct the addition in the general case is a bit annoying, but we can at least write down the zero element by copying the proofs above. Namely, we should start with $A[0] \leftarrow A[0] \xrightarrow{0} B[i]$, and then pushout repeatedly.

So, if $i > 1$, then the zero sequence looks like

$$0 \to B \xrightarrow{1} B \xrightarrow{0} 0 \to \cdots \to 0 \to A \xrightarrow{1} A \to 0.$$

However, if $i = 1$ then we get

$$0 \to B \to B \oplus A \to A \to 0,$$

the canonical split exact sequence. Moreover, in the Ext^1 case, note that two short exact sequences induce the same element of Ext^1 iff they both connect to another complex R as in:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & R & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

But by the five lemma, the existence of such a diagram already gives an isomorphism $P \xrightarrow{\sim} R$, and hence an isomorphism of exact sequences. Thus Ext^1 classifies short exact sequences up to isomorphism, with the zero element being the split sequence. A nice thing in the Ext^1 case which follows immediately from (b) of the above corollary is the fact that the extension class of an exact sequence $0 \to B \to C \to A \to 0$ is the image of 1_C under the map

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{A}}(A, B)$$

in the long exact sequence. In this case, we can describe the addition more clearly. Recall that in an additive category, to add two morphisms $f, g \colon X \to Y$, we use the following recipe:

$$X \xrightarrow{\Delta} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla} Y.$$

For addition of extensions of A by B, we do a similar thing. First take the sum of the two sequences as in the bottom of the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \dashrightarrow & Q & \longrightarrow & A & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \nabla} & \llcorner & \uparrow & & \| & & \\ 0 & \longrightarrow & B \oplus B & \longrightarrow & P & \dashrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow & \lrcorner & \downarrow{\scriptstyle \Delta} & & \\ 0 & \longrightarrow & B \oplus B & \longrightarrow & E \oplus F & \longrightarrow & A \oplus A & \longrightarrow & 0 \end{array}$$

Then pushout upwards along the addition map ∇. We claim the resulting sequence at the top is the sum of E and F. Indeed, this data induces a commutative diagram:

$$\begin{array}{ccc} \mathrm{Hom}(A, A) & \longrightarrow & \mathrm{Ext}^1(A, B) \\ \| & & \uparrow \\ \mathrm{Hom}(A, A) & \longrightarrow & \mathrm{Ext}^1(A, B \oplus B) \\ \downarrow & & \| \\ \mathrm{Hom}(A, A \oplus A) & \longrightarrow & \mathrm{Ext}^1(A, B \oplus B) \end{array}$$

Since Hom and Ext^1 preserve sums in the second variable, the images of $(1, 0)$ and $(0, 1)$ in the bottom right Ext^1 respectively are $[E]$ and $[F]$, and the left-lower vertical map sends $1 \mapsto (1, 1)$. Therefore the upper horizontal map sends $1 \mapsto [E] + [F]$ as desired.

Example 1.2.29. (This is misplaced.) Let $\mathcal{R}$ be a right preadapted class in $\mathcal{A}$. Let us prove two easy related statements.

1. If every object of $\mathcal{A}$ has a bounded resolution in $\mathcal{R}$, then every complex in $\mathrm{Ch}^b(\mathcal{A})$ has a bounded right resolution in $\mathcal{R}$. If every object of $\mathcal{A}$ has $\mathcal{R}$-resolution of length d, then every complex in $\mathrm{Ch}^{[a,b]}(\mathcal{A})$ has a right resolution in $\mathrm{Ch}^{[a,b+d]}(\mathcal{R})$.

Let $A^\bullet \in \mathrm{K}^+(\mathcal{A})$ be bounded, so that $A^i = 0$ for $i > N$, and take a resolution $A \to X$ where $X \in \mathrm{K}^+(\mathcal{R})$. The map $A^N \to X^N$ composed with d_X^N factors through $A^{N+1} = 0$, so it factors as $A^N \to \ker d_X^N \hookrightarrow X^N$. In particular, $A \to X$ factors as $A \to \tau_{\le N} X \to X$.

Since X can only have cohomology in degrees $\le N$, the second map is a quasi-isomorphism, so the first one is too. $\tau_{\le N} X$ has all but its last term $\ker d_X^N$ in $\mathcal{R}$. Take an exact sequence

$$0 \to \ker d_X^N \to Y^0 \to \cdots \to Y^d \to 0$$

with $Y^i \in \mathcal{R}$, and then form the new complex

$$Y \coloneqq (\cdots \to X^{N-2} \to X^{N-1} \to Y^0 \to Y^1 \to \cdots)$$

where the middle differential is $X^{N-1} \to \ker d_X^N \to Y^0$. Then Y has top degree $N + d$, and the obvious map $\tau_{\le N} \to Y$ is a quasi-isomorphism because Y is exact in $[N+1, d+N]$ and has Nth cohomology $\ker d_X^N / \operatorname{im} d_X^{N-1}$, as desired.

Here's a less direct statement on the level of cohomology. Now we'll assume that $\mathcal{R}$ is a right adapted class for a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$. If A is an object of $\mathcal{A}$, its RF-dimension is the largest i for which $R^iF(A) \neq 0$ (or ∞). If $A^\bullet$ is in $\mathrm{Ch}^+(\mathcal{A})$, then the RF-amplitude of A is the interval $[a, b]$ in which $RF(A)$ has nonzero cohomology.

2. If every object of $\mathcal{A}$ has finite RF-dimension, then every complex in $\mathrm{Ch}^b(\mathcal{A})$ has bounded RF-amplitude. If every object of $\mathcal{A}$ has RF-dimension $\le d$, then every complex in $\mathrm{Ch}^{[a,b]}(\mathcal{A})$ has RF-amplitude inside $[a, b+d]$.

To see this, let A be a bounded complex, and consider the triangle

$$\tau_{\le n-1} A \to \tau_{\le n} A \to H^n(A)[-n] \to \tau_{n-1} A[1]$$

in the derived category of $\mathcal{A}$. Applying the triangulated functor RF to this, we deduce that if $\tau_{\le n-1} A$ had cohomology concentrated in $[a, b_0 + d]$, and $H^n(A)[0]$ has no cohomology after degree d, then $\tau_{\le n} A$ is concentrated in $[a, b_0 + 1 + d]$. Thus we conclude by induction.

1.3. Homological algebra revisited

1.3.1. Basic notions. We now have quite a big toolkit to do homological algebra, and yet we have gotten no real results. Now we come to this. Let $\mathcal{A}$ be an abelian category, and suppose we have a short exact sequence

$$0 \to A \to B \to C \to 0$$

of chain complexes. If $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor with an adapted class $\mathcal{R}$, then RF exists and is triangulated on the derived categories. In particular if A, B, C are bounded below, then $A \to B \to C \to A[1]$ is an exact triangle, so $RF(A) \to RF(B) \to RF(C) \to RF(A)[1]$ is an exact triangle. In particular, there is a long exact sequence in cohomology:

$$\cdots \to H^i(RF(A)) \to H^i(RF(B)) \to H^i(RF(C)) \to H^{i+1}(RF(A)) \to \cdots$$

Likewise, if F was left exact, and A, B, C were bounded above, then the left derived functor $LF\colon D^-(\mathcal{A}) \to D^-(\mathcal{B})$ exists and we have an exact triangle

$$LF(A) \to LF(B) \to LF(C) \to LF(A)[1]$$

in $D^-(\mathcal{B})$, producing a similar long exact sequence in cohomology. In practice, we usually care most about objects of $\mathcal{A}$ themselves, and not chain complexes. In the situation above, we define the *classical right derived functors* by

$$R^iF \coloneqq \mathcal{A} \hookrightarrow D^+(\mathcal{A}) \xrightarrow{RF} D^+(\mathcal{B}) \xrightarrow{H^i} \mathcal{B}.$$

So, if $0 \to A \to B \to C \to 0$ is an exact sequence of objects in $\mathcal{A}$, then we get a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to \cdots$$

of right-derived functors, natural in the exact sequence. The first few terms of this are justified by the following lemma.

Lemma 1.3.1. *In the situation above, $R^iF = 0$ for $i < 0$, and there is a natural isomorphism $F \xrightarrow{\sim} R^0F$.*

Proof. For A in $\mathcal{A}$, e.g. from Lemma 1.2.15 we have a quasi-isomorphism $A[0] \to X^\bullet$ with $X^i = 0$ for $i < 0$ and X^i in $\mathcal{R}$ for all i. We have the commutative diagram on the left below,

$$\begin{array}{ccc} F(A)[0] & \longrightarrow & RF(A[0]) \\ \downarrow & & \downarrow\cong \\ F(X) & \xrightarrow{\cong} & RF(X) \end{array} \qquad \begin{array}{ccc} F(A) & \longrightarrow & R^0F(A) \\ \downarrow & & \cong\downarrow \\ H^0(F(X)) & \xrightarrow{\cong} & H^0(RF(X)) \end{array}$$

and on the right is what we get by applying H^0. First, if we apply H^i for $i < 0$, then $H^i(F(X)) = 0$ because $F(X^i) = 0$ for $i < 0$. The natural map $F \to R^0F$ is the top of the right-hand square. Since F is left exact, the map

$$F(A) \to H^0(F(X)) = \ker(F(d_X^0)) = F(\ker(d_X^0))$$

is given by applying F to the isomorphism $A \to \ker(d_X^0)$ (since $A \to X$ is a quasi-isomorphism), so is an isomorphism. □

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor, and suppose its right derived functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists. We say an object A of $\mathcal{A}$ is *F-acyclic* if $R^iF(A) = 0$ for all $i \neq 0$. F-acyclics behave nicely in long exact sequences:

Lemma 1.3.2. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and $\mathcal{R}$ an F-adapted class so that RF exists.*

(a) If $0 \to A \to B \to C \to 0$ is an exact sequence of objects of $\mathcal{A}$ and A is F-acyclic, then B is F-acyclic iff C is.

(b) Given a (bounded below) long exact sequence

$$0 \to A^0 \to A^1 \to A^2 \to \cdots$$

of F-acyclic objects, the corresponding sequence

$$0 \to F(A^0) \to F(A^1) \to F(A^2) \to \cdots$$

is exact as well.

The corresponding statements for right exact functors hold as well.

Proof. (a) The LES gives an exact sequence $0 \to R^iF(B) \to R^iF(C) \to 0$ for $i > 0$.

(b) Break up the sequence into short exact sequences:

$$0 \to A^0 \to A^1 \to C^1 \to 0, \qquad 0 \to C^i \to A^{i+1} \to C^{i+1} \to 0 \quad i \geq 1.$$

Since $R^1F(A^0) = 0$, F preserves the first exact sequence. Moreover, since A^0, A^1 are F-acyclic, (a) shows that C^1 is F-acyclic. But inductively, if C^i is F-acyclic, then C^{i+1} is too by the second SES and (a). Thus applying F to all of the exact sequences above we get exact sequences, and then we can put them back together. □

Next we prove a powerful result about F-acyclics. The first proof is by trickery, so we provide another direct proof afterwards. We'll give a third proof in a special case once we construct Cartan-Eilenberg resolutions.

Theorem 1.3.3. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and $\mathcal{R}$ an F-adapted class, so that its right derived functor $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists. For each complex $X^\bullet$ of F-acyclic objects, the natural map*

$$F(X) \to RF(X) \qquad \text{in } D^+(\mathcal{B})$$

is an isomorphism. In fact, if $\overline{\mathcal{R}}$ is the class of all F-acyclic objects, then $\overline{\mathcal{R}}$ is the unique largest class of objects adapted to F, and it is closed under taking summands, quotients (of F-acyclics by F-acyclics), and extensions.

Proof 1. The second statement, particularly that $\overline{\mathcal{R}}$ is an adapted class, implies the first immediately by uniqueness of the derived functor. $\overline{\mathcal{R}}$ contains 0 and is closed under biproducts. Part (b) of the lemma above was exactly the statement that F takes acyclic complexes in $\mathrm{K}^+(\overline{\mathcal{R}})$ to acyclic complexes.

Now, if $\mathcal{R}'$ is any F-adapted class, then for C in $\mathcal{R}'$, $C[0]$ is in $\mathrm{K}^+(\mathcal{R}')$, so $F(C)[0] \xrightarrow{\sim} RF(C[0])$ in the derived category, hence C is F-acyclic. Thus $\mathcal{R}' \subset \overline{\mathcal{R}}$. In particular, $\mathcal{R} \subset \overline{\mathcal{R}}$, so every object admits a mono to an object in $\overline{\mathcal{R}}$. Lastly,

the statement about quotients and extensions follows from the previous lemma, and the statement about summands follows from the fact that if $A \oplus B$ is acyclic, then $0 = R^iF(A \oplus B) = R^iF(A) \oplus R^iF(B)$ for $i > 0$. □

In most cases, we will have either injective or projective resolutions of objects. Here is the main set of lemmas telling us that they behave well. To make this especially simple, we first define the Hom complex.

Given $A^\bullet$ and $B^\bullet$ in $\mathrm{Ch}(\mathcal{A})$, we define the complex $\underline{\mathrm{Hom}}_{\mathcal{A}}(A, B)$ by

$$\underline{\mathrm{Hom}}^i_{\mathcal{A}}(A, B) \coloneqq \prod_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(A^p, B^{p+i}),$$

(a complex of abelian groups), where the differential is given by

$$df = d_B f - (-1)^i f \circ f_A,$$

or more precisely $(df)^p = d_B^{p+i} \circ f^p - (-1)^i f^{p+1} \circ d_A^p$. An element of $\underline{\mathrm{Hom}}^0(A, B)$ is a sequence of maps $f^p \colon A^p \to B^p$ for all p; if $df = 0$, then $df = fd$, i.e. f is a chain map. An element of $\underline{\mathrm{Hom}}^{-1}(A, B)$ is a sequence of degree -1 maps $h^p \colon A^p \to B^{p-1}$, and if a chain map f is of the form dh, then $f = df + fd$, i.e. f is nullhomotopic. In other words,

$$\ker d^0 = \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, B) \qquad \mathrm{im}\, d^{-1} = \{f : A \to B : f \sim 0\},$$

so $H^0(\underline{\mathrm{Hom}}(A, B)) = \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, B)$. One checks easily that

$$\underline{\mathrm{Hom}}(A, B)[n] = \underline{\mathrm{Hom}}(A, B[n]) = \underline{\mathrm{Hom}}(A[-n], B).$$

Now we construct a bicomplex $C^{\bullet,\bullet}$ whose product totalization is the Hom complex. Define $C^{p,q} \coloneqq \mathrm{Hom}_{\mathcal{A}}(A^{-p}, B^q)$, and let

$$d_v(\phi) = d_B^q \circ \phi \qquad d_h(\phi) = (-1)^{p+q+1}\phi \circ d_A^{-p-1}.$$

Then in each direction we have $d^2 = 0$, and the vertical and horizontal differentials anticommute. It is then clear from the definition that

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(A, B) = \mathrm{Tot}^{\Pi}(C).$$

Therefore we can apply our spectral sequence associated to a product totalization. Recall that in this case (e.g. for the first sequence) $C^{p,n-p} = 0$ for $p \ll 0$. This translates to asking for $\mathrm{Hom}(A^p, B^{n+p}) = 0$ for $p \gg 0$, e.g. if A or B is bounded above. For the second sequence, we get the condition $\mathrm{Hom}(A^{q-n}, B^q)$ for $q \ll 0$, which is satisfied e.g. if A or B is bounded below. Here, we are using the fact that arbitrary products are exact in the category of modules. Thus:

Lemma 1.3.4. *Suppose A or B is a bounded above complex in $\mathrm{Ch}(\mathcal{A})$. Then there is a spectral sequence*

$$E_2^{p,q} = H_h^p(H_v^q(\mathrm{Hom}(A^*, B^*))) \implies H^{p+q}(\underline{\mathrm{Hom}}_{\mathcal{A}}(A, B)).$$

If A or B is bounded below, then we have a spectral sequence

$$E_2^{p,q} = H_v^p(H_h^q(\mathrm{Hom}(A^*, B^*))) \implies H^{p+q}(\underline{\mathrm{Hom}}_{\mathcal{A}}(A, B)).$$

In both cases, the filtration on the resulting object is described by the totalization above and Thm. 1.1.4.

Recall that an object I (resp. P) of $\mathcal{A}$ is called *injective* (resp. projective) if $\mathrm{Hom}_{\mathcal{A}}(-, I)$ (resp. $\mathrm{Hom}_{\mathcal{A}}(P, -)$) is an exact functor.

Corollary 1.3.5. *Let $I^\bullet$ be a bounded below complex of injective objects of $\mathcal{A}$.*

(a) If $A^\bullet$ is an acyclic complex, then $\underline{\mathrm{Hom}}_{\mathcal{A}}(A, I)$ is acyclic.

(b) If $q\colon B \to C$ is a quasi-isomorphism, then

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C, I) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(B, I)$$

is an isomorphism of abelian groups.

(c) Any quasi-isomorphism $q\colon I \to A$ has a one-sided homotopy inverse, i.e. a quasi-isomorphism $r\colon A \to I$ such that $rq \sim 1_I$. In particular, a quasi-isomorphism between bounded-below complexes of injective objects is a homotopy equivalence.

(d) The natural map

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, I) \to \mathrm{Hom}_{D(\mathcal{A})}(A, I)$$

is an isomorphism of abelian groups for any A in $D(\mathcal{A})$.

The corresponding statements hold for a bounded-above complex of projectives.

Proof. If $A^\bullet$ is acyclic, then since I^n is injective for each n, the sequence

$$\cdots \leftarrow \mathrm{Hom}(A^{i-1}, I^n) \leftarrow \mathrm{Hom}(A^i, I^n) \leftarrow \mathrm{Hom}(A^{i+1}, I^n) \leftarrow \cdots$$

is exact for all $n \in \mathbb{Z}$. Thus $H_h^q(\mathrm{Hom}(A^*, I^n)) = 0$ for all n, so by the spectral sequence of Lemma 1.3.4, $\underline{\mathrm{Hom}}(A, I)$ is acyclic. Since cones commute with functors, this means that for any quasi-isomorphism $B \to C$, the map

$$\underline{\mathrm{Hom}}(C, I) \to \underline{\mathrm{Hom}}(B, I)$$

is a quasi-isomorphism. Taking H^0 gives the statement of (b). For (c), we have an isomorphism of abelian groups

$$- \circ q\colon \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, I) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(I, I),$$

so there is a map $r\colon A \to I$ such that $rq \sim 1_I$. Since q and 1_I are quasi-isomorphisms, r is as well. So, q is a split monomorphism in $\mathrm{K}(\mathcal{A})$ and r is a split epimorphism. In the case that $A = I'$ is also a bounded below complex of injectives, r satisfies the same hypotheses and hence is a split monomorphism itself. Thus r is a homotopy equivalence, so q is too.

For the last result, any map $A \to I$ is represented by a right roof

$$A \to X \xleftarrow{q} I$$

for some quasi-isomorphism q. By (c), there is a one-sided homotopy inverse $r\colon X \to I$, so we get an equivalent roof (commutative in $\mathrm{K}(\mathcal{A})$):

$$\begin{array}{ccccc} A & \longrightarrow & X & \xleftarrow{q} & I \\ & \searrow & \downarrow{\scriptstyle r} & \swarrow{\scriptstyle 1_I} & \\ & & I & & \end{array}$$

Then the map $A \to I$ on the bottom right is equal to the one we started with, and induced from $\mathrm{K}(\mathcal{A})$. Similarly, if $a\colon A \to I$ becomes zero in $D(\mathcal{A})$, then there's

a quasi-isomorphism $q\colon I \to X$ such that the composite $A \xrightarrow{a} I \xrightarrow{q} X$ is 0. Then $rq \sim 1_I$ shows that $a \sim 0$. □

The above result will be central to the rest of these notes. Let us start to deduce some consequences. If A is an object of $\mathcal{A}$, then an *injective resolution* is a quasi-isomorphism $A[0] \to I$ where I is a complex of injectives with $I^i = 0$ for $i < 0$. Likewise a projective resolution is one of the form $P \to A[0]$ with $P^i = 0$ for $i > 0$.

Proposition 1.3.6. *Let $\mathcal{A}$ be an abelian category. Let $A, B \in \operatorname{Ob}\mathcal{A}$.*

(a) An object I is injective iff $\operatorname{Ext}^1_{\mathcal{A}}(-, I) = 0$ iff $\operatorname{Ext}^i_{\mathcal{A}}(-, I) = 0$ for all $i > 0$.

(b) An object P is projective iff $\operatorname{Ext}^1_{\mathcal{A}}(P, -) = 0$ iff $\operatorname{Ext}^i_{\mathcal{A}}(P, -) = 0$ for all $i > 0$.

(c) If $B[0] \to I^\bullet$ is an injective resolution, then there is a natural isomorphism

$$\operatorname{Ext}^i_{\mathcal{A}}(A, B) \xrightarrow{\sim} H^i(\operatorname{Hom}_{\mathcal{A}}(A, I^\bullet))$$

in A. Likewise, if $P^\bullet \to A$ is a projective resolution, then there is a natural isomorphism

$$H^i(\operatorname{Hom}_{\mathcal{A}}(P^\bullet, B)) \xrightarrow{\sim} \operatorname{Ext}^i_{\mathcal{A}}(A, B)$$

in B.

Proof. The injective and projective statements are dual, so we prove the injective versions. If I is injective, then $I[i]$ is a bounded below complex of injectives, so

$$\operatorname{Hom}_{D^*(\mathcal{A})}(A[0], I[i]) = \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(A[0], I[i]) = 0,$$

since for $i > 0$ every chain map $A[0] \to I[i]$ is zero.

(c) If $B \to I^\bullet$ is a qis, then we have isomorphisms

$$\begin{aligned}\operatorname{Hom}_{D(\mathcal{A})}(A[0], B[i]) &\xrightarrow{\sim} \operatorname{Hom}_{D(\mathcal{A})}(A[0], I[i]) \\ &\cong \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(A[0], I[i]) \\ &\cong H^i(\underline{\operatorname{Hom}}_{\mathcal{A}}(A[0], I)),\end{aligned}$$

but since $A[0]$ is a 0-complex, we have $\underline{\operatorname{Hom}}^i(A[0], I) = \operatorname{Hom}(A, I^i)$ and the differentials coincide as well. □

Example 1.3.7. It is not too difficult to show that elements of $\operatorname{Ext}^1(C, A)$ are in natural correspondence with extensions of C by A. Regardless, if $0 \to A \to B \to C \to 0$ is such an extension, then we define the *extension class* to be the image of 1_C in $\operatorname{Ext}^1_{\mathcal{A}}(C, A)$ under the induced map:

$$\cdots \to \operatorname{Hom}_{\mathcal{A}}(C, B) \to \operatorname{Hom}_{\mathcal{A}}(C, C) \xrightarrow{\delta} \operatorname{Ext}^1_{\mathcal{A}}(C, A) \to \cdots$$

Note that the extension splits iff the extension class is zero, because the latter condition is (by exactness) equivalent to the existence of a section of $B \to C$. So, e.g., I is injective iff every SES with I as the first term splits.

Example 1.3.8. Consider the exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ over $\mathbb{Z}$. This is a projective resolution of $\mathbb{Z}/2$, so to compute $\operatorname{Ext}^*(\mathbb{Z}/2, \mathbb{Z})$ we apply $\operatorname{Hom}(-, \mathbb{Z})$ to $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$ to get the same thing back, and the cohomology of this is

$$\operatorname{Ext}^0(\mathbb{Z}/2, \mathbb{Z}) = 0 \qquad \operatorname{Ext}^1(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \qquad \operatorname{Ext}^i(\mathbb{Z}/2, \mathbb{Z}) = 0, i > 0.$$

Since the sequence above was not split, its extension class was the unique nonzero element of $\operatorname{Ext}^1(\mathbb{Z}/2, \mathbb{Z})$.

Continuing in our development of the theory, we consider the role of injectives/projectives for computing derived functors.

Proposition 1.3.9. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. Let $\mathcal{R}$ be an F-adapted class, so that RF exists.*

(a) Injective objects of $\mathcal{A}$ are F-acyclic.
(b) If $I^\bullet$ is a bounded-below complex of injective objects, then the natural map $F(I) \to RF(I)$ in $D^+(\mathcal{B})$ is an isomorphism. In particular, if $A^\bullet \to I^\bullet$ is an injective resolution, then canonically $RF(A) \cong F(I)$.

In general, the class $\mathcal{I}_\mathcal{A}$ of injective objects is adapted to any left exact functor F if (and only if) $\mathcal{A}$ has enough injectives. The corresponding statements hold for projectives.

Proof. (b) immediately follows from (a). For the statement about adapted classes, consider an exact sequence

$$0 \to I^0 \to I^1 \to \cdots$$

(bounded below) of injectives, and break it up into short exact sequences

$$0 \to I^0 \to I^1 \to C^1 \to 0 \qquad 0 \to C^i \to I^{i+1} \to C^{i+1} \to 0.$$

Applying $\operatorname{Ext}^i(A, -)$ to the first sequence shows that $\operatorname{Ext}^i(-, C^1)$, i.e. C^1 is injective. Doing it to the second sequence shows that C^{i+1} is injective if C^i is. Hence by induction, all of the sequences split, so they're all preserved by F, and we can recombine. Thus if $\mathcal{A}$ has enough injectives $\mathcal{I}_\mathcal{A}$ is F-adapted for any F.

For (a), embed an injective $0 \to I \to X \to Y \to 0$ for X in $\mathcal{R}$. Then the sequence splits, so I is a direct summand of X, which is F-acyclic, so I is F-acyclic. □

Remark 1.3.10. Let $\mathcal{A}$ have enough injectives. Then $\mathcal{I}_\mathcal{A}$ is a preadapted class, so in the composite

$$\mathrm{K}^+(\mathcal{I}_\mathcal{A}) \to \mathrm{K}^+(\mathcal{I}_\mathcal{A})[S_{\mathcal{I}_\mathcal{A}}^{-1}] \to D^+(\mathcal{A})$$

the second map is an equivalence. But the first map is also an equivalence, because all quasi-isomorphisms between bounded above complexes of injectives are homotopy equivalences. Because of this, much of the work of the previous (and this) section would be a lot easier if we made this assumption, but we went to all this trouble because in algebraic geometry one doesn't have enough projectives.

Example 1.3.11. Let $\mathcal{A}$ be a *semisimple* abelian category, which means that every exact sequence splits, so equivalently every object is projective, or equivalently every object is injective. Thus $\mathrm{K}^+(\mathcal{A}) \xrightarrow{\sim} D^+(\mathcal{A})$ is an equivalence, and likewise $\mathrm{K}^-(\mathcal{A}) = D^-((A))$. For a complex $C^\bullet$, let $H^\bullet(C)$ denote the complex:

$$\cdots \to H^{i-1}(C) \xrightarrow{0} H^i(C) \xrightarrow{0} H^{i+1}(C) \to \cdots$$

By choosing splittings $H^i(C) \oplus \operatorname{im} d^{i-1} = \ker d^i$, including into the kernel gives a map $H^\bullet(C) \to C^\bullet$ which is a quasi-isomorphism. Thus every object in the derived category is isomorphic to a complex with zero differentials. In the bounded above or bounded below case, we know this must be a homotopy equivalence. By definition, for such a complex a nullhomotopic map is necessarily zero, so this gives an equivalence $\mathrm{K}^+(\mathcal{A}) = D^+(\mathcal{A})$ with $\mathbb{Z}$-graded bounded above complexes, and likewise in the bounded below case.

If we look more closely we can get this for the full derived category. The map $H^\bullet(C) \to C$ constructed above is a termwise injection, with some acyclic kernel A as in

$$0 \to H^\bullet(C) \to C \to A \to 0.$$

If we additionally split $0 \to \ker d^i \to C^i \to \operatorname{im} d^i \to 0$, we get isomorphisms

$$C^i = H^i(C) \oplus \operatorname{im} d^{i-1} \oplus \operatorname{im} d^i \equiv H^i(C) \oplus A,$$

where A is the exact sequence $\operatorname{im} d^{i-1} \oplus \operatorname{im} d^i \to \operatorname{im} d^i \oplus \operatorname{im} d^{i+1} \to \cdots$ that is the identity on the common factor and zero otherwise. Thus A is precisely the cone of the complex

$$\cdots \to \operatorname{im} d^{i-1} \xrightarrow{0} \operatorname{im} d^i \xrightarrow{0} \operatorname{im} d^{i+1} \to \cdots.$$

Since the sequence is termwise split, there is an exact triangle $H(C) \to C \to A \to H(C)[1]$ already in $\mathrm{K}(\mathcal{A})$ with $A \cong 0$, so $H(C) \to C$ is a homotopy equivalence. (The termwise split detail will be explained later.)

The point of this example is that we will often end up with an object of $D(k)$, i.e. the derived category of k-vector spaces, and this tells us that two such objects are isomorphic iff they have isomorphic cohomology.

1.3.2. Classical derived functors. Let us briefly revisit the classical theory of derived functors using our new language. In the last section, we defined Ext without any reference to derived functors, so let us explain the connection now.

In the previous section, Cor. 1.3.5(a) showed that if $I^\bullet \in \mathrm{K}^+(\mathcal{I}_\mathcal{A})$, then there is a well-defined functor

$$\underline{\mathrm{Hom}}_\mathcal{A}(-, I) \colon D(\mathcal{A})^{\mathrm{op}} \to D(\mathbb{Z}),$$

where $D(\mathbb{Z}) = D(\mathbf{Ab})$. Likewise, if P was a bounded above complex of projectives, then $\underline{\mathrm{Hom}}(P, -)$ was well-defined on $D(\mathcal{A})$. If $I \to I'$ is a chain map, then we get a corresponding natural transformation $\underline{\mathrm{Hom}}_\mathcal{A}(-, I) \to \underline{\mathrm{Hom}}_\mathcal{A}(-, I')$ on complexes, and thus as functors on the derived category above. Such a transformation is independent of the homotopy class of $I \to I'$, because the hom complex is already well-defined on the homotopy categories. So this construction gives a functor

$$\underline{\mathrm{Hom}}_\mathcal{A} \colon D(\mathcal{A})^{\mathrm{op}} \times \mathrm{K}^+(\mathcal{I}_\mathcal{A}) \to D(\mathbb{Z})$$

Thus, if $\mathcal{A}$ has enough injectives, then by choosing an inverse $D^+(\mathcal{A}) \to \mathrm{K}^+(\mathcal{I}_\mathcal{A})$ we get a functor

$$\mathrm{RHom}_\mathcal{A} \colon D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathbb{Z}).$$

The choice of an inverse gives, for each $B \in \mathrm{K}^+(\mathcal{A})$, a natural map

$$\underline{\mathrm{Hom}}_\mathcal{A}(-, B) \to \mathrm{RHom}_\mathcal{A}(-, B)$$

of functors, and hence a natural map $\underline{\mathrm{Hom}}_\mathcal{A}(A, B) \to \mathrm{RHom}_\mathcal{A}(A, B)$ in $D(\mathbb{Z})$ for all $A \in D(\mathcal{A})^{\mathrm{op}}$ and $B \in D^+(\mathcal{A})$. By definition, it is an isomorphism if B is a bounded-below complex of injectives.

Likewise, if there are enough projectives in $\mathcal{A}$ then we get another well-defined functor

$$\mathrm{RHom}_\mathcal{A} \colon D^-(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(\mathbb{Z})$$

with the same name. Note that we haven't really done anything complicated here: to compute $\mathrm{RHom}(A, B)$, say if the right entry is bounded above, take an injective resolution $B \to I$, then $\mathrm{RHom}(A, B) = \underline{\mathrm{Hom}}(A, I)$, the Hom-complex. The 'overlapping' case works as follows: we will stick to the case where we are assuming enough injectives, however.

$$\begin{array}{ccccc}
\mathrm{RHom}(A, B) & \xrightarrow{\cong} & \mathrm{RHom}(A, I) & = & \underline{\mathrm{Hom}}(A, I) \\
\cong\downarrow & & \downarrow\cong & & \cong\downarrow \\
\mathrm{RHom}(P, B) & \xrightarrow{\cong} & \mathrm{RHom}(P, I) & = & \underline{\mathrm{Hom}}(P, I) \\
\uparrow\downarrow\cong & & \| & & \\
\underline{\mathrm{Hom}}(P, B) & \xrightarrow{\cong} & \underline{\mathrm{Hom}}(P, I) & &
\end{array}$$

In the diagram, since we're taking the injective definition, our definition for $\mathrm{RHom}(A, B)$ is to first take an injective resolution $B \to I$ and compute the Hom complex. Now if $P \to A$ is any quasi-isomorphism, then the manipulations on the right side are valid. If $P \to A$ is a bounded above projective resolution (e.g. even if A is unbounded in both directions), then the bottom steps are valid. This shows that as long as we replace one of the entries, we can compute using whatever resolution we want.

Retain the assumption of enough injectives. As we have already seen, taking cohomology of RHom computes maps in the derived category:

$$\begin{aligned}
H^i(\mathrm{RHom}(A, B)) &= \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A, I[i]) \\
&= \mathrm{Hom}_{D(\mathcal{A})}(A, I[i]) \\
&= \mathrm{Hom}_{D(\mathcal{A})}(A, B[i])
\end{aligned}$$

(for possibly unbounded A, and bounded below B). Likewise, if we have enough projectives, the same thing works via the first variable for bounded above A and possibly unbounded B.

Proposition 1.3.12. *Suppose $\mathcal{A}$ has enough injectives. For any object A of $\mathcal{A}$, the functor*

$$\mathrm{RHom}(A, -) \colon D^+(\mathcal{A}) \to D^+(\mathbb{Z})$$

is canonically the right derived functor of $\mathrm{Hom}(A,-)\colon \mathcal{A}\to \mathbf{Ab}$*. Its classical right derived functors agree with* Ext*:*

$$H^i \circ \mathrm{RHom}(A,-) \cong R^i\mathrm{Hom}(A,-) \cong \mathrm{Ext}^i_{\mathcal{A}}(A,-).$$

Likewise, a similar statement holds if $\mathcal{A}$ *has enough projectives.*

Proof. Since $\mathcal{A}$ has enough injectives, the supposed right derived functor is well-defined. The natural map $\mathrm{Hom}(A,-)\to R\mathrm{Hom}(A,-)$ is an isomorphism on bounded above complexes of injectives, and we claim the same is true for $\mathrm{RHom}(A,-)\to \mathrm{Hom}(A,-)$. Indeed, putting in a complex I of injectives we obtain $\underline{\mathrm{Hom}}(A[0],I) = \mathrm{Hom}(A,I^*)$ as desired. The second statement is a consequence of the fact that we showed Ext^i could be computed by an injective resolution in the second variable. □

The tensor product is a bit more intuitive. The fundamental lemma we need is something we proved a long time ago:

Lemma 1.3.13. *Let* $\mathcal{A}$ *be endowed with a compatible monoidal structure, and let* P *be a bounded above complex of flat objects. Assume that taking countable direct sums in* $\mathcal{A}$ *is exact. Then* $P\otimes A\cong A\otimes P$ *is acyclic for any acyclic complex* A*.*

Proof. This was Example 1.1.6. □

Since the tensor product complex commutes with cones (check this), the functor $P\otimes(-)$ preserves quasi-isomorphisms if P is a bounded above complex of flats. In particular, for each such P we get functors

$$P\otimes(-), (-)\otimes P\colon D(\mathcal{A})\to D(\mathcal{A})$$

on the full derived category. Now assume $\mathcal{R}$ denotes the class of flat objects, and assume $\mathcal{A}$ has enough flat objects so that $\mathcal{R}$ is a preadapted class. Then we have a triangulated equivalence

$$\mathrm{K}^-(\mathcal{R})[S_{\mathcal{R}}^{-1}]\to D^-(\mathcal{A}).$$

The functor $\mathrm{K}^-(\mathcal{R})\times D(\mathcal{A})\to D(\mathcal{A})$ (and the flipped one) satisfies the property that for P a bounded above acyclic complex of flats, and $B\in D^-(\mathcal{A})$, $P\otimes B$ is acyclic. Indeed, take a quasi-isomorphism $Q\to B$ with $Q\in \mathrm{K}^-(\mathcal{R})$, then $P\otimes Q\to P\otimes B$ is a quasi-isomorphism, but P is acyclic so $P\otimes Q$ is acyclic. It follows that we get a functor:

$$(-)\otimes^{\mathbf{L}}(-)\colon D^-(\mathcal{A})\times D^-(\mathcal{A})\to D^-(\mathcal{A}).$$

Let us further assume that tensoring with a fixed object is right exact (e.g. our category is closed monoidal). Then we have:

Lemma 1.3.14. *Keep the notation and assumptions as above. For* X,Y,Z *in* $D^-(\mathcal{A})$*:*

(a) We have natural isomorphisms $X\otimes^{\mathbf{L}} Y\cong Y\otimes^{\mathbf{L}} X$*.*
(b) We have natural isomorphisms $(X\otimes^{\mathbf{L}} Y)\otimes^{\mathbf{L}} Z\cong X\otimes^{\mathbf{L}}(Y\otimes^{\mathbf{L}} Z)$*.*

(c) If A is an object of $\mathcal{A}$, then the functor $A \otimes^{\mathbf{L}} (-)\colon D^-(\mathcal{A}) \to D^-(\mathcal{A})$ is the left derived functor of $A \otimes (-)\colon \mathcal{A} \to \mathcal{A}$, a functor for which $\mathcal{R}$ is an adapted class. The same holds the other way around.

PROOF. (a) If $P^\bullet \to X^\bullet$ is a flat resolution, then

$$X \otimes^{\mathbf{L}} Y \cong P \otimes Y \cong Y \otimes P \cong Y \otimes^{\mathbf{L}} X.$$

For (b), let $P \to X$ and $Q \to Z$ be flat resolutions, then by functoriality

$$(X \otimes^{\mathbf{L}} Y) \otimes^{\mathbf{L}} Z \cong (P \otimes^{\mathbf{L}} Y) \otimes^{\mathbf{L}} Z \cong (P \otimes^{\mathbf{L}} Y) \otimes^{\mathbf{L}} Q.$$

The latter expression is equal to $(P \otimes Y) \otimes Q$ which is $P \otimes (Y \otimes Q)$, and then we are done by the same argument backwards.

For (c), note that indeed $\mathcal{R}$ is an adapted class: a special case of what we have said above is that tensoring with A sends bounded above acyclic complexes of flats to acyclics. □

In light of this, we denote the higher left derived functors by

$$\operatorname{Tor}^i_{\mathcal{A}}(A, -) = L^i(A \otimes -) \qquad \operatorname{Tor}^i_{\mathcal{A}}(-, B) = L^i(- \otimes B),$$

both of which coincide with $H^i(A \otimes^{\mathbf{L}} B)$.

Lemma 1.3.15. *In the situation above, for all A, B in $\mathcal{A}$:*

(a) $\operatorname{Tor}^i_{\mathcal{A}}(A, B) = 0$ for $i > 0$, and $\operatorname{Tor}^0(A, B) = A \otimes B$
(b) $\operatorname{Tor}^i_{\mathcal{A}}(A, B) \cong \operatorname{Tor}^i_{\mathcal{A}}(B, A)$ for all A, B.
(c) An object A is flat iff $\operatorname{Tor}^i_{\mathcal{A}}(A, -) = \operatorname{Tor}^i_{\mathcal{A}}(-, A) = 0$ for $i < 0$, (iff $\operatorname{Tor}^{-1}_{\mathcal{A}}(A, -) = 0$). Consequently, any projective object is flat.

PROOF. (a) is because Tor's are the classical left derived functors of a functor, and (b) is a direct consequence of Lemma 1.3.14(a). For (c), if Tor^{-1} is zero then tensoring with A is exact by the LES. Conversely, an object of the adapted class for F is always F-acyclic. □

Example 1.3.16. Suppose $0 \to A \to B \to C \to 0$ is an exact sequence in $\mathcal{A}$ and C is flat. For any X in $\mathcal{A}$,

$$0 = \operatorname{Tor}^{i-1}(X, C) \to \operatorname{Tor}^i(X, A) \to \operatorname{Tor}^i(X, B) \to \operatorname{Tor}^i(X, C) = 0$$

is exact, so the natural map $\operatorname{Tor}^i(-, A) \to \operatorname{Tor}^i(-, B)$ is an isomorphism for all $i < 0$. Therefore B is flat iff C is flat.

Moreover, for any such X, the first part of this sequence is

$$0 = \operatorname{Tor}^{-1}(X, C) \to X \otimes A \to X \otimes B \to X \otimes C \to 0,$$

so tensoring any object with the sequence is exact if C is flat. Note that we are using the symmetry of Tor in a crucial way here.

Remark 1.3.17. If we didn't yet know that projectives were flat, but we had enough projectives, then we note that projectives are automatically an adapted class for $A \otimes (-)$ for all A in $\mathcal{A}$, so we could still define the derived functors $\operatorname{Tor}^*(A, -)$ and say $\operatorname{Tor}^*(-, A)'$. But, it's *not* true that projectives would have to be flat, and thus it's not true that these two functors would coincide.

Example 1.3.18. Let's compute $\mathbb{Z}/m \otimes^{\mathbf{L}} \mathbb{Z}/n$ in the derived category $D^-(\mathbb{Z})$. We can pick one of them to resolve, but for fun let's resolve both simultaneously and take the tensor product complex:

$$\begin{aligned}\mathbb{Z}/m \otimes^{\mathbf{L}} \mathbb{Z}/n &= \big(0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to 0\big) \otimes \big(0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0\big) \\ &= \big(0 \to \mathbb{Z} \xrightarrow{(m \quad -n)^T} \mathbb{Z}^2 \xrightarrow{(n \quad m)} \mathbb{Z} \to 0\big),\end{aligned}$$

where the rightmost $\mathbb{Z}$ has degree 0. If $m, n \neq 0$ and $d = \gcd(m, n)$, then the cohomology of this complex is

$$\operatorname{Tor}^0(\mathbb{Z}/m, \mathbb{Z}/n) = \operatorname{Tor}^{-1}(\mathbb{Z}/m, \mathbb{Z}/n) = \mathbb{Z}/d \qquad \operatorname{Tor}^{-i}(\mathbb{Z}/m, \mathbb{Z}/n) = 0, i > 2.$$

Example 1.3.19. Let k be a field, consider the ring $A = k[\epsilon]/(\epsilon^2)$, and view $k = A/(\epsilon)$ as an A-module. Consider the sequence

$$\cdots \to A \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \to A/(\epsilon) = k \to 0.$$

The kernel of the last map is (ϵ), which is the image of the previous map. But $\epsilon\colon A \to A$ has a kernel: $\epsilon(a + b\epsilon) = a\epsilon$, so its kernel is (ϵ). Thus this repeats and we get an infinite free resolution as above. Therefore,

$$\begin{aligned}k \otimes^{\mathbf{L}} k &= k \otimes \big[\cdots \to A \xrightarrow{\epsilon} A \xrightarrow{\epsilon} A \to 0\big] \\ &= \big[\cdots \to k \xrightarrow{0} k \xrightarrow{0} k \to 0\big],\end{aligned}$$

because ϵ acts by zero on k. Thus $k \otimes^{\mathbf{L}} k = \bigoplus_{i \geq 0} k[i]$ has cohomology in every degree ≤ 0. As we will discuss, we have just computed the derived self-intersection of a point inside a double point. This also gives an example of a module with infinite Tor-dimension, and hence infinite projective dimension.

Now let us study what we can say given two adjoint functors $F\colon \mathcal{A} \to \mathcal{B} : G$ with $F \dashv G$. Given complexes $A^\bullet$ and $B^\bullet$, we have a natural isomorphism

$$\operatorname{Hom}_{\mathcal{B}}(F(A^i), B^{n+i}) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{A}}(A^i, G(B^{n+i})),$$

and naturality in both components means that this commutes with the differential. It follows that we get a (natural) isomorphism

$$\underline{\operatorname{Hom}}^\bullet_{\mathcal{B}}(F(A), B) \xrightarrow{\sim} \underline{\operatorname{Hom}}^\bullet_{\mathcal{A}}(A, G(B))$$

of Hom complexes. If we take zero-cycles or H^0, we obtain

$$\operatorname{Hom}_{\operatorname{Ch}(\mathcal{B})}(F(A), B) \cong \operatorname{Hom}_{\operatorname{Ch}\mathcal{A}}(A, G(B))$$

and

$$\operatorname{Hom}_{\mathrm{K}(\mathcal{B})}(F(A), B) \cong \operatorname{Hom}_{\mathrm{K}(\mathcal{A})}(A, G(B))$$

respectively, so we expect these adjunctions to pass to the derived category. If we had been working with unbounded derived functors, this would be a lot easier. In our setting, we will not be able to get this in full generality, because our derived functors aren't even defined on the same categories. However, if we assume e.g. that $\mathcal{B}$ has enough injectives, then we can say something substantive.

THEOREM 1.3.20. *Let $F\colon \mathcal{A} \rightleftarrows \mathcal{B}\colon G$ be an adjunction between abelian categories $\mathcal{A}$ and $\mathcal{B}$. Assume F has a (left) adapted class $\mathcal{R}_F$ and $\mathcal{B}$ has enough injectives, so that the derived functors*

$$LF\colon D^-(\mathcal{A}) \to D^-(\mathcal{B}) \qquad RG\colon D^+(\mathcal{B}) \to D^+(\mathcal{A})$$

both exist. Then for $A \in D^-(\mathcal{A})$ and $B \in D^+(\mathcal{B})$ there is a natural isomorphism

$$\mathrm{Hom}_{D(\mathcal{B})}(LF(A), B) \cong \mathrm{Hom}_{D(\mathcal{A})}(A, RG(B)).$$

If $\mathcal{A}$ has either enough injectives or enough projectives, then there is a natural isomorphism

$$\mathrm{RHom}_{\mathcal{B}}(LF(A), B) \cong \mathrm{RHom}_{\mathcal{A}}(A, RG(B))$$

in $D(\mathbb{Z})$ for $A \in D^-(\mathcal{A})$ and $B \in D^+(\mathcal{B})$ as well.

PROOF. Let $X \to A$ be a qis with $X \in \mathrm{K}^-(\mathcal{R}_F)$ and $B \to I$ a qis with I a bounded below complex of injectives. Then

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{B})}(LF(A), B) &= \mathrm{Hom}_{D(\mathcal{B})}(F(X), I)\\ &= \mathrm{Hom}_{\mathrm{K}(\mathcal{B})}(F(X), I)\\ &= \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X, G(I)).\end{aligned}$$

Note that if C is an acyclic complex in $\mathrm{K}^-(\mathcal{R}_F)$, then

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(C, G(I)) = \mathrm{Hom}_{\mathrm{K}(\mathcal{B})}(F(C), I) = 0$$

because $F(C)$ is acyclic and I is K-injective. Hence by Lemma 1.2.16, we have

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X, G(I)) = \mathrm{Hom}_{D(\mathcal{A})}(X, G(I)) = \mathrm{Hom}_{D(\mathcal{A})}(A, RG(B)),$$

so this proves the first statement. For the second one, note that we need enough injectives or projectives in $\mathcal{A}$ simply to define RHom. Since LF preserves bounded above-ness and RG preserves bounded below-ness, the two sides of the desired isomorphism are defined.

Almost all of the steps we did before still work:

$$\begin{aligned}\mathrm{RHom}_{\mathcal{B}}(LF(A), B) &= \mathrm{RHom}_{\mathcal{B}}(F(X), I)\\ &= \underline{\mathrm{Hom}}_{\mathcal{B}}(F(X), I)\\ &= \underline{\mathrm{Hom}}_{\mathcal{A}}(X, G(I)),\end{aligned}$$

and we have the canonical map

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(X, G(I)) \to \mathrm{RHom}_{\mathcal{A}}(X, G(I)) = \mathrm{RHom}_{\mathcal{A}}(A, RG(B)).$$

Combining these gives a natural map

$$\mathrm{RHom}_{\mathcal{B}}(LF(A), B) \to \mathrm{RHom}_{\mathcal{A}}(A, RG(B))$$

which we claim is an isomorphism. We can check this after applying H^i for all $i \in \mathbb{Z}$, which gives the map

$$\mathrm{Hom}_{D(\mathcal{B})}(LF(A), B[i]) \to \mathrm{Hom}_{D(\mathcal{A})}(A, RG(B)[i]).$$

Since $RG(B)[i] = RG(B[i])$, it is easy to see that this recovers the isomorphism proved in the first part. □

Remark 1.3.21. Here is an even easier case than the above where we can say something much stronger. Suppose we have an adjunction $F\colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ as before but F is an exact functor and $\mathcal{B}$ has enough injectives. Then F descends to $D^*(\mathcal{A}) \to D^*(\mathcal{B})$, and in particular $D^+(\mathcal{A}) \to D^+(\mathcal{B})$. Moreover, G preserves injective objects because

$$\mathrm{Hom}_{\mathcal{A}}(-, G(I)) = \mathrm{Hom}_{\mathcal{A}}(F(-), I)$$

is a composition of exact functors if I is injective in $\mathcal{B}$. Moreover, if $I^\bullet \in \mathrm{K}^+(\mathcal{I}_{\mathcal{B}})$, we have $F(RG(I^\bullet)) \leftarrow FG(I^\bullet) \to I^\bullet$ on the homotopy category with the left map an isomorphism, and thus we obtain a natural map $F(RG(B)) \to B$ natural in $B \in D^+(\mathcal{B})$. Therefore this induces a natural map

$$\mathrm{Hom}_{D^+(\mathcal{A})}(A, RG(B)) \to \mathrm{Hom}_{D^+(\mathcal{A})}(FA, F(RG(B))) \to \mathrm{Hom}_{D^+(\mathcal{B})}(FA, B)$$

on Homs in the derived category, which we claim is an isomorphism. Indeed to check this we may assume $B = I^\bullet \in \mathrm{K}^+(\mathcal{I}_{\mathcal{B}})$, and $A = A^\bullet$, and then it follows from

$$\begin{aligned}\mathrm{Hom}_{D^+(\mathcal{A})}(A^\bullet, G(I^\bullet)) &= \mathrm{Hom}_{\mathrm{K}^+(\mathcal{A})}(A^\bullet, G(I^\bullet)) \\ &\cong \mathrm{Hom}_{\mathrm{K}^+(\mathcal{B})}(F(A^\bullet), I^\bullet) \\ &\cong \mathrm{Hom}_{D^+(\mathcal{B})}(F(A^\bullet), I^\bullet),\end{aligned}$$

since $G(I^\bullet)$ is also a bounded below complex of injectives. Therefore in this situation we get an actual adjunction $F \dashv RG$ on the bounded below categories induced by units and counits. Moreover, the units/counits also induce maps between RHom, so it is clear by the same proof as above that we will have a natural isomorphism

$$\mathrm{RHom}_{\mathcal{A}}(A, RG(B)) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{B}}(F(A), B)$$

between RHom's.

To conclude this section, we are going to state an important result which holds in the case of enough injectives, which is the existence of a *Cartan-Eilenberg* resolution. From this we will deduce a very important spectral sequence which relates the classical higher derived functors to the cohomology of the derived functor applied to a complex. A special case of this will be the Grothendieck spectral sequence. The existence theorem, whose proof is purely classical, will be discussed in the next section.

Definition 1.3.22. Let $\mathcal{A}$ be an abelian category and $C^\bullet$ a bounded below complex. A *Cartan-Eilenberg resolution of* C is a bicomplex $I^{\bullet,\bullet}$ equipped with a morphism $\epsilon\colon C^\bullet \to I^{\bullet,0}$ of complexes satisfying the following properties:

(a) $I^{p,q} = 0$ for $q < 0$, and there is k such that $I^{p,q} = 0$ for $p < k$ and all q.
(b) The natural maps $C^p[0] \to I^{p,\bullet}$ are injective resolutions for all $p \in \mathbb{Z}$.
(c) The natural maps

$$\ker d_C^p[0] \to \ker d_h^{p,\bullet} \qquad \mathrm{im}\, d_C^{p-1}[0] \to \mathrm{im}\, d_h^{p-1,\bullet}$$

are injective resolutions for all $p \in \mathbb{Z}$.

(d) The maps $H^p(C) \to H^p_h(I^{\bullet,\bullet})$ are injective resolutions for all $p \in \mathbb{Z}$.

A dual definition is made in the projective/bounded above case.

The point is to have an object-wise injective resolution which remains an injective resolution after taking cohomology. The picture is as follows:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & I^{p-1,1} & \longrightarrow & I^{p,1} & \longrightarrow & I^{p+1,1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & I^{p-1,0} & \longrightarrow & I^{p,0} & \longrightarrow & I^{p+1,0} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \longrightarrow & C^{p-1} & \longrightarrow & C^{p} & \longrightarrow & C^{p+1} & \longrightarrow & \cdots
\end{array}$$

So, each vertical sequence provides an injective resolution of C^p. Once we take horizontal cohomology everywhere, so that there are only vertical differentials remaining, we will obtain an injective resolution of $H(C^p)$. Note also that (a) in the definition implies that the bicomplex is a shifted first-quadrant complex.

THEOREM 1.3.23. *Let $\mathcal{A}$ be an abelian category with enough injectives. Then every bounded below complex admits a Cartan-Eilenberg resolution. Likewise, if $\mathcal{A}$ has enough projectives, then the same holds for bounded above complexes. Moreover, it can be chosen functorially.*

The idea of the proof is simple, namely split up the complex into pairs of short exact sequences, choose compatible injective resolutions of each term, and then glue this data together. Since we will need some standard lemmas, we push this to the next section.

If we're given a Cartan-Eilenberg resolution $C \to I^{\bullet,0}$, the totalization behaves nicely as well. First, since it's a shifted first-quadrant complex, the totalization $\operatorname{Tot} I$ involves only finite direct sums and therefore always exists. Moreover, we have a canonical map

$$C^n \xrightarrow{\epsilon} I^{n,0} \hookrightarrow \bigoplus_{p+q=n} I^{p,q}$$

which commutes with the differentials, hence a chain map $\widetilde{\epsilon}\colon C \to \operatorname{Tot} I$. Note that each term in $\operatorname{Tot} I$ is an injective object. Now take vertical cohomology, so we can apply the first spectral sequence associated to a bicomplex. Since the resolutions are exact at degrees 1 and beyond, only the first row remains, and the canonical maps $C^p \to E_1^{p,0}$ are isomorphisms. Thus $C^\bullet \cong E_1^{\bullet,0}$, and the sequence degenerates at E_2:

$$E_2^{p,0} = H^p(C) \qquad E_2^{p,q} = 0 \quad q > 0.$$

The edge maps induce $\widetilde{\epsilon}\colon H^p(C) \xrightarrow{\sim} H^p(\operatorname{Tot} I)$ an isomorphism, hence the natural map $\widetilde{\epsilon}$ is an injective resolution of C.

THEOREM 1.3.24. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories with an adapted class $\mathcal{R}$, so that RF exists. Suppose $\epsilon\colon C^\bullet \to I^{\bullet,0}$ is a Cartan-Eilenberg resolution of a bounded below complex C. The two spectral sequences associated to the bicomplex $F(I^{\bullet,\bullet})$ are bounded, and satisfy*

$$E_{h,1}^{p,q} = R^q F(C^p) \implies H^{p+q}(RF(C))$$
$$E_{v,2}^{p,q} = R^p F(H^q(C)) \implies H^{p+q}(RF(C))$$

each converging to degree-wise finite filtrations.

PROOF. It is clear from the definition that $F(\operatorname{Tot} I) = \operatorname{Tot}(F(I))$ since F is additive. We have just shown that $C \to \operatorname{Tot} I$ is an injective resolution, and both sequences converge to the cohomology of $\operatorname{Tot} F(I) = F(\operatorname{Tot} I) = RF(C)$, so both RHS's above are correct. For the first sequence, if we apply F and simply take vertical cohomology, this is just taking the cohomology of F applied to an injective resolution of C^p, which is exactly $R^q F(C^p)$.

Now let's take horizontal cohomology first. Our E_0 page has the portion $F(I^{p-1,q}) \to F(I^{p,q}) \to F(I^{p+1,q})$, and the underlying I differentials come with an exact sequence

$$0 \to \operatorname{im} d_h^{p-1,q} \to \ker d_h^{p,q} \to H_h^p(I^{\bullet,q}) \to 0,$$

where our assumption is that all three of these terms are injective. In particular the sequence splits, so applying F keeps it exact. Therefore, the E_1 page looks like:

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \\
\cdots \longrightarrow & F(J^{p-1,q+1}) & F(J^{p,q+1}) & F(J^{p+1,q+1}) & \longrightarrow \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
\cdots \longrightarrow & F(J^{p-1,q}) & F(J^{p,q}) & F(J^{p+1,q}) & \longrightarrow \cdots
\end{array}$$

Here, $J^{p,q} = H_h^p(I^{\bullet,q})$, so as q varies these are the terms in a vertical injective resolution for $H^p(C)$. Hence, when we take vertical cohomology to get to E_2, the (p,q)th spot will become $R^q F(H^p(C))$. The indices of this second spectral sequence get flipped, so this gives the statement in the theorem. □

We will often directly use the first spectral sequence in the theorem, even when we are interested in eventually computing classical derived functors themselves. The reason is that we can try to find a resolution $P^\bullet \to A$ of an object A, and then the theorem gives us a spectral sequence

$$E_1^{p,q} = R^q F(P^p) \implies H^{p+q}(RF(P)) = R^{p+q} F(A).$$

This is especially nice because we even know the (horizontal) differentials on the first page, namely they are $R^q F$ applied to the differentials in P. This technique is useful because we usually can't write down injective resolutions, but we can write down e.g. locally free resolutions.

We can also use the first sequence when we want hypercohomology, which we'll do when discussing Hodge-to-de-Rham. The second sequence gives us the Grothendieck spectral sequence.

Corollary 1.3.25. *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors of abelian categories such that $\mathcal{R}_F$ is an adapted class for F, and $\mathcal{B}$ has enough injectives. Assume that for each $X \in \mathcal{R}_F$, $F(X)$ is G-acyclic. Then there is a bounded spectral sequence*

$$E_2^{p,q} = R^pG\big(H^q(RF(A))\big) \implies H^{p+q}\big(R(G \circ F)(A)\big)$$

for every A in $\mathrm{Ch}^+(\mathcal{A})$ (converging to a finite filtration). In particular, if A is an object of $\mathcal{A}$, then

$$E_2^{p,q} = R^pG\big(R^qF(A)\big) \implies R^{p+q}(GF)(A)$$

as a special case.

Proof. Since $\mathcal{B}$ has enough injectives, RG exists and we may define it with the maximal adapted class $\mathcal{R}_G$ of all G-acyclic objects in $\mathcal{B}$. Then $F(\mathcal{R}_F) \subset \mathcal{R}_G$, so by Thm. 1.2.22 we can identify $RG \circ RF = R(G \circ F)$. Let $A \in \mathrm{Ch}^+(\mathcal{A})$. Then $RF(A)$ in $D^+(\mathcal{B})$ has a Cartan-Eilenberg resolution by Thm. 1.3.23 since $\mathcal{B}$ has enough injectives, so applying the second sequence of Thm. 1.3.24 to $C = RF(A)$, we are done. □

Example 1.3.26. Let us demonstrate an example showing that, in spite of the Grothendieck spectral sequence, knowing $R^qF(A)$ does not determine $R^n(GF)(A)$. We will instead do it using left derived functors. First some elementary observations: if $R \to S$ is a ring map, then the functor

$$F = (-) \otimes_R S\colon \mathrm{Mod}(R) \to \mathrm{Mod}(S)$$

preserves free modules, which are an adapted class for tensor products on both sides. Using this and the fact that restriction of scalars is exact, it follows that $LF(A)|_R = A \otimes^{\mathbf{L}}_R S$ for any $A \in D^-(R)$.

Now let $S = k[\epsilon]/(\epsilon^2)$ which is a quotient of $R = S[t]$ by t. Moreover, k is a quotient of S by (ϵ). Consider the functors

$$F = (-) \otimes_R S\colon \mathrm{Mod}(R) \to \mathrm{Mod}(S) \qquad G = (-) \otimes_S k\colon \mathrm{Mod}(S) \to \mathrm{Mod}(k).$$

Lastly, consider the R-modules $M = R/(\epsilon - t)$ and $N = R/(t, \epsilon) = k$. The consideration about free modules above shows that the composition satisfies the hypotheses for composing derived functors, and the Grothendieck spectral sequence.

Now let's compute: t is a nonzerodivisor on R, so $S = [R \xrightarrow{t} R]$ in $D^-(R)$. Thus the underlying R-modules of $LF(M)$ and $LF(N)$ are

$$M \otimes^{\mathbf{L}}_R S = [M \xrightarrow{t} M] \cong [S \xrightarrow{\epsilon} S]$$

because $k[\epsilon, t]/(\epsilon^2, t - \epsilon) = k[\epsilon]/(\epsilon^2)$ as S-modules. On the other hand,

$$N \otimes^{\mathbf{L}}_R S = [k \xrightarrow{0} k] = k \oplus k[1].$$

So, $L^0F(M) = L^{-1}F(M) = k$ and lower degrees are zero, and the same is true for N. However, we showed previously that $LG(k)$ has cohomology in every degree, so the same holds for $LG(LF(N))$. On the other hand, our presentation for $M \otimes^{\mathbf{L}}_R S$ is already a free resolution, so applying LG gives $k \xrightarrow{0} k$, which has cohomology in only two degrees.

Let us see what is happening in the Grothendieck spectral sequence, which by construction has the same E_2-page in both cases:

$\cdots$	-3	-2	-1	0	
$\cdots$	k	k	k	k	0
$\cdots$	k	k	k	k	-1

The differentials go 2 spaces to the right and 1 down, and since this is a spectral sequence of k-vector spaces the differentials between nonzero terms are each either 0 or isomorphisms. In either case the sequence will degenerate on E_3. Now we can see that in the case of M, all of the nonzero differentials have to be isomorphisms so that we end up with only k at $(-1, 0)$ and $(0, 0)$. For N, all of the differentials are 0, so this is already E_∞ and we get the expected two-dimensional cohomology in all strictly negative degrees.

1.4. Appendix: Classical homological algebra

This appendix will have a few short sections discussing some basic results from the classical theory (in contrast to the previous section, which recast the classical theory in our new language). In the last section, we will discuss any results from commutative algebra which are helpful in establishing the cohomology theory which we would like to set up in the next chapter.

1.4.1. More on resolutions. In the previous section we stated Thm. 1.3.23 on the existence of Cartan-Eilenberg resolutions. In this section, we will complete the proof of this and introduce some useful homological techniques along the way.

A short exact sequence $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ of complexes is *termwise split* if $0 \to A^i \to B^i \to C^i \to 0$ is a split exact sequence for each $i \in \mathbb{Z}$. These behave especially nicely because the connecting map in the long exact sequence already exists in the chain complex category.

To see this, choose a splitting $B^i = A^i \oplus C^i$ and let $s^i \colon C^i \to B^i$ and $\pi^i \colon B^i \to A^i$ be the splitting maps. Then we claim that the family of maps

$$\delta^n := \pi^{n+1} \circ d_B^n \circ s^n \colon C \to A^{n+1}$$

defines a morphism $\delta \colon C \to A[1]$ of complexes. To see this, first compute the matrix of the differential $A^i \oplus C^i \to A^{i+1} \oplus C^{i+1}$ of B: on the A^i component, we have $d_B i = i d_A$ i.e. the canonical injection. On the C^i component the map is $d_B s$, which has components

$$p(d_B s) = d_C p s = d_C \qquad \pi d_B s = \delta.$$

This shows that

$$d_B = \begin{pmatrix} d_A & \delta \\ 0 & d_B \end{pmatrix}.$$

Doing matrix multiplication, $d^2 = 0$ implies that

$$0 = d_A \delta + \delta d_C = -d_{A[1]} \delta + \delta d_C,$$

so indeed $\delta \colon C \to A[1]$ is a chain map. From the explicit differential, we immediately conclude that $B = \mathrm{Cone}(-\delta)[-1]$ compatible with A, so there is an exact triangle

$$C \xrightarrow{-\delta} A[1] \to \mathrm{Cone}(-\delta) \cong B[1] \to C[1]$$

in $\mathrm{K}(\mathcal{A})$. Shifting and rotating this we obtain $A \to B \to C \xrightarrow{\delta} A[1]$, thus a termwise split sequence gives an exact triangle in $\mathrm{K}(\mathcal{A})$. Conversely, we know that all triangles are of this form up to isomorphism, because $0 \to B \to \mathrm{Cone}\, f \to A[1] \to 0$ is a termwise split exact sequence.

In particular, if we pass to the derived category our triangle has the same form, so applying the cohomological functor H^0 gives the usual long exact sequence with the connecting map being induced by δ. Of course, this is non-unique and depends exactly on the choice of splitting.

Lemma 1.4.1. *Let $\mathcal{A}$ be an abelian category.*

(a) *If $q\colon B \to C$ is a quasi-isomorphism which is a termwise epimorphism, then $\ker d^i_B \to \ker d^i_C$ is an epimorphism for all i.*
(b) *If $i\colon A \to B$ is a quasi-isomorphism which is a termwise monomorphism, and $I \in \mathrm{Ch}^+(\mathcal{I}_\mathcal{A})$, then every map $A \to I$ extends to B.*

Proof. (a) Consider the commutative diagrams with exact rows:

$$\begin{array}{ccccc} B^i & \longrightarrow & \operatorname{im} d^i_B & \longrightarrow & 0 \\ {\scriptstyle q}\downarrow & & {\scriptstyle q}\downarrow & & \\ C^i & \longrightarrow & \operatorname{im} d^i_C & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccccc} \operatorname{im} d^{i-1}_B & \longrightarrow & \ker d^i_B & \longrightarrow & H^i(B) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow\cong & & \downarrow \\ \operatorname{im} d^{i-1}_C & \longrightarrow & \ker d^i_C & \longrightarrow & H^i(C) & \longrightarrow & 0 \end{array}$$

The left-hand diagram shows that the map on coboundaries is an epi for all i, and then the diagram on the right proves the claim by the epimorphism 4-lemma.

(b) Then $\underline{\mathrm{Hom}}_\mathcal{A}(B, I) \to \underline{\mathrm{Hom}}_\mathcal{A}(A, I)$ is a quasi-isomorphism, and it is a termwise epi because $\mathrm{Hom}(C^i, I^{k+i}) \to \mathrm{Hom}(B^i, I^{k+i})$ is surjective. Thus this induces a surjection on 0-cocycles by (a), i.e.

$$\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(B, I) \to \mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(A, I)$$

is surjective, so every map lifts as desired. □

We are going to apply this to the following powerful resolution extension lemma, which is the key step in finding Cartan-Eilenberg resolutions.

Lemma 1.4.2. *Let $\mathcal{A}$ be an abelian category, with $0 \to A \to B \to C \to 0$ a short exact sequence in $\mathrm{Ch}^+(\mathcal{A})$. Let*

$$A \to I_1 \qquad C \hookrightarrow I_3$$

be injective resolutions with $I_1, I_3 \in \mathrm{Ch}^+(\mathcal{I}_\mathcal{A})$, and assume the latter is a monomorphism. Then there exists $I_2 \in \mathrm{Ch}^+(\mathcal{I}_\mathcal{A})$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \vdots\downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1 & \dashrightarrow & I_2 & \dashrightarrow & I_3 & \longrightarrow & 0 \end{array}$$

of complexes in $\mathrm{Ch}^+(\mathcal{A})$ with exact rows, such that $B \to I_2$ is an injective resolution. If I_1, I_3 are concentrated in $[a, b]$, then we may assume I_2 is as well.

If the given sequence was split, then we could just take the direct sum of the injective resolutions. We can't reduce to this case, but we can reduce to the case that it is termwise split, where we can take the direct sum termwise, and then modify the differential.

Proof (of Lemma 1.4.2). First take the pushout in the category of complexes (which is done termwise), so that we get a commutative diagram of exact sequences of complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \vdots\downarrow & & \| & & \\ 0 & \longrightarrow & I_1 & \dashrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

By triangulated 2-out-of-3 in $D^+(\mathcal{A})$, the map $B \to P$ is a quasi-isomorphism. Since I_1 is termwise injective, the bottom row is termwise split, so by choosing a splitting $P^i = I_1^i \oplus C^i$, we can take the map $\delta\colon C \to I_1[1]$ as above. This identifies $P = \mathrm{Cone}(-\delta)[-1]$. We construct a resolution for the bottom row termwise as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_1^i & \longrightarrow & I_1^i \oplus C^i & \longrightarrow & C^i & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_1^i & \longrightarrow & I_1^i \oplus I_3^i & \longrightarrow & I_3^i & \longrightarrow & 0 \end{array}$$

All we have to do is equip $I_2^i \coloneqq I_1^i \oplus I_3^i$ with a differential so that $P \to I_2$ is a chain map: Once this is done, triangulated 2/3 will say that this is a diagram of quasi-isomorphisms, and then by composing it with the initial diagram we are done.

By the lemma we just proved, the map $\delta\colon C \to I_1[1]$ extends along the quasi-isomorphic inclusion $q\colon C \to I_3$ to a map $\delta_2\colon I_3 \to I_1[1]$. In other words, $(1_{I_1[1]}, q)$ is a morphism of arrows $\delta \to \delta_2$, so this induces a map $\mathrm{Cone}(-\delta)[-1] \to \mathrm{Cone}(-\delta_2)[-1]$ which componentwise is exactly the middle map of the diagram above. □

At last, we finish the proof that Cartan-Eilenberg resolutions exist in an abelian category $\mathcal{A}$ with enough injectives.

Proof (of Thm. 1.3.23). We use B and Z notation for coboundaries and cocycles. Assume for simplicity that our complex C has $C^i = 0$ for $i < 0$, and we inductively construct compatible resolutions for $i \geq 0$. For simplicity, in this proof we use the word 'resolution' to mean injective resolution of an object of $\mathcal{A}$ which extends only to the right (so is termwise injective).

For the ith step, we'll use the exact sequences

$$0 \to Z^{i-1} \to C^{i-1} \to B^i \to 0 \qquad 0 \to B^i \to Z^i \to H^i(C) \to 0,$$

and assume that we are given a resolution $Z^{i-1}[0] \to J_Z^{i-1,\bullet}$ from the previous step. Choose a resolution $B^i[0] \to J_B^{i,\bullet}$, and using our lemma extend this to a diagram of extensions like so:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_Z^{i-1,\bullet} & \dashrightarrow & I^{i-1,\bullet} & \dashrightarrow & J_B^{i,\bullet} & \longrightarrow & 0 \\ & & \uparrow & & \vdots\uparrow & & \uparrow & & \\ 0 & \longrightarrow & Z^{i-1} & \longrightarrow & C^{i-1} & \longrightarrow & B^i & \longrightarrow & 0 \end{array}$$

Of course, if $i = 0$ then we take the left two vertical maps and the resolution to be zero. To construct the next J_Z resolution, choose a resolution $H^i(C)[0] \to J_H^{i,\bullet}$ and extend over the second sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_B^{i,\bullet} & \dashrightarrow & J_Z^{i,\bullet} & \dashrightarrow & J_H^{i,\bullet} & \longrightarrow & 0 \\ & & \uparrow & & \vdots\uparrow & & \uparrow & & \\ 0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i(C) & \longrightarrow & 0 \end{array}$$

Now we are done: pasting the successive diagrams of exact sequences of the first kind together gives a commutative diagram with vertical maps $\epsilon\colon C^i[0] \to I^{i,\bullet}$ injective resolutions, so that when we take coboundaries, cocycles, and then cohomology, we get the injective resolutions J^i_B, J^i_Z, J^i_H. This gives us a bicomplex up to our sign convention, so we should flip the sign of the vertical differential in the odd columns. □

1.4.2. Cohomological δ-functors. Here we quickly discuss the classical topic of δ-functors, and then relate it to what we defined as a δ-functor from an abelian category to a triangulated category.

Definition 1.4.3. A *(cohomological) δ-functor* $T\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is a sequence $(T^i)_{i\geq 0}$ of additive functors $T^i\colon \mathcal{A} \to \mathcal{B}$ equipped with maps

$$\delta = \delta_{A\to B\to C}\colon T^i(C) \to T^{i+1}(A) \qquad i \geq 0$$

for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, such that:

(a) For every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the sequence

$$0 \to T^0(A) \to T^0(B) \to T^0(C) \xrightarrow{\delta} T^1(A) \to \cdots$$

is exact in $\mathcal{B}$.

(b) Given a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

of short exact sequences in $\mathcal{A}$, the corresponding diagram

$$\begin{array}{ccc} T^i(C) & \xrightarrow{\delta} & T^{i+1}(A) \\ \downarrow & & \downarrow \\ T^i(C') & \xrightarrow{\delta} & T^{i+1}(A') \end{array}$$

in $\mathcal{B}$ commutes for all $i \geq 0$.

Note that in particular if T is a δ-functor, then T^0 has to be left exact. The primary examples of δ-functors come from the triangulated δ-functors in the sense of Def. 1.2.7. Namely, if $\mathcal{A} \to \mathcal{C}$ is a δ-functor from an abelian category to a triangulated category $\mathcal{C}$, and $H\colon \mathcal{C} \to \mathcal{B}$ is a cohomological functor, then

$$T^i := \mathcal{A} \to \mathcal{C} \xrightarrow{H\circ[i]} \mathcal{B}$$

is a cohomological δ-functor, essentially by definition. So, for example:

(a) Since $\mathrm{Ch}^*(\mathcal{A}) \to D^*(\mathcal{A})$ are δ-functors, whenever we have a cohomological functor $D^*(\mathcal{A}) \to \mathcal{B}$ we get a δ-functor $\mathrm{Ch}^*(\mathcal{A}) \to \mathcal{B}$.

(b) Since the inclusion $\mathcal{A} \to D^*(\mathcal{A})$ is a δ-functor, any cohomological functor on the derived category gives a δ-functor $\mathcal{A} \to \mathcal{B}$. For example, using the functor $\mathrm{Hom}_{D^*(\mathcal{A})}(A[0], -)$ produces $\{\mathrm{Ext}^i_{\mathcal{A}}(A, -)\}$ from $\mathcal{A}$ to **Ab**.

(c) In particular, if $F\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor of abelian categories admitting an adapted class, so that $RF\colon D^+(\mathcal{A}) \to D^+(\mathcal{B})$ exists, then the classical derived functors

$$(R^iF)_{i\geq 0} \qquad R^iF\colon \mathcal{A} \to \mathcal{B}$$

form a δ-functor, because they are the following composite

$$\mathcal{A} \to D^+(\mathcal{A}) \xrightarrow{RF} D^+(\mathcal{B}) \xrightarrow{H^0\circ[i]} \mathcal{B},$$

where the composition of the first two functors is a triangulated δ-functor since the first one is, and RF is triangulated.

Definition 1.4.4. Let T and S be cohomological δ-functors from $\mathcal{A}$ to $\mathcal{B}$.

(a) A morphism $f\colon T \to S$ is a sequence of natural transformations $f^i\colon T^i \to S^i$ such that the diagram

$$\begin{array}{ccc} T^i(C) & \xrightarrow{\delta_T} & T^{i+1}(A) \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^i} \\ S^i(C) & \xrightarrow{\delta_S} & S^{i+1}(A) \end{array}$$

commutes for every exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$.

(b) The δ-functor T is called *universal* if for every δ-functor S and morphism $f^0\colon T^0 \to S^0$, there is a unique morphism $f\colon T \to S$ extending f^0.

Clearly, a universal δ-functor is uniquely determined by its first component, meaning that if T, S are universal, and $T^0 \cong S^0$, then there's a unique isomorphism $T \cong S$ of δ-functors extending the given one. In the special case of the δ-functor associated to classical derived functors, we have *some* kind of universal property but it is a universality among exact functors of triangulated categories.

Sometimes, it is enough to check things on the level of classical derived functors to get information about the derived category, and to construct maps between such functors we don't want to have to start out with the full data of a triangulated functor. For example, if A, B are objects of $\mathcal{A}$ and there is a map $A \to B$ such that $R^iF(A) \to R^iF(B)$ is an isomorphism, then we already know that $RF(A[0]) \to RF(B[0])$ is an isomorphism (because it is one on cohomology). We need one final definition before proving the main result.

Definition 1.4.5. An additive functor $F\colon \mathcal{A} \to \mathcal{B}$ of abelian categories is *effaceable* if for each A in $\mathcal{A}$ there exists a monomorphism $i\colon A \to X$ for some X in $\mathcal{A}$ such that $F(i) = 0$.

The following is the main theorem on δ-functors.

Theorem 1.4.6. *Let T be a δ-functor from $\mathcal{A}$ to $\mathcal{B}$, and suppose that T^i is effaceable for $i > 0$. Then T is a universal δ-functor.*

Proof. Let S be another δ-functor and assume we have a natural transformation $f^0\colon T^0 \to S^0$. Suppose inductively that we have constructed a natural map $f^n\colon T^n \to S^n$. To define f^{n+1} on an object A, let $A \to X$ be an *effacement* (of

degree $n+1$), i.e. a mono as in the definition of effaceable. We have a diagram of exact sequences

$$\begin{array}{ccccccc} T^nX & \longrightarrow & T^nC & \longrightarrow & T^{n+1}A & \xrightarrow{0} & T^{n+1}X \\ {\scriptstyle f^n}\downarrow & & {\scriptstyle f^n}\downarrow & & \vdots\downarrow & & \\ S^nX & \longrightarrow & S^nC & \longrightarrow & S^{n+1}A & \longrightarrow & S^{n+1}X \end{array}$$

(without the dotted line). The 0 means that $T^{n+1}A$ is the cokernel of the map $T^nX \to T^nC$, and any dotted map filling in the square would have to factor through the cokernel of $S^nX \to S^nC$, so it is uniquely determined. We are going to define this as f^{n+1} and demonstrate the required properties, but note that this morphism is already unique for any extension of f^n to a morphism of δ-functors.

1. f^{n+1} *is independent of the chosen effacement.* Suppose we are given two (degree $n+1$) effacements $A \to X$ and $A \to X'$. Form the pushout P, then we still have a mono $A \to P$, and let $D = P/A$ so that we have a morphism
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & D & \longrightarrow & 0 \end{array}$$
of exact sequences. If we consider the associated long exact sequences, we get a commutative diagram:
$$\begin{array}{ccccccc} T^nX & \longrightarrow & T^nC & \longrightarrow & T^{n+1}A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ ? & & ? & & ? & & ? \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ S^nP & \longrightarrow & S^nD & \longrightarrow & S^{n+1}A & \longrightarrow & S^{n+1}P \end{array}$$
There are two natural choices for the middle row: either we apply T to the bottom row, or we apply S to the top row. The first choice, since T is a δ-functor, shows that $A \to P$ is also an effacement. That f^n is a natural transformation is exactly the statement that the first two vertical composites $T^nX \to S^nP$ and $T^nC \to S^nD$ are the same no matter which middle row we take. It follows from the same argument above that any two maps $T^{n+1}A \to S^{n+1}A$ filling in the diagram are equal.

   For the two choices of middle rows, the possible composites are
$$T^{n+1}A \xrightarrow{1} T^{n+1}A \to S^{n+1}A \qquad T^{n+1}A \to S^{n+1}A \xrightarrow{1} S^{n+1}A,$$
where in the first case the second map is induced by the effacement $A \to P$, and in the second the first map is induced by the effacement $A \to X$. Therefore these two effacements induce the same morphism f^{n+1}. Since the same argument applies to $0 \to A \to X' \to X'/A \to 0$ instead of the current top row, this shows that $A \to X$ and $A \to X'$ induce the same map f^{n+1}.
2. f^{n+1} *commutes with* δ. Start with a sequence $0 \to A \to B \to C \to 0$, and choose an effacement $B \to X$. The composite $A \to X$ is again

a monomorphism, so writing $D = X/A$ we have a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & D & \longrightarrow & 0 \end{array}$$

One portion of the long exact sequence is the diagram

$$\begin{array}{ccc} T^{n+1}A & \longrightarrow & T^{n+1}B \\ \| & & \downarrow 0 \\ T^{n+1}A & \longrightarrow & T^{n+1}X \end{array}$$

which shows that $A \to X$ is also an effacement for X. We have another commutative diagram as follows:

$$\begin{array}{ccccc} T^nC & \longrightarrow & T^nD & \longrightarrow & T^{n+1}A \\ \downarrow & & \downarrow & & \downarrow \\ S^nC & \longrightarrow & S^nD & \longrightarrow & S^{n+1}A \end{array}$$

The right-hand square is the definition of f^{n+1} using $A \to X$, and the left-hand square is naturality of f^n. However, since we have a morphism of exact sequences, the composites of both rows are exactly the δ maps for T and S, which is what we wanted.

3. f^{n+1} *is natural.* This is similar to Step 1: Given an effacement $A \to X$ and any map $A \to B$, push out along the map to obtain a morphism of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & \ulcorner & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & P & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

The long exact sequence gives us:

$$\begin{array}{ccccccc} T^nX & \longrightarrow & T^nC & \longrightarrow & T^{n+1}A & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ ? & & ? & & ? & & \\ \downarrow & & \downarrow & & \downarrow & & \\ S^nP & \longrightarrow & S^nC & \longrightarrow & S^{n+1}B & & \end{array}$$

As before, we have two ways to fill this in, either applying T to the bottom row or S to the top row. In either case, naturality of f^n shows that the two maps $T^nX \to S^nP$ and $T^nC \to S^nC$ have to coincide, so any two extensions filling in $T^{n+1}A \to S^{n+1}B$ are the same. But these two possibilities are exactly:

$$T^{n+1}A \to T^{n+1}B \xrightarrow{f^{n+1}} S^{n+1}B \qquad T^{n+1}A \xrightarrow{f^{n+1}} S^{n+1}B \to S^{n+1}B$$

Filling in the diagram in the second case is by definition of f^{n+1}, but in the first case we are applying part (2) to deduce that the bottom right square commutes. It follows that the above maps are equal, which is what we wanted.

We have constructed f^{n+1} given f^n, which completes the induction step, and we have already explained why it is uniquely determined by f^n. In particular, all the f^i are uniquely determined by f^0, so this shows T is universal. □

Corollary 1.4.7. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories, $F\colon \mathcal{A} \to \mathcal{B}$ a left exact functor which admits an adapted class. The classical higher derived functors $(R^iF)_{i\geq 0}\colon \mathcal{A} \to \mathcal{B}$ form a universal δ-functor.*

In particular, if $T = (T^i)_{i\geq 0}$ is a universal δ-functor from $\mathcal{A}$ to $\mathcal{B}$, and the functor T^0 admits an adapted class (e.g. $\mathcal{A}$ has enough injectives), then the canonical map $T \to (R^iT^0)_{i\geq 0}$ is an isomorphism.

Proof. The second statement is simply uniqueness of universal δ-functors. To check universality, we need to show that the higher derived functors are effaceable, which is clear because members of an adapted class are F-acyclic. □

Remark 1.4.8. Note from the proof here that derived functors with an adapted class satisfy a much stronger condition than effaceability: we found a mono to a single object which made all higher functors vanish, whereas we only needed it to send the map to zero, and the effacement was allowed to depend on the degree. Thus, this gives a general criterion to check whether a δ-functor is given by a derived functor.

Remark 1.4.9. We note in passing that, as is immediate from the definition, we can always precompose or postcompose a δ-functor with an exact functor and get a δ-functor.

1.4.3. Modules over a ring. Let R be a ring. Recall that we have made the following definitions so far:

(a) I is injective if $\mathrm{Hom}_R(-, I)$ is exact.
(b) P is projective if $\mathrm{Hom}_R(P, -)$ is exact.
(c) F is flat if $F \otimes_R (-)$ is exact.

$\mathrm{Mod}(R)$ denotes the abelian category of R-modules. We have shown that I is injective iff $\mathrm{Ext}^i_R(-, I) = 0$ for $i > 0$, and P is projective iff $\mathrm{Ext}^i_R(P, -) = 0$ for $i > 0$. Observe that if P is a free module, i.e. $P \cong R^{\oplus I}$ for some index set I, then P is projective because

$$\mathrm{Hom}_R(P, M) \cong \prod_I \mathrm{Hom}_R(R, M) \cong \prod_I M,$$

and it is clear that this functor is exact: if $M \to N \to 0$ is a surjection, then we can lift a sequence $(y_i)_{i \in I}$ termwise to a sequence in M. Moreover, every module M admits a surjection from a free module, namely $R^{(M)} \to M \to 0$, so $\mathrm{Mod}(R)$ has enough projectives.

Note that $R^{\oplus I} \otimes M \cong (R \otimes M)^{\oplus I} \cong M^{\oplus I}$, and this functor is clearly exact, because if $M \subset N$ and a sequence $(x_i)_{i \in I}$ is zero in M, then it is zero in N too. Thus $\mathrm{Mod}(R)$ has enough flats by virtue of having enough frees, so Tor is well-defined and satisfies the standard properties. In particular, projective modules are flat. In this section we will use the notation $\mathrm{Tor}^R_i(M, N) \coloneqq \mathrm{Tor}^{-i}_{\mathrm{Mod}(R)}(M, N)$. We have also shown that F is flat iff $\mathrm{Tor}^R_i(F, -) = 0$ for $i > 0$.

In such a setting we have also demonstrated the following equivalent conditions for injectivity, projectivity, and flatness; I is injective iff every short exact sequence of the form $0 \to I \to M \to M'' \to 0$ splits, P is projective iff every short exact sequence of the form $0 \to M' \to M \to P \to 0$ splits, and F is flat iff for every short exact sequence $0 \to M' \to M \to F \to 0$, the sequence

$$0 \to M' \otimes N \to M \otimes N \to F \otimes N \to 0$$

is exact. From this we deduce that a module P is projective iff it is a direct summand of a free module, where the second direction follows from $\mathrm{Ext}^1(P \oplus P', M) = \mathrm{Ext}^1(P, M) \oplus \mathrm{Ext}^1(P', M)$ and the fact that free modules are projective. The first goal of this section is the following theorem:

THEOREM 1.4.10. *Let R be a ring. Then* $\mathrm{Mod}(R)$ *has enough injectives. Moreover, injective embeddings can be chosen functorially.*

We will deduce this by first studying abelian groups. To do this, we will use a few times the right adjoint to restriction of scalars: recall that if $R \to S$ is a ring map, then we have a canonical functor

$$\mathrm{Hom}_R(S, -) \colon \mathrm{Mod}(R) \to \mathrm{Mod}(S),$$

where for an R-module M we view $\mathrm{Hom}_R(S, M)$ as an S-module by pre-multiplication in the first factor. The adjunction gives isomorphisms

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(N, M)$$

of restricted R-modules, or of S-modules by doing the same thing as above on the RHS. Here are some basic properties:

Lemma 1.4.11. *Let R be a ring.*

(a) A product of injective modules is injective.
(b) If $R \to S$ is a flat map, and E is injective over S, then $E|_R$ is injective over R.
(c) Let $R \to S$ be an epimorphism of rings and E an S-module. If $E|_R$ is injective, then E is injective.
(d) If $R \to S$ is a ring map and E is injective over R, then $\mathrm{Hom}_R(S, E)$ is injective over S.

Proof. (a) Note that $\mathrm{Hom}(-, \prod_i E_i) = \prod_i \mathrm{Hom}(-, E_i)$, and we already explained why taking products was exact.

(b) We have $\mathrm{Hom}_R(M, E|_R) \cong \mathrm{Hom}_S(M \otimes_R S, E)$, and both $M \mapsto M \otimes_R S$ and $\mathrm{Hom}_S(-, E)$ are exact.

(c) The forgetful map $\mathrm{Hom}_S(M, E) \to \mathrm{Hom}_R(M, E|_R)$ is of course injective, but it is surjective as well by assumption.

(d) Note that $\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, M)) = \mathrm{Hom}_R(-, M)$ as functors of S-modules, and restriction of scalars on the right is just $\mathrm{Hom}_R((-)|_R, M)$. □

A less trivial characterization is the ideal-wise criterion for injectivity:

Lemma 1.4.12. *Let E be an R-module. The following are equivalent:*

(a) E is injective.
(b) For any ideal $I \subset R$, every map $I \to E$ extends to R.
(c) $\mathrm{Ext}^1_R(R/I, E) = 0$ for every ideal $I \subset R$.

Proof. $(a) \Rightarrow (b)$ $\mathrm{Hom}(-, E)$ is supposed to preserve $0 \to I \to R$.

$(b) \Leftrightarrow (c)$ We have an exact sequence

$$\mathrm{Hom}(R, E) \to \mathrm{Hom}(I, E) \to \mathrm{Ext}^1_R(R/I, E),$$

so surjectivity of the first map is controlled by $\mathrm{Ext}^1 = 0$.

Let us add our own extra statement:

(b') *For a strict inclusion $M \subsetneq N$ and map $\phi\colon M \to E$, there exists $M' \subset N$ with $M \subsetneq M'$ and $\phi'\colon M' \to E$ extending ϕ.*

$(b') \Rightarrow (a)$ Let $M \subset N$ and $\phi\colon M \to E$. Consider the set of pairs $\{(M', \phi')\}$ with $M \subset M' \subset N$ and $\phi'|_{M'} = \phi$, which is nonempty. Order this by inclusion on the first factor and agreement of restriction on the second. Clearly if we have an ascending chain of submodules then we can extend our map to the union. We are done by Zorn's lemma, because the hypothesis guarantees that a maximal element must be N itself.

$(b) \Rightarrow (b')$ Take $x \in N \setminus M$ and consider the ideal $I = \{r \in R : rx \in M\}$. The map $I \to E$ given by $r \mapsto \phi(rx)$ can be extended to $\psi\colon R \to E$, and it is easy to check that the extension $M + \langle x \rangle \to E$ sending $a + rx \mapsto \phi(a) + \psi(r)$ works, so we are done. □

Recall that if R is a domain, then an R-module M is *divisible* if for each nonzero $r \in R$ and $x \in M$ there exists $y \in M$ such that $ry = x$.

Corollary 1.4.13. *Let R be a domain. Then an injective R-module is divisible. If R is a PID, then any divisible R-module is injective.*

PROOF. For nonzero $r \in R$, we have an exact sequence $0 \to R \xrightarrow{r} R \to R/(r) \to 0$, and thus an exact sequence:

$$\begin{array}{ccccccc} \mathrm{Hom}(R,E) & \longrightarrow & \mathrm{Hom}(R,E) & \longrightarrow & \mathrm{Ext}^1(R/(r),E) & \longrightarrow & \mathrm{Ext}^1(R,E) \\ \| & & \| & & & & \| \\ E & \xrightarrow{r\cdot(-)} & E & & & & 0 \end{array}$$

The 0 on the right is because R is projective. Divisibility is the statement that for all r, the left-hand map is surjective, i.e. $\mathrm{Ext}^1(R/(r), E) = 0$. If E is injective then this holds; conversely if R is a PID, then all ideals are either zero or of this form. But for the zero ideal, $\mathrm{Ext}^1(R, E)$ is already 0, so we conclude by Lemma 1.4.12. □

Example 1.4.14. It is clear from the definition that a quotient of a divisible module is divisible. If R is a PID, then $\mathrm{Frac}(R)$ is injective, because it is clearly divisible; thus the same is true of any R-quotient of it. In the case $R = \mathbb{Z}$, this provides the two most important injective abelian groups, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. This also gives us an example of an injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. Note that this is also a flat resolution of $\mathbb{Q}/\mathbb{Z}$.

Example 1.4.15. Here are some weird properties of the groups mentioned above. Consider the sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. It is a trivial exercise that $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}) = 0$ and $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Q})$ is an isomorphism, so we deduce $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}) = 0$. More difficult is the group $G = \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$: One description is via the sequence

$$0 \to \mathbb{Q} \to G \to \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) \to 0,$$

which splits because $\mathbb{Q}$ is injective. It is a theorem that $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z}) \cong \mathbb{R}$, then $G = \mathbb{Q} \oplus \mathbb{R} \cong \mathbb{R}$. Alternatively, one can use the simple fact that $\mathbb{Q}/\mathbb{Z} = \varinjlim_n \mathbb{Z}/n\mathbb{Z}$ as the colimit of its n-torsion parts, and then

$$\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}.$$

We can combine this with the exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \widehat{\mathbb{Z}} \to \mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

which exhibits $\mathbb{R}$ as an extension of $\mathbb{Q}/\mathbb{Z}$ by $\widehat{\mathbb{Z}}$.

Let R be a ring, and M be an R-module. Via the unique map $\mathbb{Z} \to R$, we define the R-module $M^\vee := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, which is naturally a functor $\mathrm{Mod}(R)^{\mathrm{op}} \to \mathrm{Mod}(R)$. This is exact because $\mathbb{Q}/\mathbb{Z}$ is injective over $\mathbb{Z}$. Note that $R^\vee$ is injective over R by a previous lemma.

The double dual $(M^\vee)^\vee = \mathrm{Hom}_\mathbb{Z}(M^\vee, \mathbb{Q}/\mathbb{Z})$ has for each $x \in M$ a distinguished element $\mathrm{ev}(x)\colon M^\vee \to \mathbb{Q}/\mathbb{Z}$ given by $\phi \mapsto \phi(x)$. This defines a homomorphism $\mathrm{ev}\colon M \to (M^\vee)^\vee$, which is in fact a monomorphism.

Claim: $\mathrm{ev}\colon M \to (M^\vee)^\vee$ is monic.

Indeed, if $x \neq 0$, then let $G \subset M$ be the abelian subgroup generated by x. Then either $G \cong \mathbb{Z}/n\mathbb{Z}$ or $G \cong \mathbb{Z}$. In the former case, a nonzero map $G \to \mathbb{Q}/\mathbb{Z}$ is given by choosing an n-torsion element, such as $1/n$. In the latter case, any nonzero element gives such a nonzero map. By injectivity, this extends to a nonzero $\mathbb{Z}$-map $M \to \mathbb{Q}/\mathbb{Z}$, and therefore $\mathrm{ev}(x)$ is nonzero.

Using this, we can immediately deduce the existence of enough injectives in $\mathrm{Mod}(R)$. If we cared, we could note easily that the construction is functorial.

Proof (of Thm. 1.4.10). Take a surjection $\bigoplus_I R \twoheadrightarrow M^\vee \to 0$, and apply the exact functor $(-)^\vee$ to obtain an injection

$$(M^\vee)^\vee \to \mathrm{Hom}_\mathbb{Z}(\bigoplus_I R, \mathbb{Q}/\mathbb{Z}) = \prod_I \mathrm{Hom}_\mathbb{Z}(R, \mathbb{Q}/\mathbb{Z}) = \prod_I R^\vee.$$

We just proved that M injects into the LHS, and on the other hand the RHS is a product of injective modules $R^\vee$, hence injective. Lastly, it is clear from the constructions that this is functorial. □

Now all of our derived functor theory from the previous section applies. One difference is that we now view RHom as a functor on the derived category of the ring itself, rather than to abelian groups. (...finish this discussion...).

Some finiteness conditions

Let us briefly define some conditions and prove basic properties that we will return to later. A module M is *pseudo-coherent* if it has a resolution by finite projective modules. It is *finitely presented* if there is an exact sequence $F \to F' \to M \to 0$ where F, F' are finite free, i.e. there is a sequence $0 \to K \to F \to M \to 0$ where F is finite free and K is finitely generated.

We are going to use frequently and without mention that the forgetful functor $\mathrm{Mod}(R) \to \mathbf{Set}$ creates filtered colimits. Recall that an object M is compact if $\mathrm{Hom}_R(M, -)$ preserves filtered colimits as a functor to **Set**, which by the above remark is equivalent to $\mathrm{Hom}_R(M, -)$ preserving filtered colimits. It is clear that the class of compact objects is closed under finite colimits because filtered colimits commute with finite limits (in **Set** or $\mathrm{Mod}(R)$).

Example 1.4.16. All finitely presented R-modules are compact. Indeed, observe that $\mathrm{Hom}_R(R, -) \cong \mathrm{Id}$ so R is compact, hence finite free modules are compact, and a finitely presented module is a cokernel of a map between finite free modules.

Lemma 1.4.17. *Every R-module is a filtered colimit of its finitely generated submodules. Each R-module is a filtered colimit of finitely presented modules.*

Proof. The first claim is elementary. For the second, let M be an R-module. Consider the collection of pairs (S, E) where $S \subset M$ is a finite subset of M, and - letting $K_S \coloneqq \ker(R^{\oplus S} \to M)$ - E is a finite subset of K_S.

Let $(S,E) \leq (S',E')$ if $S \subset S'$, and the image of $E \subset K_S$ under the natural map $R^{\oplus S} \to R^{\oplus S'}$ is contained in E'. For each (S,E), let $C_{S,E}$ be the cokernel of the natural map $R^{\oplus E} \to R^{\oplus S}$, so $C_{S,E}$ is finitely presented. If $(S,E) \leq (S',E')$, then by construction we get a commutative diagram:

$$\begin{array}{ccccccc} R^{\oplus E} & \longrightarrow & R^{\oplus S} & \longrightarrow & C_{S,E} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ R^{\oplus E'} & \longrightarrow & R^{\oplus S'} & \longrightarrow & C_{S',E'} & \longrightarrow & 0 \end{array}$$

The maps $R^{\oplus S} \to M$ send the relations K_S to zero, hence factor through maps $C_{S,E} \to M$. We claim $\varinjlim_{(S,E)} C_{S,E} \to M$ is an isomorphism. Taking the filtered colimit of the above diagram of exact sequences, we get the exact sequence

$$R^{\oplus K} \to R^{\oplus M} \to \varinjlim_{(S,E)} C_{S,E} \to 0$$

where K is the kernel of the map $R^{\oplus M} \to M$, so we're done. □

Lemma 1.4.18. *Let M be a finite R-module. Then there is a finite filtration*

$$M = M_0 \subset \cdots \subset M_n = M$$

such that $M_{i+1}/M_i \cong R/I_i$ for some ideals $I_i \subset R$.

Proof. We induct on the number of generators of M, the case of a cyclic module being clear. If $M = \langle x_1, \ldots, x_m \rangle$, then take a filtration by induction for $M/\langle x_1 \rangle$, which is of the form

$$0 = N_0/\langle x_1 \rangle \subset N_1/\langle x_1 \rangle \subset \cdots \subset M/\langle x_1 \rangle$$

with cyclic quotients, then we win by taking

$$0 \subset \langle x_1 \rangle \subset N_1 \subset \cdots \subset N_n \subset M$$

with the same quotients. □

We say that M is FP_n if there is an exact sequence

$$P^{-n} \to \cdots P^0 \to M \to 0$$

with each P^i finite projective. For example, if R is Noetherian (or more generally coherent), then any finitely generated module M is FP_∞, because if we have a free resolution $R^{n_k} \to \cdots \to R^{n_0} \to M$, the left-hand kernel remains finitely generated so we can extend it.

Proposition 1.4.19. *Let R be a ring.*

(a) $\mathrm{Ext}^j(M,-)$ *preserves arbitrary products for all j.*

(b) $\mathrm{Ext}^j(-,N)$ *preserves arbitrary coproducts for all j.*

(c) If M is FP_{n+1}, then the functors

$$\mathrm{Ext}^0(M,-), \mathrm{Ext}^1(M,-), \ldots, \mathrm{Ext}^n(M,-)$$

preserve filtered colimits.

(d) If M is FP_{n+1} and Q is flat, then we have

$$\mathrm{Ext}^i(M, N \otimes Q) = \mathrm{Ext}^i(M, N) \otimes Q$$

for $i = 0, \ldots, n$. If Q is finite projective and M is arbitrary, then

$$\mathrm{Ext}^i(M \otimes Q^\vee, N) = \mathrm{Ext}^i(M, N \otimes Q) = \mathrm{Ext}^i(M, N) \otimes Q$$

for every $i \geq 0$, where $Q^\vee \coloneqq \mathrm{Hom}_R(Q, R)$.

(e) If $R \to S$ is a flat map and M is FP_{n+1}, then

$$\mathrm{Ext}^i_R(M, N) \otimes_R S = \mathrm{Ext}^i_S(M \otimes_R S, N \otimes_R S)$$

for $i = 0, \ldots, n$ as S-modules.

Proof. (a) and (b): For the former, let $P^\bullet \to M$ be a projective resolution, so that $\mathrm{Ext}^*(M, N)$ is the cohomology of

$$0 \to \mathrm{Hom}(P^0, N) \to \mathrm{Hom}(P^{-1}, N) \to \cdots$$

If $N = \prod_\alpha N_\alpha$, then we can move this out of the Hom and use the fact that products are exact (so commute with cohomology). Likewise, if $N \to I^\bullet$ is an injective resolution, then $\mathrm{Ext}^*(M, N)$ is cohomology of

$$0 \to \mathrm{Hom}(M, I^0) \to \mathrm{Hom}(M, I^1) \to \cdots,$$

and taking $M = \bigoplus_\alpha M_\alpha$ we conclude in the same way as above.

(c) We may choose a projective resolution $P^\bullet \to M$ so that P^i is finite for $-i \leq n + 1$. Let N_α be a filtered diagram of modules, then to compute Ext we take cohomology of:

$$0 \to \mathrm{Hom}(P^0, N_\alpha) \to \mathrm{Hom}(P^{-1}, N_\alpha) \to \cdots$$

Since P^{-i} is finite projective for $i \leq n$, it is finitely presented hence compact, so $\mathrm{Hom}(P^{-i}, -)$ preserves filtered colimits. Since filtered colimits are exact (hence commute with cohomology), for the terms $0, \ldots, n$ we can take cohomology to obtain the result.

(d) For the first part, just as in (c) (because $(-) \otimes Q$ is exact) we reduce immediately to showing that

$$\mathrm{Hom}(M, N \otimes Q) = \mathrm{Hom}(M, N) \otimes Q$$

if Q is flat and M is finitely presented. There is an obvious natural map from the RHS to the LHS, and let $R^q \to R^r \to M \to 0$ be a presentation, and now for any P we have an exact sequence

$$0 \to \mathrm{Hom}(M, P) \to P^r \to P^q.$$

On one hand, we could set $P = N$ and tensor by Q, so the right two terms become $(Q \otimes N)^r \to (Q \otimes N)^q$. On the other hand, we could set $P = N \otimes Q$ and the right two terms become the same thing.

For the second part, again because Q is still flat, we reduce to showing that

$$\mathrm{Hom}(M \otimes Q^\vee, N) = \mathrm{Hom}(M, N \otimes Q) = \mathrm{Hom}(M, N) \otimes Q$$

with no hypotheses on M. The point is to reduce to the free case: For the first equality, note that

$$\mathrm{Hom}(M \otimes Q^\vee, N) = \mathrm{Hom}(M, \mathrm{Hom}(Q^\vee, N)),$$

and if $Q \oplus Q' = R^n$ is free, then $R^n = (R^n)^\vee = Q^\vee \oplus (Q')^\vee$ implies $Q^\vee$ is still finite projective. Thus by replacing Q with its dual we may show that $Q^\vee \otimes N = \mathrm{Hom}(Q, N)$. There's an evaluation map α_Q in the forward direction which is natural in Q; choosing Q' such that $Q \oplus Q'$ is free we see $\alpha_{Q\oplus Q'} = \alpha_Q \oplus \alpha_{Q'}$ is an isomorphism, so α_Q is an isomorphism.

For the second equality, there's a canonical map

$$\beta_Q \colon \mathrm{Hom}(M, N) \otimes Q \to \mathrm{Hom}(M, N \otimes Q)$$

which is natural in Q. If $Q \cong R^n$ were finite free, then this map would be the isomorphism

$$\mathrm{Hom}(M, N) \otimes R^n \cong \mathrm{Hom}(M, N^n) \cong \mathrm{Hom}(M, N^n).$$

So for Q finite projective there exists Q' such that $\beta_Q \oplus \beta Q'$ is an isomorphism, hence β_Q is an isomorphism.

(e) Observe that if $R \to S$ is a map, then we have

$$\mathrm{Ext}^i_R(M, N|_R) \cong \mathrm{Ext}^i_S(M \otimes_R S, N)$$

by reducing to the case of Homs and applying the extension-restriction adjunction. Thus we conclude by applying this to (d). □

As an example of how to use this, we'll prove the basic structure theorem on finite projective modules.

THEOREM 1.4.20. *Let M be an R-module.*

(a) If M is (finite) projective, then $M[S^{-1}]$ is (finite) projective over $R[S^{-1}]$ for any S.
(b) Every finite projective module over a local ring is finite free.
(c) If M is finitely presented, then M is finite projective if and only if $M_\mathfrak{p}$ is finite free.
(d) M is finite projective if and only if there exist $f_1, \ldots, f_n$ generating the unit ideal of R such that M_{f_i} is finite free.

PROOF. (a) If $M \oplus P = R^{\oplus I}$ is (finite) free, then $M[S^{-1}] \oplus P[S^{-1}] = (R[S^{-1}])^{\oplus I}$.

(b) Let $(R, \mathfrak{m})$ be local, M finite projective, and choose $\{x_1, \ldots, x_n\}$ a minimal generating set, which defines an exact sequence $0 \to K \to R^n \to M \to 0$. Thus $R^n = K \oplus M$; if $(r_1, \ldots, r_n) \in K$, then

$$r_1 x_1 + \cdots + r_n x_n = 0$$

by definition, so none of the r_i are units by minimality. In particular $K \subset \mathfrak{m}^n$, which means $K \subset \mathfrak{m}K \oplus \mathfrak{m}M$, so $K/\mathfrak{m}K = 0$. Since K is finitely generated, it is zero by Nakayama's lemma.

(c) One direction is (a) and (b), and the other is the fact that as functors

$$\mathrm{Hom}_R(M,-)_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}},(-)_{\mathfrak{p}}),$$

so if $M_{\mathfrak{p}}$ is finite projective over $A_{\mathfrak{p}}$ for all $\mathfrak{p}$, then the functor above is exact for all $\mathfrak{p}$, hence $\mathrm{Hom}_R(M,-)$ is exact.

(d) In this case $\operatorname{Spec} A = \bigcup_i D(f_i)$, so for all $\mathfrak{p}$ let M_{f_i} be free over A_{f_i} for $\mathfrak{p} \in D(f_i)$, then $M_{\mathfrak{p}} = (M_{f_i})_{\mathfrak{p}}$ is free over $A_{\mathfrak{p}}$. Conversely, suppose M is finite projective, so $M_{\mathfrak{p}}$ is finite free for each $\mathfrak{p}$. Consider a basis $A^n_{\mathfrak{p}} \to M_{\mathfrak{p}}$, which by clearing denominators we may assume is the localization of a map $A^n \to M$. The cokernel C is a finite module so its support is closed, and thus there is some f such that $A^n_f \to M_f \to 0$ is exact. Its kernel is another finite projective module, and since its support is closed there is some g such that $A^n_{fg} \to M_{fg}$ is an isomorphism. □

Basic results on flatness Now we'll discuss some basic properties of Tor, which will give some nice flatness criteria. As in the last statement, we really want to be making many of these statements in the derived category of modules.

Lemma 1.4.21. *Let R be a ring. For any M the functors*

$$\operatorname{Tor}_i^R(M,-) \qquad \operatorname{Tor}_i^R(-,M) \qquad i \geq 0$$

preserve filtered colimits.

PROOF. Let $(N_i)_{i\in I}$ be a filtered diagram. Let $P^\bullet \to M[0]$ be a flat resolution, then

$$\operatorname{Tor}_n(M, \varinjlim_i N_i) = H^{-n}(P^\bullet \otimes_R (\varinjlim_i N_i)) = \varinjlim_i H^{-n}(P^\bullet \otimes_R N_i)$$

because cohomology commutes with filtered colimits. □

As an example, we will prove the following result. I like this result because the statement is simple yet it demonstrates nice properties of classical derived functors.

Proposition 1.4.22. *Let M be an R-module. The following are equivalent:*

(a) M is flat.
(b) For all finitely generated ideals $I \subset R$, the map $I \otimes_R M \xrightarrow{\sim} IM$ is an isomorphism.
(c) $\operatorname{Tor}_1^R(R/I, M) = 0$ for all finitely generated ideals $I \subset R$.

PROOF. For an ideal I, consider the sequence $0 \to I \to R \to R/I \to 0$. Tensor this with M to obtain

$$0 = \operatorname{Tor}_1^R(R,M) \to \operatorname{Tor}_1^R(R/I,M) \to I \otimes M \to M \to M/IM \to 0.$$

The image of $I \otimes M \to M$ is IM, so (b) is equivalent to saying $I \otimes M \to M$ is injective for finitely generated I. Thus (b) is equivalent to (c) by the sequence, and clearly (a) implies (c). Assuming (c), let I be any ideal, then $I = \varinjlim_J J$ is a filtered colimit of its finitely generated submodules, so by exactness $R/I = \varinjlim_J R/J$, and hence $\operatorname{Tor}^1(R/I, M) = 0$ for all I by the lemma.

Now, for any finite module N, take a filtration

$$0 = N_0 \subset \cdots \subset N_r = N$$

with cyclic quotients by Lemma 1.4.18. For each $i \geq 0$, there is an exact sequence

$$\operatorname{Tor}_1(N_i, M) \to \operatorname{Tor}_1(N_{i+1}, M) \to \operatorname{Tor}_1(R/I, M)$$

for some ideal I. The third term is zero, and inductively assuming that $\operatorname{Tor}_1(N_i, M) = 0$ the middle term is zero. Thus $\operatorname{Tor}_1(N, M) = 0$ for every finite module N. Since any module is the filtered colimit of its finite submodules, $\operatorname{Tor}_1(N, M) = 0$ for all N, and thus M is flat. □

Corollary 1.4.23. *Let R be a Bézout domain (e.g. a PID). An R-module M is flat iff it is torsion-free.*

PROOF. A module M is torsion free iff the map $M \xrightarrow{r} M$ is injective for all nonzero $r \in R$, which is $(-) \otimes_R M$ applied to the exact sequence $0 \to R \xrightarrow{r} R \to R/(r) \to 0$. Thus

$$0 \to \operatorname{Tor}_1(R/(r)) \to M \xrightarrow{r} M$$

is exact, and all finitely generated ideals of R are principal, so we're done by the ideal-wise criterion for flatness. □

Here's an easy observation. If F is flat, then

$$M \otimes^{\mathbf{L}} (N \otimes F) = M \otimes^{\mathbf{L}} N \otimes^{\mathbf{L}} F = (M \otimes F) \otimes^{\mathbf{L}} N = (M \otimes^{\mathbf{L}} N) \otimes F.$$

Since $(-) \otimes F$ is exact, it commutes with cohomology, so

$$\operatorname{Tor}^i(M \otimes F, N) = \operatorname{Tor}^i(M, N \otimes F) = \operatorname{Tor}^i(M, N) \otimes F$$

for all $i \geq 0$. If $R \to S$ is a flat map so $(-) \otimes_R S$ is exact, then

$$(M \otimes^{\mathbf{L}}_R N) \otimes_R S = (M \otimes_R S) \otimes^{\mathbf{L}}_R (N \otimes_R S),$$

and therefore

$$\operatorname{Tor}^R_i(M, N) \otimes_R S = \operatorname{Tor}^S_i(M \otimes_R S, N \otimes_R S)$$

as S-modules. In particular,

$$\operatorname{Tor}^R_i(M, N)_{\mathfrak{p}} = \operatorname{Tor}^{R_{\mathfrak{p}}}_i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \qquad \operatorname{Tor}^R_i(M, N)_f = \operatorname{Tor}^{R_f}_i(M_f, N_f)$$

for all $f \in R$ and $\mathfrak{p} \subset R$ prime. We have the following immediate corollary:

Proposition 1.4.24. *Let M be an R-module. The following are equivalent:*

(a) M is flat.
(b) $M_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for all primes $\mathfrak{p}$ (or $\mathfrak{m}$ maximal).
(c) M_f is flat over A_f for all $f \in A$.
(d) M_{f_i} is flat over A_{f_i} for $(f_1, \ldots, f_r) = (1)$ a Zariski cover.

PROOF. Immediate from the above, because these properties all correspond to criteria for a module to be zero. □

A map $M \to N$ is *universally injective* if $M \otimes P \to N \otimes P$ is injective for all P. More commonly we say M is a *pure* submodule of N, and e.g. if $0 \to M \to N \to P \to 0$ is exact, we say it's *pure-exact* if $M \subset N$ is pure.

Lemma 1.4.25. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of R-modules. The following are equivalent:*

(a) $0 \to M' \to M \to M'' \to 0$ is pure-exact.
(b) For every finitely presented module F,

$$0 \to \operatorname{Hom}_R(F, M') \to \operatorname{Hom}(F, M) \to \operatorname{Hom}(F, M'') \to 0$$

is exact.
(c) There is a filtered family of split exact sequences

$$0 \to M'_i \to M_i \to M''_i \to 0$$

whose filtered colimit is $0 \to M' \to M \to M'' \to 0$.

PROOF. Note that (c) easily implies (a), because tensor products commute with filtered colimits and tensoring with a split sequence remains split, and (b) because F is compact so $\mathrm{Hom}_R(F,-)$ preserves the filtered colimit (and split sequences).

$(a) \Rightarrow (b)$ Let $S^\bullet$ be the exact sequence viewed as a bounded acyclic complex, so that purity is equivalent to $Q \otimes S^\bullet$ being acyclic for all Q. If F is finitely presented, then let $Q_1 \to Q_2 \to F \to 0$ be a (finite free) presentation of F, so that $\mathrm{Hom}_R(Q_i,-) \cong Q_i^\vee \otimes_R (-)$, then choose a cokernel C as

$$0 \to F^\vee \to Q_2^\vee \to Q_1^\vee \to C \to 0.$$

Now apply $\mathrm{Hom}_R(-,S^\bullet)$ to the original presentation to obtain

$$0 \to \mathrm{Hom}_R(F,S^\bullet) \to Q_2^\vee \otimes_R S^\bullet \to Q_1^\vee \otimes_R S^\bullet \to C \otimes_R S^\bullet \to 0.$$

Exactness is checked termwise, and on the right four terms is by right-exactness of tensor product and the cokernel sequence above. By assumption, the right three complexes are acyclic, so the first one is too.

$(b) \Rightarrow (c)$ Write $M'' = \varinjlim_i M_i''$ as a filtered colimit of finitely presented modules. We can pull back the exact sequence along the natural maps $M_i'' \to M_i$ to obtain a commutative diagram of exact sequences as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M_i & \dashrightarrow & M_i'' & \longrightarrow & 0 \\ & & \| & & \vdots & \lrcorner & \downarrow & & \\ 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \end{array}$$

By functoriality of pullback, the top row is a filtered diagram of exact sequences. Note that if we apply $\varinjlim_i$, the 5-lemma shows that $\varinjlim_i M_i \xrightarrow{\sim} M$ and we obtain the original diagram. To see that each of the top sequences splits, apply $\mathrm{Hom}(M_i'',-)$:

$$\begin{array}{ccccc} \mathrm{Hom}(M_i'',M_i) & \longrightarrow & \mathrm{Hom}(M_i'',M_i'') & \longrightarrow & \mathrm{Ext}^1(M_i'',M_i) \\ \downarrow & & \downarrow & & \| \\ \mathrm{Hom}(M_i'',M) & \twoheadrightarrow & \mathrm{Hom}(M_i'',M'') & \xrightarrow{0} & \mathrm{Ext}^1(M_i'',M_i) \end{array}$$

Our supposition is that the bottom left map is surjective, so that the bottom right map is zero. Thus the top right map is zero, which means that the Ext^1 class of each top sequence is 0. □

This gives us a nice way to characterize various properties of purity, but most importantly gives a new flatness result.

Corollary 1.4.26. *Let F and P be R-modules.*

- *(a) If F is finitely presented, then every pure exact sequence $0 \to M' \to M \to F \to 0$ splits.*
- *(b) P is flat iff every exact sequence $0 \to M' \to M \to P \to 0$ is pure exact.*
- *(c) A finitely presented module is flat iff it is (finite) projective.*

(d) If P is flat, then a short exact sequence $0 \to M' \to P \to M'' \to 0$ is pure exact iff M'' is flat. So, pure submodules of flat modules are flat.

PROOF. (a) If F is finitely presented and the sequence is pure, then the map $\mathrm{Hom}(F, F) \to \mathrm{Ext}^1(F, P')$ is zero, so the extension is trivial.

(b) We already know one direction. Pure exactness is trivially equivalent to $\mathrm{Tor}_1(M, N) \to \mathrm{Tor}_1(P, N) \to 0$ being surjective for all N. In particular, if we choose $M \to P$ to be a surjection from a free module, then $\mathrm{Tor}_1(P, N) = 0$ for all N so N is flat.

(c) If P is finitely presented flat, take a presentation $0 \to K \to F \to P \to 0$ with F free. Then this is pure, hence split, so P is a summand of a free module.

(d) One direction is known, and the other is by the condition in (b). □

Example 1.4.27. Let $R = C^\infty(\mathbb{R})$ be the ring of smooth functions on $\mathbb{R}$, and

$$\mathfrak{m} \coloneqq \{f \in R : f(0) = 0\} \qquad I \coloneqq \{f \in R : \exists \epsilon > 0, f(x) = 0 \forall\, |x| < \epsilon\}.$$

Consider the finite module $M = R/I$. We claim M is R-flat but not projective. Note that the R-map $R \to R_{\mathfrak{m}}$ has kernel those f which are annihilated by an element of $R - \mathfrak{m}$, i.e. f such that $gf \equiv 0$ for $g(0) \neq 0$. If f is zero in a nbhd of 0, then we can clearly arrange this, and conversely if this holds, then f is zero on the nbhd $g^{-1}(\mathbb{R} - 0)$. Thus we get an injective map $M \hookrightarrow R_{\mathfrak{m}}$, and if f/g is an element of the codomain, with $g(0) \neq 0$, then we can find a function h which agrees with f/g on a nbhd of 0, hence gets sent to f/g under localization.

M is flat because it is a localization. If M were projective, then I would be a summand of R, so $R \to I \to R$ would give an idempotent e which generates I. If $e^2 = e$, then $e(e - 1) = 0$ as functions, so $e = 0$ or 1. This is a contradiction, because $I \neq 0$ and $I \neq R$.

CHAPTER 2

Derived categories of schemes

2.1. Sheaf cohomology

In this section, we are going to discuss the basic theory of sheaf cohomology using the language of the previous section. The category we will focus on is $\mathrm{Mod}(\mathcal{O}_X)$, the category of sheaves of $\mathcal{O}_X$-modules, where $(X, \mathcal{O}_X)$ is a ringed space. In passing we will also care about $\mathrm{PMod}(\mathcal{O}_X)$, which is the category of presheaves of $\mathcal{O}_X$-modules. We will consider the derived categories $D^*(\mathcal{O}_X) \coloneqq D^*(\mathrm{Mod}(\mathcal{O}_X))$, and the derived functors of standard functors associated to maps of spaces, global sections, and tensor product. We will also construct Čech cohomology as a means for computing sheaf cohomology.

2.1.1. General setup. Let us recall the basic setup of the theory of ringed spaces. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. If $f\colon X \to Y$ is a morphism of ringed spaces, then we have the pushforward map

$$f_*\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_Y)$$

defined the same way on presheaves. We also have the inverse image map

$$f^{-1}\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(f^{-1}\mathcal{O}_Y),$$

which is an exact functor, along with the pullback map

$$f^*\colon \mathrm{Mod}(\mathcal{O}_Y) \to \mathrm{Mod}(\mathcal{O}_X)$$

obtained by base change $f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ along $f^\flat\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We have an adjunction $f^* \dashv f_*$, so f^* is right exact and f_* is left exact. If $U \subset X$ is an open subset, then we have the sections functor

$$\Gamma(U, -)\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X(U)),$$

which is left exact. Slightly more generally, we usually consider the case where $\mathcal{O}_X$ is a sheaf of algebras over some base ring R, i.e. $\mathcal{O}_X(X)$ is an R-algebra, in which case we consider the composition of the functor above with $\mathrm{Mod}(\mathcal{O}_X(U)) \to \mathrm{Mod}(R)$ by restricting scalars. The same functor as given on presheaves is by definition exact.

The categories are also naturally symmetric monoidal, with tensor products $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ given by sheafifying the sectionwise tensor products. Likewise we have an internal hom given sectionwise by

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})(U) \coloneqq \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

This satisfies a tensor-hom adjunction, an internal tensor hom adjunction, and so on.

Consider the case of an inclusion $j\colon U \hookrightarrow X$ of an open subset, where U is considered as the ringed space $(U, \mathcal{O}_X|_U)$. Then we have some more explicit formulas

$$(j_*\mathcal{F})(V) = \mathcal{F}(U \cap V) \qquad j^{-1}\mathcal{F} = \mathcal{F}|_U \qquad j^*\mathcal{F} = \mathcal{F}|_U,$$

where the second equality is to be interpreted alongside the fact that the natural map $j^{-1}\mathcal{O}_X \to \mathcal{O}_U$ is an isomorphism. Thus in this case, j^* is an exact functor, and this is exhibited by an explicit *extension by zero* functor which is left adjoint to $j^* = j^{-1}$,

$$j_!\colon \operatorname{Mod}(\mathcal{O}_U) \to \operatorname{Mod}(\mathcal{O}_X),$$

defined on $\mathcal{F}$ as the sheafification of the presheaf $j_!^p\mathcal{F}$ by:

$$(j_!^p\mathcal{F})(V) = \begin{cases} \mathcal{F}(V) & V \subset U \\ 0 & V \subsetneq U \end{cases}$$

In particular $j_!\mathcal{F}$ has stalks either $\mathcal{F}_x$ if $x \in U$, or 0 otherwise. This is witnessed by the counit map $j_!\mathcal{F}|_U \to \mathcal{F}$ which is an isomorphism in U and zero elsewhere. Note in particular that $j_!$ is an exact functor by the explicit description of its stalks.

We also mention that if $\underline{\mathbb{Z}}$ denotes the constant sheaf at $\mathbb{Z}$, viewed as a sheaf of rings on X, then $\operatorname{Mod}(X, \mathcal{O}_{\underline{\mathbb{Z}}})$ is just the category $\operatorname{Ab}(X)$ of sheaves of abelian groups on X. In this special case, we use f^{-1} and f^* interchangeably, and all of our functors go between $\operatorname{Ab}(X) \rightleftarrows \operatorname{Ab}(Y)$.

We recall in this case that more can be said for closed embeddings $i\colon Z \to X$ where the unit map $i^{-1}i_*\mathcal{F} \to \mathcal{F}$ is a stalkwise isomorphism inside Z, and $i_*\mathcal{F}|_{X-Z} = 0$. Thus in this case, i_* is an exact functor. The inverse image and pushforward functors yield an equivalence of categories between abelian sheaves on Z and abelian sheaves on X supported in Z.

To define the derived functors of all of the objects above, we want to have enough injectives and projectives. Sheaf categories unfortunately don't have enough projectives, but since all of the non-left exact functors above were defined with tensor products, it will suffice to consider flat resolutions. On the other hand, they do have enough injectives.

THEOREM 2.1.1. *The categories* $\operatorname{Mod}(\mathcal{O}_X)$ *and* $\operatorname{PMod}(\mathcal{O}_X)$ *have enough injectives. In both cases, an injective embedding can be chosen functorially. Moreover, the inclusion functor* $\operatorname{Mod}(\mathcal{O}_X) \to \operatorname{PMod}(\mathcal{O}_X)$ *preserves injectives.*

To prove this (and other things), we will make a simple observation: If $F\colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ is an adjunction between abelian categories (so $F \dashv G$), and F is an exact functor, then G preserves injectives. Indeed, if I is injective in $\mathcal{A}$, then

$$\operatorname{Hom}_{\mathcal{B}}(-, G(I)) \cong \operatorname{Hom}_{\mathcal{A}}(F(-), I)$$

is a composition of two exact functors. As an example, since sheafification is exact, we deduce immediately that the inclusion $\operatorname{Mod}(\mathcal{O}_X) \hookrightarrow \operatorname{PMod}(\mathcal{O}_X)$ preserves

injectives. A second example is that if $f\colon X \to Y$ is a flat morphism of ringed spaces, meaning that f^* is exact, then f_* preserves injectives. Another useful fact is that an arbitrary product (which exists) of injectives remains injective. Indeed,

$$\mathrm{Hom}(-, \prod_\alpha I_\alpha) = \prod_\alpha \mathrm{Hom}(-, I_\alpha)$$

as abelian groups, and arbitrary products of abelian groups are exact.

Proof (of Thm. 2.1.1). The presheaf and sheaf cases are similar in structure but the former is sectionwise and the latter is stalkwise. Let's start with presheaves: For each $U \subset X$, the sections over U functor $\Gamma_U\colon \mathrm{PMod}(\mathcal{O}_X) \to \mathrm{PMod}(\mathcal{O}_X(U))$ is exact on presheaves. Define a functor

$$Q_U\colon \mathrm{Mod}(\mathcal{O}_X(U)) \to \mathrm{PMod}(\mathcal{O}_X)$$

by sending

$$Q_U(M)(V) = \begin{cases} M & U \subset V \\ 0 & U \nsubseteq V \end{cases}$$

(note the difference in direction from extension by zero). It is easy to check that maps $\mathcal{F} \to Q_U(M)$ are in correspondence with maps $\mathcal{F}(U) \to M$, and hence that $\Gamma_U \dashv Q_U$. To conclude, fix a presheaf $\mathcal{F}$ and an inclusion $\mathcal{F}(U) \hookrightarrow I_U$ into an injective $\mathcal{O}_X(U)$-module. Then $Q_U(I_U)$ is injective by the aforementioned adjunction property, and it is equipped with a canonical map $\mathcal{F} \to Q_U(\mathcal{F}(U)) \to Q_U(I_U)$. This induces a map

$$\mathcal{F} \to \prod_{U \subset X} Q_U(I_U)$$

where the RHS is injective, so it suffices to show that this is a monomorphism. That is clear because over each open set, one of the factors in the product is the inclusion $\mathcal{F}(U) \subset I_U$.

For the sheaf case, the stalk functor $(-)_x\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_{X,x})$ is exact. If $i_x\colon \{x\} \to X$ is the inclusion, and we view $\{x\}$ as a ringed space with structure sheaf $\mathcal{O}_{X,x}$, then the stalk functor is $i_x^{-1} \cong i_x^*$, so its right adjoint is $i_{x,*}$, which is the skyscraper sheaf construction. Thus $i_{x,*}$ preserves injectives; fix $\mathcal{F}$ a sheaf and for each x choose a monomorphism $\mathcal{F}_x \to I_x$ where I_x is an injective $\mathcal{O}_{X,x}$-module.

We have a natural map $\mathcal{F} \to i_{x,*}\mathcal{F}_x \to i_{x,*}I_x$, which in particular induces a map into the product

$$\mathcal{F} \to \prod_{x \in X} i_{x,*}I_x,$$

where the RHS is injective. We can check that this is a mono sectionwise: over U, the map is

$$\mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x \to \prod_{x \in U} I_x,$$

where the first map is monic for sheaves, and the second one is a product of monomorphisms of modules.

Note that in each case, as long as we chose the injective embedding of $\mathcal{F}(U)$ or $\mathcal{F}_x$ functorially, as we may do for modules over a ring (Thm. 1.4.10), it is clear that the resulting injective embedding of $\mathcal{F}$ is functorial. □

While we're at it, let us also discuss the existence of enough flat objects, whose proof has a very similar structure to the one above. Recall that a sheaf $\mathcal{F}$ is flat if $\mathcal{F} \otimes_{\mathcal{O}_X} (-)$ is exact. Note that because tensor products commute with taking stalks, $\mathcal{F}$ is flat iff $\mathcal{F}_x$ is flat for all x. From this definition and the tensor hom adjunction, it is clear that a direct sum of flats is flat.

Lemma 2.1.2. *The category* $\mathrm{Mod}(\mathcal{O}_X)$ *has enough flat objects. The same is true for* $\mathrm{PMod}(\mathcal{O}_X)$*. In both cases, a surjection from a flat object can be chosen functorially.*

Proof. Start with $\mathrm{Mod}(\mathcal{O}_X)$: for each $j_U \colon U \hookrightarrow X$, the extension by zero $j_{U,!}\mathcal{O}_U$ is flat, because its stalks are either $\mathcal{O}_{X,x}$ or 0. By adjunction,

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{F}|_U) = \mathcal{F}(U),$$

so each section $s \in U$ gives a map $f_{U,s} \colon j_{U,!}\mathcal{O}_U \to \mathcal{F}$. On stalks, this is either the zero map, or if $x \in U$ then it is the map $\mathcal{O}_{X,x} \to \mathcal{F}_x$ sending $1 \mapsto s_x$. Consider the induced map

$$f \colon \bigoplus_{(U,s)} j_{U,!}\mathcal{O}_U \to \mathcal{F}$$

from the coproduct, where the LHS is flat. To show surjectivity, check at a point $x \in X$: each $s_x \in \mathcal{F}_x$, coming from $s \in \mathcal{F}(U)$, is in the image of the corresponding map $f_{U,s}$ on the stalk at $x \in U$, so we are done.

In the case of presheaves of $\mathcal{O}_X$-modules, repeat the same proof as above, but replace $j_{U,!}$ with $j^p_{U,!}$, and note that for presheaves, flatness is checked sectionwise. But the map we constructed above from the coproduct is clearly surjective sectionwise in the presheaf case. Lastly, it is clear that this construction is functorial in either case. □

We will study the derived functors that the preceding two results give us in more detail in the next section. In this section, we will focus on derived global sections: if $(X, \mathcal{O}_X)$ is a ringed space, and Γ_U is the sections-over-U functor, we denote the corresponding right derived functor

$$R\Gamma(U, -) \colon D^+(\mathcal{O}_X) \to D^+(\mathcal{O}_X(U)).$$

If $A^\bullet$ is a bounded above complex of $\mathcal{O}_X$-modules, we usually write

$$\mathbb{H}^i(U, A^\bullet) \coloneqq H^i(R\Gamma(U, A^\bullet))$$

and call this the *hypercohomology* of A over U. In the special case $A^\bullet = \mathcal{F}[0]$ is just an object of $\mathrm{Mod}(\mathcal{O}_X)$, we write

$$H^i(U, \mathcal{F}) \coloneqq \mathbb{H}^i(R\Gamma(U, \mathcal{F}[0]))$$

for the classical higher derived functors and call this the *sheaf cohomology* of $\mathcal{F}$ over U.

First of all, let us see that our notation is not ambiguous. Let $j\colon U \hookrightarrow X$ be an inclusion, so that j^* is the restriction functor $(-)|_U$ on $\mathcal{O}_X$-modules. Since j^* is right adjoint to the exact functor $j_!$, it preserves injectives. Suppose $V \subset U \subset X$. Then we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{O}_X) & \xrightarrow{j^*} & \mathrm{Mod}(\mathcal{O}_X|_U) \\ & \searrow^{\Gamma_V} & \downarrow \Gamma_V \\ & & \mathrm{Mod}(\mathcal{O}_X(V)) \end{array}$$

of functors. Since j^* is exact, j^* descends to the derived category, and since j^* preserves injectives, we deduce that the diagram

$$\begin{array}{ccc} D^+(\mathcal{O}_X) & \xrightarrow{j^*} & D^+(\mathcal{O}_X|_U) \\ & \searrow^{R\Gamma_V} & \downarrow R\Gamma_V \\ & & D^+(\mathcal{O}_X(V)) \end{array}$$

commutes. In particular, $R\Gamma(V,-)$ depends only on $(V,\mathcal{O}_V)$ and not the ambient space; likewise for $\mathbb{H}^*(V,-)$. Since $H^0(U,-)$ coincides with the global sections functor on sheaves, it is customary at this stage to write H^0 for sections.

Next, viewing Γ_U and Γ_V as functors on $\mathrm{Mod}(\mathcal{O}_X)$ with $V \subset U$, restriction provides a natural transformation $\Gamma_U \to \Gamma_V$, and hence a natural map $R\Gamma(U,-) \to R\Gamma(V,-)$ which is restriction on a bounded below complex of Γ-acyclics. Thus we obtain maps $\mathbb{H}^n(U,\mathcal{F}) \to \mathbb{H}^n(V,\mathcal{F})$ natural in $\mathcal{F}^\bullet$, which by the above description form a presheaf:

$$\underline{\mathbb{H}}^n(X,\mathcal{F}^\bullet) \coloneqq \big(U \mapsto \mathbb{H}^n(U,\mathcal{F}^\bullet)\big).$$

These are called the *hypercohomology presheaves*. Here is the basic observation: let $i\colon \mathrm{Mod}(\mathcal{O}_X) \hookrightarrow \mathrm{PMod}(\mathcal{O}_X)$ be the natural inclusion, which is left exact because it is right adjoint to sheafification. We have already discussed the fact that i preserves injectives.

Thus, if $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is an injective resolution, $Ri(\mathcal{F}^\bullet) = \mathcal{I}^\bullet$ as a complex of presheaves, and the nth cohomology of this is a presheaf satisfying

$$U \mapsto \ker\big(\mathcal{I}^n(U) \to \mathcal{I}^{n+1}(U)\big)/\mathrm{im}\,\big(\mathcal{I}^{n-1}(U) \to \mathcal{I}^n(U)\big),$$

but the RHS is $H^n(\Gamma(U,\mathcal{I}^\bullet)) = \mathbb{H}^n(U,\mathcal{F}^\bullet)$, which shows that we have a canonical identification

$$H^n(Ri(\mathcal{F}^\bullet)) = \underline{\mathbb{H}}^n(X,\mathcal{F}^\bullet),$$

and in particular $R^n i(\mathcal{F}) = \underline{H}^n(X,\mathcal{F})$. In the rest of this section, we will consider the theory of flasque sheaves. We will consider applications of flasque sheaves in the section on derived functors of sheaves.

Let $\mathcal{F}$ be an object of $\mathrm{Mod}(\mathcal{O}_X)$. $\mathcal{F}$ is called *flasque* if for each $V \subset U$, the restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is surjective. Equivalently, all restriction maps $\mathcal{F}(X) \to \mathcal{F}(V)$ are surjective, because we can lift to $\mathcal{F}(X)$ first and then restrict down to $\mathcal{F}(U)$.

Flasqueness is a very non-geometric condition on a sheaf, since we usually expect to obtain more 'functions' when we restrict to a smaller open. We care

about flasque sheaves mainly because they are Γ-acyclic, so we can compute e.g. hypercohomology on flasque resolutions. The following is the main structure theorem on flasque sheaves:

Lemma 2.1.3. *Let X be a ringed space.*

1. *A finite direct sum of flasque sheaves is flasque.*
2. *If $\mathcal{F}$ is flasque, then $\mathcal{F}|_U$ is flasque for every $U \subset X$.*
3. *If $f\colon X \to Y$ is a morphism of ringed spaces and $\mathcal{F}$ is flasque, then $f_*\mathcal{F}$ is flasque.*

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of $\mathcal{O}_X$-modules.

4. *If $\mathcal{F}'$ is flasque, then*
$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$$
is exact for every $U \subset X$.
5. *If $\mathcal{F}'$ and $\mathcal{F}$ are flasque, then $\mathcal{F}''$ is flasque.*
6. *In the situation of (5), if $f\colon X \to Y$ is a morphism of ringed spaces, then $0 \to f_*\mathcal{F}' \to f_*\mathcal{F} \to f_*\mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_Y$-modules.*

Proof. (1) A direct sum of surjections is a surjection. (2) Flasqueness of $\mathcal{F}|_U$ is included in the definition of flasqueness of $\mathcal{F}$. (3) The restriction maps $H^0(U, f_*\mathcal{F}) \to H^0(V, f_*\mathcal{F})$ are just the restriction maps
$$H^0(f^{-1}(U), \mathcal{F}) \to H^0(f^{-1}(V), \mathcal{F})$$
of $\mathcal{F}$, which are surjective. (5) assuming (4): We have a commutative diagram involving the restriction maps
$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}''(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}''(V) \end{array}$$
where the horizontal maps are surjective by (4), and the left-hand vertical map is surjective. Thus the right-hand one is too. (6) This is a sequence of flasque sheaves by (3), hence it is exact over every open set by (5), thus exact as $\mathcal{O}_Y$-modules.

(4) We may take $U = X$; let $s \in \mathcal{F}''(X)$ be a section and consider the collection $\{(U,t)\}$ of opens $U \subset X$ and sections $t \in \mathcal{F}(U)$ such that $t \mapsto s|_U$. Order this collection by setting $(U,t) \leq (U',t')$ if $U \subset U'$ and $t'|_U = t$. The fact that $\mathcal{F} \to \mathcal{F}''$ is surjective means that the opens in this collection cover X. Also, given a totally ordered subset $\{(U_\alpha, t_\alpha)\}$ of this collection, the sections t_α glue to a section t over $\bigcup_\alpha U_\alpha$, by definition of the ordering.

Thus there is a maximal element (U,t) by Zorn's lemma, so assume for contradiction $U \neq X$. Then we can find an element (U',t') such that $U' \supsetneq U$, so $t - t' \mapsto s - s = 0$ over $U \cap U'$, which means there exists $r \in \mathcal{F}'(U \cap U')$ such that $r \mapsto t - t'$. By flasqueness, extend r to a section over U', and consider its image $i(r) \in \mathcal{F}(U)$. Then t and $t' + i(r)$ have the same restriction to $U \cap U'$, so they glue to a section of $\mathcal{F}$ over $U \cup U'$. The image of this section in $\mathcal{F}''(U \cap U')$ is s, because the composite $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is zero, which contradicts the maximality. $\square$

Corollary 2.1.4. *Let X be a ringed space. An injective $\mathcal{O}_X$-module is flasque. Moreover, the flasque $\mathcal{O}_X$-modules form an adapted class for Γ, and f_* for any morphism $f\colon X \to Y$. In particular, any flasque $\mathcal{O}_X$-module is Γ- and f_*-acyclic.*

Proof. Let $U \subset X$ and $j\colon U \hookrightarrow X$ be the inclusion, then $j_!\mathcal{O}_U \to \mathcal{O}_X$ is a monomorphism, so

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{O}_U, \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{O}_U, \mathcal{I}) = \mathcal{I}(U) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{I}) = \mathcal{I}(X)$$

is surjective for any injective sheaf $\mathcal{I}$, i.e. $\mathcal{I}$ is flasque.

To show the second claim, note that the flasque sheaves are a (right) preadapted class by Lemma 2.1.3(1) and what we just proved. Thus, let $\mathcal{F}^\bullet$ be a bounded below acyclic complex of flasque $\mathcal{O}_X$-modules; we have to show Γ and f_* preserve acyclicity.

Assume $\mathcal{F}^i = 0$ for $i < n$, and split into exact sequences

$$0 \to \mathcal{G}^i \to \mathcal{F}^{i+1} \to \mathcal{G}^{i+1} \to 0$$

where $\mathcal{G}^n = \mathcal{F}^n$ and the lower $\mathcal{G}^i$ are zero. Since $\mathcal{G}^n$ is flasque, by induction and Lemma 2.1.3(5), all $\mathcal{G}^i$ are flasque. Thus Γ and f_* preserve each sequence above, by Lemma 2.1.3(4) and (6), and hence they both preserve acyclicity of the original complex. □

Here's an example of how to use this, which we will see more of in two sections. Suppose $i\colon Z \to X$ is a morphism of ringed spaces which is a 'closed topological embedding', meaning the underlying map of spaces is a closed embedding. Then we know that i_* is an exact functor, because $(i_*\mathcal{F})_x$ is either 0, or $\mathcal{F}_z$ if $x = i(z)$. Thus this descends to a functor on the derived categories $i_*\colon D^*(\mathcal{O}_Z) \to D^*(\mathcal{O}_X)$.

Moreover, this comes with a natural ring homomorphism $\mathcal{O}_X(X) \to \mathcal{O}_Z(Z)$, along which restriction of scalars is exact. Thus this gives a map $D^*(\mathcal{O}_Z(Z)) \to D^*(\mathcal{O}_X(X))$ on the derived categories, which is simply termwise restriction of scalars. We can use this to formulate a basic 'change of scalars' result:

Lemma 2.1.5. *Let Z and X be ringed spaces and $i\colon Z \to X$ be a closed topological embedding. Then the diagram*

$$\begin{array}{ccc} D^+(\mathcal{O}_Z) & \xrightarrow{i_*} & D^+(\mathcal{O}_X) \\ {\scriptstyle R\Gamma}\downarrow & & \downarrow{\scriptstyle R\Gamma} \\ D^+(\mathcal{O}_Z(Z)) & \longrightarrow & D^+(\mathcal{O}_X(X)) \end{array}$$

commutes (up to natural isomorphism). In particular, there is a canonical identification

$$\mathbb{H}^n(Z, \mathcal{F}^\bullet) = \mathbb{H}^n(X, i_*\mathcal{F}^\bullet)$$

of hypercohomology for all $n \geq 0$.

Proof. We claim both directions of the square are the derived functors of the composite $\mathrm{Mod}(\mathcal{O}_Z) \to \mathrm{Mod}(\mathcal{O}_Z(Z)) \to \mathrm{Mod}(\mathcal{O}_X(X))$. One way is clear because

the second functor is exact, so all modules are acyclic for it. For the other way, note that the composite is equal to

$$\mathrm{Mod}(\mathcal{O}_Z) \xrightarrow{i_*} \mathrm{Mod}(\mathcal{O}_X) \to \mathcal{O}_X(X)$$

But flasque sheaves form an adapted class for both i_* and $R\Gamma$, and i_* preserves flasque sheaves, so we are done by the results on composition of derived functors. □

Remark 2.1.6. Note that if $f\colon X \to Y$ is any map of ringed spaces, then the exact same proof can be repeated with f instead of i, except pushing forward may not be exact, so we have to replace the top functor with

$$Rf_*\colon D^+(\mathcal{O}_X) \to D^+(\mathcal{O}_Y)$$

and we'll get a similar commutative diagram. In this case,

$$\mathbb{H}^n(X, \mathcal{F}^\bullet) = \mathbb{H}^n(Y, Rf_*\mathcal{F}^\bullet)$$

for all $n \geq 0$. We will study the properties of derived pushforward in the next two sections.

Example 2.1.7. Suppose we have a morphism $\mathcal{O}_1 \to \mathcal{O}_2$ of sheaves of rings on X. Then this gives a morphism of ringed spaces $(X, \mathcal{O}_1) \to (X, \mathcal{O}_2)$, which on underlying spaces is the identity map. Thus the lemma applies, and shows that the hypercohomology of a complex of $\mathcal{O}_2$-modules viewed as an $\mathcal{O}_1(X)$-module, is the same as its hypercohomology by first viewing it as a complex of $\mathcal{O}_1$-modules. If we take $\underline{\mathbb{Z}} \to \mathcal{O}_X$, then this shows that the underlying abelian group of cohomology as $\mathcal{O}_X$-modules agrees with abelian sheaf cohomology.

2.1.2. Čech cohomology. Recall that in a category with enough injectives, we defined

$$\mathrm{RHom}_{\mathcal{A}}\colon D(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D(\mathbb{Z})$$

via the equivalence $\mathrm{K}^+(\mathcal{I}_{\mathcal{A}}) \xrightarrow{\sim} D^+(\mathcal{A})$, and showed that it can be computed via an injective resolution in the second factor, or a projective resolution in the first factor. If we consider $\mathcal{A} = \mathrm{Mod}(\mathcal{O}_X)$ or $\mathrm{PMod}(\mathcal{O}_X)$, the hom complex functor $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(-, \mathcal{I})$ factors through $\mathrm{Mod}(\mathcal{O}_X(X))$ because each Hom group is an $\mathcal{O}_X(X)$-module, so we can consider RHom as a functor

$$\mathrm{RHom}_{\mathcal{O}_X}\colon D(\mathcal{O}_X)^{\mathrm{op}} \times D^+(\mathcal{O}_X) \to D(\mathcal{O}_X(X)),$$

and likewise for the derived categories of $\mathrm{PMod}(\mathcal{O}_X)$. This is because any relevant exactness/acyclicity is checked on the underlying abelian group. Via the exact restriction of scalars functor $D(\mathcal{O}_X(X)) \to D(\mathbb{Z})$, the diagram

$$\begin{array}{ccc} D(\mathcal{O}_X)^{\mathrm{op}} \times D^+(\mathcal{O}_X) & \longrightarrow & D(\mathcal{O}_X(X)) \\ & \searrow & \downarrow \\ & & D(\mathbb{Z}) \end{array}$$

commutes (and again, likewise for $\mathrm{PMod}(\mathcal{O}_X)$). This is because, if we fix $\mathcal{I}$ a bounded below complex of injectives, the functor is just the one induced by $\underline{\mathrm{Hom}}(-,\mathcal{I})$ on the localizations. Note that in particular, if $A, B \in D^+(\mathcal{O}_X)$, then

$$H^0(\mathrm{RHom}_{\mathcal{O}_X}(A,B)) = \mathrm{Hom}_{D^+(\mathcal{O}_X)}(A,B)$$

naturally has the structure of an $\mathcal{O}_X(X)$-module.

Almost all of the properties of RHom carry over even when we consider it as an $\mathcal{O}_X(X)$-module. For example, for a fixed sheaf $\mathcal{F}$, $\mathrm{RHom}(\mathcal{F},-)$ viewed as a functor $D^+(\mathcal{O}_X) \to D^+(\mathcal{O}_X(X))$ is still the right derived functor of the left exact functor

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X(X)).$$

We prove this the same way as before: the right derived functor on the derived category $R\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)$ exists, and on the equivalent category $\mathrm{K}^+(\mathcal{I}_{\mathrm{Mod}(\mathcal{O}_X)})$ it is isomorphic to $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},-)$, but the same is true for $\underline{\mathrm{Hom}}(\mathcal{F}[0],-) = \mathrm{Hom}(\mathcal{F},-)$. Likewise, the same holds for $\mathrm{PMod}(\mathcal{O}_X)$. Combining the above remarks shows that hypercohomology of a complex $\mathcal{F}^\bullet$ is just maps $\mathcal{O}_X \to \mathcal{F}^\bullet$ in the derived category. Also, by this discussion we view $\mathrm{Ext}^i(\mathcal{F},\mathcal{G})$ as an $\mathcal{O}_X(X)$-module.

Now, let $\mathcal{U} = \{U_i\}$ be an open cover of the ringed space X. For indices $i_0, \dots, i_p$, we use the notation

$$U_{i_0 \cdots i_p} \coloneqq U_{i_0} \cap \cdots \cap U_{i_p}.$$

If $\mathcal{F}$ is an object of $\mathrm{PMod}(\mathcal{O}_X)$, we define

$$\check{C}^p(\mathcal{U},\mathcal{F}) \coloneqq \prod_{i_0,\dots,i_p} \mathcal{F}(U_{i_0\cdots i_p}) \qquad p \geq 0.$$

We make this into a complex in $\mathrm{Ch}^+(\mathcal{O}_X(X))$ by defining the differential as

$$d(s)_{(i_0,\dots,i_{p+1})} = \sum_{k=0}^{p+1} (-1)^k s_{i_0 \cdots \widehat{i_k} \cdots i_{p+1}},$$

where on the RHS we implicitly consider the restriction of the section s to the open $U_{i_0\cdots i_{p+1}}$. Here is $d^2 = 0$:

$$\begin{aligned}
d^2(s)_{i_0\cdots i_{p+2}} &= \sum_{l=0}^{p+2} (-1)^l d(s)_{i_0 \cdots \widehat{i_l} \cdots i_{p+2}} \\
&= \sum_{l=0}^{p+2} \Big(\sum_{k<l} (-1)^{k+l} s_{i_0 \cdots \widehat{i_k} \cdots \widehat{i_l} \cdots i_{p+2}} + \sum_{k \geq l} (-1)^{k+l} s_{i_0 \cdots \widehat{i_l} \cdots \widehat{i_{k+1}} \cdots i_{p+2}} \Big) \\
&= \sum_{k<l} (-1)^{k+l} s_{i_0 \cdots \widehat{i_k} \cdots \widehat{i_l} \cdots i_{p+2}} + \sum_{k>l} (-1)^{k-1+l} s_{i_0 \cdots \widehat{i_l} \cdots \widehat{i_k} \cdots i_{p+2}} \\
&= \sum_{k<l} (-1)^{k+l} s_{i_0 \cdots \widehat{i_k} \cdots \widehat{i_l} \cdots i_{p+2}} - \sum_{k<l} (-1)^{k+l} s_{i_0 \cdots \widehat{i_k} \cdots \widehat{i_l} \cdots i_{p+2}} \\
&= 0.
\end{aligned}$$

For the third equality, we re-indexed k to $k+1$, and for the fourth we flipped k and l. Thus, $\check{C}^\bullet(\mathcal{U},\mathcal{F})$ is a complex, called the *Čech complex of $\mathcal{F}$ with respect to $\mathcal{U}$*. We write

$$\check{H}^p(\mathcal{U},\mathcal{F}) \coloneqq H^p(\check{C}^p(\mathcal{U},\mathcal{F}))$$

for its cohomology, called the *Čech cohomology of $\mathcal{F}$ with respect to $\mathcal{U}$*. Čech cohomology behaves well with respect to products. Indeed, since taking sections of presheaves (or sheaves) commutes with arbitrary products, and products commute with products, we have

$$\check{C}^\bullet(\mathcal{U},\prod_\alpha \mathcal{F}_\alpha) = \prod_\alpha C^\bullet(\mathcal{U},\mathcal{F}_\alpha).$$

Since arbitrary products are exact in the category of modules, we conclude $\check{H}^n(\mathcal{U},\prod_\alpha \mathcal{F}_\alpha) = \prod_\alpha \check{H}^n(\mathcal{U},F_\alpha)$ for all n.

Our next goal is to relate this cohomology to the sheaf cohomology defined above, which will be through a spectral sequence. Čech cohomology is not itself a derived functor on $\mathrm{Mod}(\mathcal{O}_X)$, but it is one on $\mathrm{PMod}(\mathcal{O}_X)$. The result is the following: Let $j_i\colon U_i \to X$ be the inclusion, and consider the presheaf extension-by-zero functors $j^p_{i,!}$. We write $\mathcal{O}^{\mathrm{pre}}_{\mathcal{U}}$ for the image presheaf of the canonical map

$$\bigoplus_i j^p_{i,!}\mathcal{O}_{U_i} \to \mathcal{O}_X$$

induced by the counits $j^p_{i,!}\mathcal{O}_{U_i} \to \mathcal{O}_X$, i.e. the sectionwise image. Then:

Proposition 2.1.8. *The diagram of functors*

$$\begin{array}{ccc} \mathrm{PMod}(\mathcal{O}_X) & \xrightarrow{\check{C}(\mathcal{U},-)} & \mathrm{Ch}^+(\mathrm{Mod}(\mathcal{O}_X(X)) \\ \downarrow & & \downarrow \\ D^+(\mathrm{PMod}(O_X)) & \xrightarrow[\mathrm{RHom}^{\mathrm{pre}}_{\mathcal{O}_X}(\mathcal{O}_{\mathcal{U}},-)]{} & D^+(\mathcal{O}_X(X)) \end{array}$$

commutes up to canonical natural isomorphism. Moreover, there is a canonical identification $\mathrm{Hom}^{\mathrm{pre}}_{\mathcal{O}_X}(\mathcal{O}_{\mathcal{U}},-) = \check{H}^0(\mathcal{U},-)$. *Therefore*

$$R\check{H}^0(\mathcal{U},-) = \mathrm{Ext}^{\mathrm{pre},n}_{\mathcal{O}_X}(\mathcal{O}_{\mathcal{U}},-) = \check{H}^n(\mathcal{U},-)$$

for all $n \geq 0$.

Let us demonstrate this in two simple steps. First, since we are working in the presheaf category, $\mathcal{O}_X$ is projective, because it is projective sectionwise. So, $\mathcal{O}_U$ is projective in $\mathrm{PMod}(\mathcal{O}_U)$, and since $j_!$ is a right adjoint to restriction, which is exact, $j^p_!\mathcal{O}_U$ is projective.

Now, let $j_{i_0\cdots i_p}$ be the inclusion of $U_{i_0\cdots i_p}$. Write

$$K^{-n}(\mathcal{U}) \coloneqq \bigoplus_{i_0,\ldots,i_n} j^p_{i_0\cdots i_n,!}\mathcal{O}_{U_{i_0\cdots i_n}},$$

so each $K^{-n}(\mathcal{U})$ is projective. Moreover,

$$\begin{aligned}\mathrm{Hom}^{\mathrm{pre}}_{\mathcal{O}_X}(K^{-n}(\mathcal{U}),\mathcal{F}) &= \prod_{i_0,\dots,i_n} \mathrm{Hom}^{\mathrm{pre}}_{\mathcal{O}_X}(j^p_{i_0\cdots i_n,!}\mathcal{O}_{U_{i_0\cdots i_n}},\mathcal{F}) \\ &= \prod_{i_0,\dots,i_n} \mathrm{Hom}^{\mathrm{pre}}(\mathcal{O}_{U_{i_0\cdots i_n}},\mathcal{F}|_{U_{i_0\cdots i_n}}) \\ &= \prod_{i_0,\dots,i_n} \mathcal{F}(U_{i_0\cdots i_n}) = \check{C}^p(\mathcal{U},\mathcal{F})\end{aligned}$$

for any presheaf $\mathcal{F}$, by adjunction. Since the Čech differential $\check{C}^n(\mathcal{U},-) \to \check{C}^{n+1}(\mathcal{U},-)$ is natural, it corresponds by Yoneda to a map $K^{-n-1}(\mathcal{U}) \to K^{-n}(\mathcal{U})$, which also satisfies $d^2 = 0$ by Yoneda. We modify this differential by multiplying the $-p$th one by $(-1)^p$, so that

$$\underline{\mathrm{Hom}}(K^\bullet(\mathcal{U}),\mathcal{F}) = \check{C}^\bullet(\mathcal{U},\mathcal{F})$$

holds as complexes (according to our convention for Hom complexes). Since $K^\bullet$ is a bounded above complex of projectives, we may replace the left-hand side with $\mathrm{RHom}(K(\mathcal{U}),\mathcal{F})$ after passing to the derived category. It remains to show that $\mathrm{K}(\mathcal{U})$ is isomorphic in the derived category to $\mathcal{O}^{\mathrm{pre}}_{\mathcal{U}}$.

This is the second step: form the augmented complex

$$\cdots \to K^{-2}(\mathcal{U}) \to K^{-1}(\mathcal{U}) \to K^0(\mathcal{U}) = \bigoplus_i j^p_{i,!}\mathcal{O}_{U_i} \to \mathcal{O}_{\mathcal{U}} \to 0$$

We claim this is an exact sequence of presheaves; if this holds, then by applying $\mathrm{Hom}(-,\mathcal{F})$ and using one-sided exactness, we deduce that $\mathrm{Hom}(\mathcal{O}_{\mathcal{U}},\mathcal{F}) \cong \check{H}^0(\mathcal{U},\mathcal{F})$ as both sides are kernels of the same map. To prove the claim, we may show that evaluating the sequence at each open set W produces a contractible complex of $\mathcal{O}_X(W)$-modules.

Proof (of Prop. 2.1.8). This proof is essentially copied from Lemma 01EM in the Stacks Project. As above, fix an open set W, and let I be the collection of indices $\{i\}$ such that $W \subset U_i$, so that by definition of the functors $j^p_!$, the augmented complex above becomes:

$$\cdots \to \bigoplus_{i_0,\dots,i_{p+1}\in I} \mathcal{O}_X(W) \to \bigoplus_{i_0,\dots,i_p\in I} \mathcal{O}_X(W) \to \cdots$$

The differential here is

$$d(s)_{i_0\cdots i_p} = \sum_{j=0}^{p+1}\sum_{i\in I}(-1)^j s_{i_0\cdots i_{j-1}ii_j\cdots i_p}.$$

Now, choose an element $\ell \in I$ which we will use to define the contracting homotopy. Set:

$$h(s)_{i_0\cdots i_{p+1}} = \begin{cases} s_{i_1\cdots i_{p+1}} & i_0 = \ell \\ 0 & i_0 \neq \ell \end{cases}$$

Then it is an elementary check that

$$dh + hd(s)_{i_0\dots i_p} = s_{i_0\cdots i_p}$$

by splitting the sum into components where $i_0 = \ell$ and $i_0 \neq \ell$, where the two sums cancel as in the proof that $d^2 = 0$. □

Here are some immediate deductions from this result. In fact, these can be deduced in a more direct way by explicitly constructing homotopies, which we will do e.g. in the case of the alternating Čech complex.

Example 2.1.9. Let $\mathcal{U}$ be an open cover of X, and suppose that $X \in \mathcal{U}$ is itself one of the elements of the cover. Then for each presheaf $\mathcal{F}$, the Cech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F})$ just has a zeroth cohomology group $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ with higher cohomology all zero. Indeed, in this case the presheaf $\mathcal{O}_{\mathcal{U}}^{\mathrm{pre}}$ is (by definition) all of $\mathcal{O}_X$, and $\mathrm{Hom}^{\mathrm{pre}}(\mathcal{O}_X, -)$ is the global sections functor which is exact on presheaves. This will be the key observation in the proof that quasicoherent sheaves on affine schemes have vanishing higher cohomology, which is because of the fact that standard affine covers of local rings have an element which is the whole space already.

In fact, in this case $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \simeq \mathcal{F}(X)[0]$ already in the homotopy category: indeed, we just showed that

$$K(\mathcal{U}) \simeq \mathcal{O}_{\mathcal{U}}[0] = \mathcal{O}_X[0]$$

in the homotopy category, so applying $\underline{\mathrm{Hom}}^{\mathrm{pre}}_{\mathcal{O}_X}(-, \mathcal{F})$ gives homotopy equivalences

$$\check{C}(\mathcal{U}, \mathcal{F}) = \underline{\mathrm{Hom}}^{\mathrm{pre}}_{\mathcal{O}_X}(K(\mathcal{U}), \mathcal{F}) \simeq \underline{\mathrm{Hom}}^{\mathrm{pre}}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \mathcal{F}(X)[0]$$

as desired.

Example 2.1.10. One also might want to know the relation between Čech complexes when we add an open set. Let $\mathcal{U}$ be an open cover of X, and $\mathcal{U} \cup V = \mathcal{U} \cup \{V\}$ be the cover obtained from adding V. Also let $\mathcal{U} \cap V = \{U \cap V : U \in \mathcal{U}\}$ be the open cover of V obtained by restricting $\mathcal{U}$. From the definition it is clear that we have a canonical inclusion $\mathcal{O}_{\mathcal{U}}^{\mathrm{pre}} \subset \mathcal{O}_{\mathcal{U} \cup V}^{\mathrm{pre}}$ of presheaves.

This data gives a natural map $\mathcal{O}_{\mathcal{U}}^{\mathrm{pre}} \oplus j^p_{V,!}\mathcal{O}_V \to \mathcal{O}_{\mathcal{U} \cup V}^{\mathrm{pre}}$, which is surjective by definition of the relevant objects, and we wish to compute the kernel. Fix an open set $W \subset X$ and consider the map on sections. The codomain is $\mathcal{O}_X(W)$ if W is contained in V or an element of $\mathcal{U}$, and zero otherwise. The domain is $\mathcal{O}_X(W)$ if W is contained in an element of $\mathcal{U}$ but not V, zero if W is contained in neither, $\mathcal{O}_X(W) \oplus \mathcal{O}_X(W)$ if W is contained in an element of $\mathcal{U}$ and V, and $\mathcal{O}_X(W)$ if it's contained in V and not an element of $\mathcal{U}$.

Based on this explicit description, it is easy to see that we have an exact sequence

$$0 \to j^p_{V,!}\mathcal{O}^{\mathrm{pre}}_{\mathcal{U} \cap V} \to \mathcal{O}^{\mathrm{pre}}_{\mathcal{U}} \oplus j^p_{V,!}\mathcal{O}_V \to \mathcal{O}^{\mathrm{pre}}_{\mathcal{U} \cup V} \to 0$$

of presheaves on X, and hence an exact triangle in the derived category. Applying $\mathrm{RHom}^{\mathrm{pre}}_{\mathcal{O}_X}(-, \mathcal{F})$ gives an exact triangle

$$\check{C}(\mathcal{U} \cup V, \mathcal{F}) \to \check{C}(\mathcal{U}, \mathcal{F}) \oplus \mathcal{F}(V)[0] \to \check{C}(\mathcal{U} \cap V, \mathcal{F}|_V) \to \check{C}(\mathcal{U} \cup V, \mathcal{F})[1]$$

in the derived category of $\mathcal{O}_X(X)$. For the second term, we used the fact that $j^p_{V,!}\mathcal{O}_V$ was a projective object, hence RHom could be computed using Hom. This

is called the Čech Mayer-Vietoris sequence. Thus we get a long exact sequence which looks like this:

$$0 \to \check{H}^0(\mathcal{U} \cup V, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{F}) \oplus \mathcal{F}(V) \to \check{H}^0(\mathcal{U} \cap V, \mathcal{F}) \to$$
$$\check{H}^1(\mathcal{U} \cup V, \mathcal{F}) \to \check{H}^1(\mathcal{U}, \mathcal{F}) \to \check{H}^1(\mathcal{U} \cap V, \mathcal{F}) \to \check{H}^2(\mathcal{U} \cup V, \mathcal{F}) \to \cdots$$

As a sample application, let's show by induction on n that a cover $\mathcal{U}$ consisting of n opens has vanishing Čech cohomology in degree $\geq n$. In the case $n = 1$ we can apply the previous result that the Cech complex only has $\check{H}^0$. Assuming the result for covers with n elements (for all spaces and presheaves), let $\mathcal{U} = \{U_1, \ldots, U_n\}$, $V = U_{n+1}$, then for all $i \geq 0$ there are exact sequences

$$\check{H}^i(\mathcal{U} \cap U_{n+1}, \mathcal{F}|_{U_{n+1}}) \to \check{H}^{i+1}(\mathcal{U} \cup U_{n+1}, \mathcal{F}) \to \check{H}^{i+1}(\mathcal{U}, \mathcal{F}).$$

Since $\mathcal{U}$ and $\mathcal{U} \cap U_{n+1}$ are covers consisting of n elements, for all $i \geq n$ the left- and right-hand terms in the sequence above are zero. Thus the middle term is zero, proving the claim. We will prove shortly a much stronger result by exhibiting a homotopy equivalent complex which has only n terms in it.

We now use what we have shown to relate Čech cohomology to sheaf cohomology, which was the point of introducing it. We'll give two different methods of doing this, the latter of which will illustrate a slightly more general technique.

THEOREM 2.1.11. *Let X be a ringed space and $\mathcal{U}$ an open cover of X.*

(a) There is a natural map $\check{C}(\mathcal{U}, -) \to R\Gamma(X, -)$ in $D^+(\mathcal{O}_X(X))$.

(b) For each $\mathcal{O}_X$-module $\mathcal{F}$, there is a bounded spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F})$$

converging to a finite filtration on the RHS, which is natural in $\mathcal{F}$.

In particular, if the restrictions of $\mathcal{F}$ to all finite intersections of elements of the open cover $\mathcal{U}$ are Γ-acyclic, then the map $\check{C}(\mathcal{U}, \mathcal{F}) \to R\Gamma(X, \mathcal{F})$ is an isomorphism in $D^+(\mathcal{O}_X(X))$.

PROOF. Consider the composition of functors

$$\mathrm{Mod}(\mathcal{O}_X) \xrightarrow{i_X} \mathrm{PMod}(\mathcal{O}_X) \xrightarrow{\check{H}^0(\mathcal{U}, -)} \mathrm{Mod}(\mathcal{O}_X(X)).$$

First observe that $\check{H}^0(\mathcal{U}, \mathcal{F})$ is the kernel in

$$0 \to \check{H}^0(\mathcal{U}, \mathcal{F}) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

where the second map is the same one as in the sheaf condition, by inspection of the Čech differential. Thus this gives a natural map $H^0(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{F})$ induced by restriction, which is an isomorphism iff $\mathcal{F}$ satisfies the sheaf condition for the cover $\mathcal{U}$. So, $\check{H}^0(\mathcal{U}, -) \circ i_X = \Gamma(X, -)$.

We know $\mathrm{Mod}(\mathcal{O}_X)$ and $\mathrm{PMod}(\mathcal{O}_X)$ have enough injectives and that i_X preserves them, so the theorem on a composition of derived functors says that

$R\Gamma = R\check{H}^0(\mathcal{U}, -) \circ Ri_X$. The natural map $i_X \to Ri_X$ provides a natural transformation

$$\check{C}(\mathcal{U}, -) = R\check{H}^0(\mathcal{U}, -) \circ i_X \to R\check{H}^0(\mathcal{U}, -) \circ Ri_X = R\Gamma.$$

The spectral sequence is exactly the Grothendieck spectral sequence for a composition of right derived functors, and our explicit computation of $R^p i_X$ as cohomology presheaves. For the last remark, we have

$$E_1^{p,q} = \check{C}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})),$$

so by assumption for $q > 0$ and $p \geq 0$, this is zero. Thus the spectral sequence degenerates at E_2, with only the first row

$$E_2^{p,0} = \check{H}^p(\mathcal{U}, \underline{H}^0(\mathcal{F})) = \check{H}^p(\mathcal{U}, \mathcal{F})$$

being nonzero. Thus $\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$ is an isomorphism, so the map of (a) is a quasi-isomorphism. We are talking about the same map because of the description of edge maps in the Grothendieck spectral sequence. □

The spectral sequence in the theorem above is called the Čech-to-derived spectral sequence. We will give a different proof and slight generalization shortly. First, some other results on Čech cohomology:

Example 2.1.12. If $\mathcal{F}$ is a flasque $\mathcal{O}_X$-module, then $\check{H}^n(\mathcal{U}, \mathcal{F}) = 0$ for $n > 0$ and any open cover $\mathcal{U}$ (and $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$, since $\mathcal{F}$ is a sheaf). Indeed, any restriction of a flasque sheaf is flasque, so by the theorem its Čech cohomology agrees with its sheaf cohomology.

Lemma 2.1.13. *Let $\mathcal{F}'$ be an $\mathcal{O}_X$-module such that $\check{H}^1(\mathcal{U}, \mathcal{F}') = 0$ for every cover $\mathcal{U}$ in a cofinal system of open covers of X. Then for any exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of $\mathcal{O}_X$-modules,*

$$0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X) \to 0$$

is an exact sequence of $\mathcal{O}_X(X)$-modules.

In this context, a *cofinal system of open covers* is a collection of open covers such that every open cover has a refinement in the collection.

Proof. We just have to check surjectivity of $\mathcal{F}(X) \to \mathcal{F}''(X)$, so let $s \in \mathcal{F}''(X)$ be a section. Then there's an open cover $\{U_i\}$ and sections $t_i \in \mathcal{F}(U_i)$ such that $t_i \mapsto s|_{U_i}$, and by restricting we may assume that this cover is in the given cofinal system. For each i, j, $t_j - t_i \mapsto s - s = 0$, so there is a unique $r_{ij} \in \mathcal{F}'(U_{ij})$ such that $r_{ij} \mapsto t_j - t_i$. Thus $r = (r_{ij})_{ij} \in \check{C}^1(\{U_i\}, \mathcal{F}')$, and we claim it is a cocycle. Indeed,

$$d(r)_{ijk} = r_{jk} - r_{ik} + r_{ij} = 0$$

holds because of injectivity of $\mathcal{F}' \to \mathcal{F}$ and

$$d(r)_{ijk} \mapsto t_k - t_j - (t_k - t_i) + t_j - t_i = 0$$

over $\mathcal{F}(U_{ijk})$. Hence by assumption there exists $q_i \in \mathcal{F}'(U_i)$ such that $q_j - q_i = r_{ij}$ for all i, j. Letting f be the map $\mathcal{F}' \to \mathcal{F}$, consider the sections $t_i - f(q_i)$. Then

$$t_j - f(q_j) - \big(t_i - f(q_i)\big) = t_j - t_i - f(r_{ij}) = 0$$

over U_{ij}, so these sections glue to an element of $\mathcal{F}(X)$. This maps to s because it does so on an open cover. □

Example 2.1.14. Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module such that for all $U \subset X$ and open covers $\mathcal{U}$ of U, $\check{H}^p(\mathcal{U}, \mathcal{F}|_U) = 0$ for $p > 0$. Then $H^p(X, \mathcal{F}) = 0$ for $p > 0$. We prove this by induction on p. Choose an exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ with $\mathcal{I}$ injective, so that $H^i(X, \mathcal{G}) \xrightarrow{\sim} H^{i+1}(X, \mathcal{F})$ for $i > 0$. Moreover, since $H^1(X, \mathcal{I}) = 0$, Lemma 2.1.13 shows that $H^1(X, \mathcal{F}) = 0$, which is the base case. For the general case, Lemma 2.1.13 shows that this sequence is in fact an exact sequence of presheaves, and since Čech cohomology is a derived functor on presheaves, $\check{H}^i(\mathcal{U}, \mathcal{G}|_U) = \check{H}^{i+1}(\mathcal{U}, \mathcal{F}|_U) = 0$ for $i > 0$ since $\mathcal{I}$ is flasque. Thus by the inductive hypothesis applied to $\mathcal{G}$, we are done.

A very slight generalization of the argument above will give us affine cohomology vanishing for quasi-coherent sheaves, and gives a more practical criterion for checking cohomology vanishing.

Lemma 2.1.15. *Let X be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Assume there exists a basis $\mathcal{B}$ such that for each $U \in \mathcal{B}$, there is a cofinal system of open coverings Cov_U of U such that:*

(a) For $\mathcal{U} \in \mathrm{Cov}_U$, $U_{i_0 \cdots i_p} \in \mathcal{B}$.

(b) For $\mathcal{U} \in \mathrm{Cov}_U$, $\check{H}^p(\mathcal{U}, \mathcal{F}|_U) = 0$ for $p > 0$.

Then $H^p(U, \mathcal{F}) = 0$ for $p > 0$ and all $U \in \mathcal{B}$.

Proof. We proceed by induction on $p > 0$ over all $\mathcal{F}$ simultaneously. Choose an exact sequence $0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0$ where $\mathcal{I}$ is injective. First of all, this gives an exact sequence

$$\mathcal{I}(U) \to \mathcal{G}(U) \to H^1(U, \mathcal{F}) \to H^1(U, \mathcal{I}) = 0,$$

and since the first Čech cohomology vanishes on a cofinal system of open covers of U, we conclude $H^1(U, \mathcal{F}) = 0$ by Lemma 2.1.13. This is the base case. For $p > 1$, note moreover that $H^{p-1}(U, \mathcal{G}) \xrightarrow{\sim} H^p(U, \mathcal{F})$, so it suffices to show that $H^{p-1}(U, \mathcal{G}) = 0$, for which it suffices to show that $\mathcal{G}$ satisfies the same hypotheses as $\mathcal{F}$ in the assumptions.

To check this, for each U and $\mathcal{U} \in \mathrm{Cov}_U$, the sequence of sheaves induces a sequence

$$(*) \qquad 0 \to \check{C}^\bullet(\mathcal{U}, \mathcal{F}|_U) \to \check{C}^\bullet(\mathcal{U}, \mathcal{I}|_U) \to \check{C}^\bullet(\mathcal{U}, \mathcal{G}|_U) \to 0$$

of complexes which we claim is exact. Since products of modules are exact, this reduces to showing that $0 \to \mathcal{F}(V) \to \mathcal{I}(V) \to \mathcal{G}(V) \to 0$ is exact, where V is of the form $U_{i_0 \cdots i_p}$. But then $V \in \mathcal{B}$, so this follows again from Lemma 2.1.13 as before.

The exact sequence $*$ of complexes gives an exact sequence

$$\check{H}^n(\mathcal{U}, \mathcal{I}|_U) \to \check{H}^n(\mathcal{U}, \mathcal{G}|_U) \to \check{H}^{n+1}(\mathcal{U}, \mathcal{F}|_U)$$

in Čech cohomology for every $n \geq 1$. The left term is 0 because $\mathcal{I}|_U$ is flasque, and the right term is zero by assumption. Therefore the middle term is zero, so $\mathcal{G}$ satisfies the same hypotheses as $\mathcal{F}$. □

Now we come to the second method for relating Čech cohomology to derived-functor cohomology. The idea is to sheafify (not literally) the Čech complex into a complex of sheaves.

Continue the assumptions that X is a ringed space and $\mathcal{U}$ is an open cover of X. Let $\mathcal{F}$ be a presheaf, and define the complex of presheaves

$$\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \coloneqq \prod_{i_0,\dots,i_p} j_{i_0\cdots i_p,*}\mathcal{F}|_{U_{i_0\cdots i_p}} = \prod_{\ell(I)=p} j_{I,*}\mathcal{F}|_{U_I},$$

where we are using the obvious convention for the second notation, noting that I runs over tuples of the form $I = (i_0, \dots, i_p)$ and its length ℓ is p, not $p+1$. We define the differential component-wise as before: to give a map $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U}, \mathcal{F})$ we have to give maps $\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{F})|_{U_I} \to \mathcal{F}|_{U_I}$ for each I of length p, and restricting commutes with the product so we define

$$d(s)_{i_0\cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_0\cdots\widehat{i_k}\cdots i_{p+1}}$$

sectionwise. Moreover, this is natural in $\mathcal{F}$, so if $\mathcal{F}^\bullet$ is a complex of presheaves, we can define a bicomplex $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ by the same formula levelwise, except that we should twist the vertical differentials by $(-1)^q$ to agree with our convention for bicomplexes. Finally, note that there is no ambiguity about sheaves/presheaves because arbitrary products in the two categories agree.

Lemma 2.1.16. *With notation as above, let $\mathcal{U}$ be an open cover, $\mathcal{F}$ be an $\mathcal{O}_X$-module, and $\mathcal{F}^\bullet$ a bounded above complex of $\mathcal{O}_X$-modules.*

(a) As defined above, $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F})$ is indeed a complex, and $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F}^\bullet)$ is indeed a bicomplex.

(b) The identities

$$\Gamma(V, \check{\mathcal{C}}(\mathcal{U}, \mathcal{F})) = \check{C}(\mathcal{U} \cap V, \mathcal{F}|_V) \qquad \check{\mathcal{C}}(\mathcal{U}, \mathcal{F}^\bullet)|_V = \check{\mathcal{C}}(\mathcal{U} \cap V, \mathcal{F}^\bullet|_V)$$

both hold for every open subspace $V \subset X$.

(c) The natural morphism of complexes $\mathcal{F}[0] \to \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a quasi-isomorphism. More generally, the natural morphism of complexes

$$\mathcal{F}^\bullet \to \operatorname{Tot} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$$

is a quasi-isomorphism.

In light of (c), we usually call $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F})$ or $\check{\mathcal{C}}(\mathcal{U}, \mathcal{F}^\bullet)$ a Čech resolution of $\mathcal{F}$ or $\mathcal{F}^\bullet$ respectively.

Proof. (a-b) Note that the first claim in (b) holds by definition of j_*. Thus the second claim of (b) holds for a single sheaf, and so it's clear for a complex of sheaves. For (a), the second claim follows from the first, and the first follows from (b) applied to every open and the fact that the regular Čech complex is a complex.

(c) The natural map $\mathcal{F} \to \prod_i j_{i,*}\mathcal{F}|_{U_i}$ is the one induced by the restriction maps. We need to show that

$$0 \to \mathcal{F} \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

is an exact sequence of $\mathcal{O}_X$-modules. It suffices to show this after restricting to an open cover, namely $\mathcal{U}$, so by the second part of (b) we may assume that $X \in \mathcal{U}$. Then, by the first part of (b), evaluation on every open V produces an exact sequence, because the augmented Čech complex is exact when the cover contains the whole space. This proves the first part of (c).

For the second part, the natural map above extends termwise to a map $\mathcal{F}^\bullet \to \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}^\bullet) \hookrightarrow \operatorname{Tot} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$. On the other hand, we have the spectral sequence

$$E_1^{p,q} = H_v^q H_h^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^q)) \implies H^{p+q}(\operatorname{Tot} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)).$$

The LHS is 0 for $p > 0$, and $H^q(\mathcal{F}^\bullet)$ by the first part, so this coincides with the cohomology of the totalization, and the induced map

$$H^q(\mathcal{F}^\bullet) \to H^q(\operatorname{Tot} \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)) \cong H^q(\mathcal{F}^\bullet)$$

is an isomorphism. □

This gives us a very easy way to formulate the Čech to sheaf cohomology map. Namely, we have the natural transformation $\Gamma \to R\Gamma$, and thus a commutative diagram:

$$\begin{array}{ccc} \Gamma(X, \mathcal{F}) & \longrightarrow & C^\bullet(\mathcal{U}, \mathcal{F}) \\ \downarrow & \swarrow & \downarrow \\ R\Gamma(X, \mathcal{F}) & \xrightarrow{\cong} & R\Gamma(X, \check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})) \end{array}$$

To check that this coincides with the comparison map we defined earlier, we have to show that

$$\begin{array}{ccccc} \check{C}(\mathcal{U}, \mathcal{F}) & = & = & = & \Gamma(\check{\mathcal{C}}(\mathcal{U}, \mathcal{F})) \\ \cong\downarrow & & & & \downarrow \\ R\check{H}^0(\mathcal{F}) & \longrightarrow & R\check{H}^0 \circ Ri_X(\mathcal{F}) & \longrightarrow & R\Gamma(\check{\mathcal{C}}(\mathcal{U}, \mathcal{F})) \end{array}$$

commutes. We first claim that $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ is a complex of $\check{H}^0(\mathcal{U}, -)$-acyclic objects for any presheaf $\mathcal{F}$. Indeed, since Čech cohomology commutes with products, we may show that for $j\colon V \to X$ an inclusion,

$$\check{H}^p(\mathcal{U}, j_{V,*}\mathcal{F}|_V) = 0 \qquad p > 0$$

where V is contained in an element of $\mathcal{U}$; this is clear because $\check{C}(\mathcal{U}, j_{V,*}\mathcal{F}|V) = \check{C}(\mathcal{U} \cap V, \mathcal{F}|_V)$, and the latter complex only has zeroth cohomology since V is an element of $\mathcal{U} \cap V$. To apply this fact in showing that the two comparison maps

agree, we consider the following diagram:

$$\begin{array}{ccccc} \check{H}^0(\mathcal{F}) & \longrightarrow & R\check{H}^0 i_X(\mathcal{F}) & \longrightarrow & R\Gamma(X,\mathcal{F}) \\ \downarrow & & \downarrow & & \downarrow\cong \\ \check{H}^0(\check{\mathcal{C}}(\mathcal{U},\mathcal{F})) & \xrightarrow{\cong} & R\check{H}^0(\check{\mathcal{C}}(\mathcal{U},\mathcal{F})) & \longrightarrow & R\Gamma(X,\check{\mathcal{C}}(\mathcal{U},\mathcal{F})) \end{array}$$

Since $\mathcal{F}$ is a sheaf, the outer composites agree with the natural maps $\Gamma \to R\Gamma$ because of the uniqueness statement in the universal property of derived functors. The bottom left map is an isomorphism because we just showed that $\check{\mathcal{C}}(\mathcal{U},\mathcal{F})$ was a bounded below complex of $R\check{H}^0(\mathcal{U},-)$. So, what we have to show is that the identification of $R\check{H}^0 i_X(\mathcal{F})$ with $\check{C}(\mathcal{U},\mathcal{F})$ under the map to $R\check{H}^0(\check{\mathcal{C}}(\mathcal{U},\mathcal{F}))$ agrees under the displayed isomorphism with $\check{H}^0(\check{\mathcal{C}}(\mathcal{U},\mathcal{F})) = \check{C}(\mathcal{U},\mathcal{F})$. This is evident from the bifunctoriality of RHom and the projective resolution of $\mathcal{O}_{\mathcal{U}}^{\mathrm{pre}}$.

The Čech resolution gives a more natural way to describe the Čech-to-derived spectral sequence, and we will state a slightly more general version of it.

Proposition 2.1.17. *Let X be a ringed space, $\mathcal{F}^\bullet \in \mathrm{Ch}^+(\mathcal{O}_X)$, and $\mathcal{U}$ an open cover of X.*

(a) There is a natural bounded spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{\mathbb{H}}(\mathcal{F}^\bullet)) \implies \mathbb{H}^{p+q}(X,\mathcal{F}^\bullet)$$

converging to a bounded filtration on the hypercohomology.

(b) Suppose $f\colon X \to Y$ is a morphism, and that $\mathcal{U}$ is finite. Denote the composite $U_{i_0\cdots i_p} \hookrightarrow X \to Y$ by $f_{i_0\cdots i_p}$. There is a bounded spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0,\ldots,i_p} H^q(Rf_{i_0\cdots i_p,*}\mathcal{F}^\bullet|_{U_{i_0\cdots i_p}}) \implies H^{p+q}Rf_*(\mathcal{F}^\bullet)$$

converging to a bounded filtration on the derived pushforwards. Here the E_1 differential is the Čech differential on the higher pushforwards.

Proof. (a) This is just the spectral sequence associated to a bicomplex, where we take vertical cohomology first. Namely, choose an injective resolution $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$, and consider $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet)$. Since restriction to an open set is exact, the right adjoint j_* preserves injectives. Since arbitrary products of injectives are injective, it follows that $\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{I}^\bullet)$ is a bicomplex of injective objects, and hence so is the totalization.

By Lemma 2.1.16, this means that we have isomorphisms

$$R\Gamma(X,\mathcal{F}^\bullet) \xrightarrow{\sim} \Gamma(X,\mathcal{I}^\bullet) \xrightarrow{\sim} \Gamma(X, \operatorname{Tot}\check{\mathcal{C}}(\mathcal{U},\mathcal{I}^\bullet)) = \operatorname{Tot}\check{C}(\mathcal{U},\mathcal{I}^\bullet)$$

and likewise on open subsets. Computing vertical cohomology for $\check{C}^\bullet(\mathcal{U},\mathcal{I}^\bullet)$ first, we obtain

$$\begin{aligned} H_v^q(\check{C}^p(\mathcal{U},\mathcal{I}^\bullet)) &= \prod_{i_0,\cdots,i_p} H^q(\mathcal{I}^\bullet(U_{i_0\cdots i_p})) \\ &= \prod_{i_0,\cdots,i_p} \mathbb{H}^q(U_{i_0\cdots i_P},\mathcal{F}^\bullet), \end{aligned}$$

and so the corresponding spectral sequence will give

$$\check{H}^p(\mathcal{U}, \underline{\mathbb{H}}^q(\mathcal{F}^\bullet)) \implies H^{p+q} \operatorname{Tot} \check{C}^\bullet(\mathcal{U}, \mathcal{I}^\bullet),$$

but we showed above that the RHS is $R^{p+q}\Gamma(X, \mathcal{F}^\bullet)$ as desired.

(b) We argue the same way as before, except that while products of R-modules are exact, products of $\mathcal{O}_X$-modules aren't. Once again consider $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$, which is a bicomplex of injectives, and therefore

$$Rf_*\mathcal{F} \xrightarrow{\sim} f_*\mathcal{I}^\bullet \xrightarrow{\sim} f_* \operatorname{Tot}(\check{\mathcal{C}}(\mathcal{U}, \mathcal{I}^\bullet)) = \operatorname{Tot} f_*\check{\mathcal{C}}(\mathcal{U}, \mathcal{I}^\bullet).$$

This will take care of the RHS of the spectral sequence. For the LHS, note that

$$f_*\check{\mathcal{C}}^p(\mathcal{U}, \mathcal{I}1) = \bigoplus_{i_0,\dots,i_p} f_{i_0,\dots,i_p,*}\mathcal{I}^q|_{U_{i_0\cdots i_p}},$$

so taking vertical cohomology and using the fact that sums commute with cohomology, we obtain the desired spectral sequence. □

To conclude this section, we discuss an often-used variant of the Čech complex, which is more convenient when actually computing with small open covers. This is the ordered Čech complex, denoted $\check{C}^\bullet_{\text{ord}}(\mathcal{U}, \mathcal{F})$, where $\mathcal{U} = \{U_i : i \in I\}$ is an open cover of X and I is an ordered set. Then we define

$$\check{C}^p_{\text{ord}}(\mathcal{U}, \mathcal{F}) \coloneqq \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0\cdots i_p}),$$

and so $\check{C}^p_{\text{ord}}(\mathcal{U}, \mathcal{F}) = 0$ if $|\mathcal{U}| \leq p$. We use the same differential as in the standard Čech complex, which still works because if $(i_0, \dots, i_p)$ is in strictly increasing order, then $(i_0, \dots, \widehat{i_k}, \dots, i_p)$ is also in strictly increasing order. Thus we have a new chain complex, and this admits a natural map $\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet_{\text{ord}}(\mathcal{U}, \mathcal{F})$ which forgets the components that aren't in strictly increasing order. The main point is:

Lemma 2.1.18. *Let X be a ringed space, $\mathcal{U}$ an ordered open cover of X, and $\mathcal{F}$ an object of* $\operatorname{PMod}(\mathcal{O}_X)$*. The natural map*

$$\check{C}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{C}^\bullet_{\text{ord}}(\mathcal{U}, \mathcal{F})$$

is a homotopy equivalence.

Proof. Recall from Prop. 2.1.8 that we defined $\mathcal{O}^{\text{pre}}_{\mathcal{U}}$, constructed a projective resolution $K^\bullet(\mathcal{U}) \to \mathcal{O}^{\text{pre}}_{\mathcal{U}}[0]$, and saw that $\underline{\operatorname{Hom}}(K^\bullet(\mathcal{U}), \mathcal{F}) = \check{C}^\bullet(\mathcal{U}, \mathcal{F})$. To prove this lemma, we can make an almost identical definition

$$K^{-n}_{\text{ord}} = \bigoplus_{i_0 < \cdots < i_n} j^p_{i_0\cdots i_n,!}\mathcal{O}_{U_{i_0\cdots i_n}},$$

and it is just as clear from earlier that $\underline{\operatorname{Hom}}^{\text{pre}}_{\mathcal{O}_X}(K^\bullet_{\text{ord}}(\mathcal{U}), \mathcal{F}) = \check{C}^\bullet_{\text{ord}}(\mathcal{U}, \mathcal{F})$. It is also clear that the canonical projection as in the lemma statement is induced (after applying $\underline{\operatorname{Hom}}(-, \mathcal{F})$) by a canonical inclusion $K^\bullet_{\text{ord}}(\mathcal{U}) \hookrightarrow K^\bullet(\mathcal{U})$. It suffices to show that this map is a homotopy equivalence. Since these are complexes of projectives, we may show it is a quasi-isomorphism.

The key point is that $K^0(\mathcal{U}) = K^0_{\mathrm{ord}}(\mathcal{U})$, and so the corresponding definition for $\mathcal{O}^{\mathrm{pre}}_{\mathcal{U}}$ is the same in the ordered and unordered case. Therefore, we just need to show that the augmented complex

$$\cdots \to K^{-2}_{\mathrm{ord}}(\mathcal{U}) \to K^{-1}_{\mathrm{ord}}(\mathcal{U}) \to K^0_{\mathrm{ord}}(\mathcal{U}) \to \mathcal{O}^{\mathrm{pre}}_{\mathcal{U}} \to 0$$

is exact. We now basically use the same proof as in Prop. 2.1.8, whereby we fix an open $W \subset X$, and consider the set I' of indices for which $W \subset U_i$. Over W, the augmented complex becomes

$$\cdots \to \bigoplus_{i_0 < i_1 \in I'} \mathcal{O}_X(W) \to \bigoplus_{i_0 \in I'} \mathcal{O}_X(W) \to \mathcal{O}_X(W) \to 0,$$

(assuming $I' \neq \varnothing$). We can use almost the same cone contraction as earlier, except that placing a fixed index at the beginning might not retain the ordering. Thus, choose $\ell \in I'$, and define a homotopy by

$$h(s)_{j_0 \cdots j_{p+1}} = \begin{cases} (-1)^r s_{j_0 \cdots \widehat{j_r} \cdots j_{p+1}} & \exists j_r = \ell \\ 0 & \text{otherwise} \end{cases}.$$

Now, just as in the earlier proof, $hd + dh = 0$ follows from breaking the sum into terms including components with index ℓ and excluding them, and those including ℓ cancel out. □

This gives another proof that if $\mathcal{U}$ is a cover with n elements, then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for $p \geq n$. Note also that since the ordered Čech cohomology is identified with the unordered one, we also have properties such as a cover being ordered Čech-acyclic if one of the opens is the whole space. Using this, we can make the construction sheafy:

Remark 2.1.19. We may make a corresponding definition $\check{\mathcal{C}}^\bullet_{\mathrm{ord}}(\mathcal{U}, \mathcal{F})$ for the sheaf Čech complex by only including terms with strictly increasing indices. Then the same argument we used earlier shows that the augmented complex $\mathcal{F}[0] \to \check{\mathcal{C}}^\bullet_{\mathrm{ord}}(\mathcal{U}, \mathcal{F})$ is exact, provided $\mathcal{F}$ is a sheaf. Moreover, the corresponding projection map $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F}) \to \check{\mathcal{C}}^\bullet_{\mathrm{ord}}(\mathcal{U}, \mathcal{F})$ is therefore a quasi-isomorphism.

2.2. The derived category $D(\mathcal{O}_X)$

2.2.1. The main derived functors. In the previous sections we focused on $R\Gamma$ and Čech cohomology. Now we will describe the other derived functors, such as

$$Rf_* \qquad Lf^* \qquad (-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \qquad \mathbf{R}\mathcal{H}om \qquad \mathcal{E}xt$$

to name a few. We will discuss their relation to $R\Gamma$, and some basic compatibilities that they have with each other. Often, we can say a lot more if we consider certain objects in $D^+(\mathcal{O}_X)$ that satisfy more hypotheses, such as 'pseudo-coherence' or 'perfectness', which we will study more in the next section. Examples are the consideration of duals, projection formulae, and so on.

The most important derived functor is Rf_* for $f\colon X \to Y$ a morphism of ringed spaces. We showed in Remark 2.1.6 that if $f\colon X \to Y$ is such a morphism, then there is a commutative diagram

$$\begin{array}{ccc} D^+(\mathcal{O}_X) & \xrightarrow{Rf_*} & D^+(\mathcal{O}_Y) \\ {\scriptstyle R\Gamma}\downarrow & & \downarrow{\scriptstyle R\Gamma} \\ D^+(\mathcal{O}_X(X)) & \longrightarrow & D^+(\mathcal{O}_Y(Y)) \end{array}$$

where the bottom map is the restriction of scalars functor induced by $\mathcal{O}_X(X) \to \mathcal{O}_Y(Y)$. Thus, the natural map $f_* \to Rf_*$ induces a comparison map

$$R\Gamma(Y, f_*\mathcal{F}^\bullet) \to R\Gamma(Y, Rf_*\mathcal{F}^\bullet) = R\Gamma(X, \mathcal{F}^\bullet)$$

which is an isomorphism, e.g., if $\mathcal{F}^\bullet$ is a bounded above complex of f_*-acyclics. Here is a much more general statement:

THEOREM 2.2.1. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms of ringed spaces.*

(a) There is a canonical identification

$$Rg_* \circ Rf_* = R(g \circ f)_*$$

of pushforwards $D^+(\mathcal{O}_X) \to D^+(\mathcal{O}_Z)$.

(b) For $A \in D^+(\mathcal{O}_X)$, there is a bounded spectral sequence

$$E_2^{p,q} = H^p(Rg_* H^q(Rf_* A)) \implies H^{p+q}(R(g \circ f)_* A)$$

converging to a finite filtration on the RHS. In particular, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then we have a spectral sequence

$$R^p g_*(R^q f_* \mathcal{F}) \implies R^{p+q}(gf)_* \mathcal{F}.$$

This is called the Leray spectral sequence.

PROOF. This is a composition of derived functors and the corresponding Grothendieck spectral sequence. We have to check that f_* sends e.g. injectives to g_*-acyclic sheaves. Indeed, injectives are flasque, g_* preserves flasqueness, and flasque sheaves are g_* acyclic. □

Sheaf cohomology is a special case of derived pushforward. If X is a ringed space, the unique map $p\colon X \to *$ extends to a map of ringed spaces by viewing $(*, \mathcal{O}_X(X))$ as a ringed space. Then $\mathrm{Mod}(\mathcal{O}_*) = \mathrm{Mod}(\mathcal{O}_X(X))$, and the functor p_* coincides with Γ.

Other useful results are the 'hypercohomology spectral sequences', for which we apply Thm. 1.3.24. If $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules, this gives us bounded spectral sequences

$$E_1^{p,q} = R^q f_* \mathcal{F}^p \implies H^{p+q} Rf_*(\mathcal{F}^\bullet)$$

and, for $R\Gamma$,

$$E_1^{p,q} = H^q(X, \mathcal{F}^p) \implies \mathbb{H}^{p+q}(X, \mathcal{F}^\bullet).$$

This gives a way to convert results about cohomology of a single sheaf to results about arbitrary objects of the derived category. We also have the corresponding secondary sequences on cohomology:

$$E_2^{p,q} = R^p f_*(H^q A) \implies H^{p+q} Rf_*(A)$$

and

$$E_2^{p,q} = H^p(X, H^q(A)) \implies \mathbb{H}^{p+q}(X, A)$$

for $A \in D^+(\mathcal{O}_X)$. In applications to algebraic geometry, we will often want to argue about complexes which have quasicoherent or coherent cohomology, and this spectral sequence lets us translate sheaf level statements about such things to bounded below statements.

The following result, called the *Mayer-Vietoris sequence*, is a good demonstration of how to formulate results in the derived category:

Proposition 2.2.2. *Let $f\colon X \to Y$ be a morphism of ringed spaces, and $A \in D^+(\mathcal{O}_X)$. Let $U, V \subset X$ be open subsets, and denote by f_U, f_V, and $f_{U\cap V}$ the composites of f with the inclusions of U, V, and $U \cap V$ into X respectively. Then there is a distinguished triangle*

$$Rf_* A \xrightarrow{(1\ 1)^T} Rf_{U,*} A|_U \oplus Rf_{V,*} A|_V \xrightarrow{(1\ -1)} Rf_{U\cap V,*} A|_{U\cap V} \to Rf_* A[1]$$

in $D^+(\mathcal{O}_Y)$ which is natural in A.

Proof. Choose a quasi-isomorphism $A \xrightarrow{\sim} \mathcal{I}^\bullet$ with $\mathcal{I}^\bullet \in \mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$, then the (augmented) ordered Čech complex for $\mathcal{I}^\bullet$ with respect to U, V, is the exact sequence of complexes:

$$0 \to \mathcal{I}^\bullet \xrightarrow{(1\ 1)^T} j_{U,*}\mathcal{I}^\bullet|_U \oplus j_{V,*}\mathcal{I}^\bullet|_V \xrightarrow{(1\ -1)} j_{U\cap V,*}\mathcal{I}^\bullet_{U\cap V} \to 0$$

This gives an exact triangle

$$A \to Rj_{U,*} A|_U \oplus Rj_{V,*} A|_V \to Rj_{U\cap V,*} A \to A[1]$$

in $D^+(\mathcal{O}_X)$. Now apply Rf_* to this. □

The following observation justifies the name: take f to be the identity, apply $R\Gamma$, and then the long exact sequence in cohomology is

$$\cdots \to \mathbb{H}^i(X, A) \to \mathbb{H}^i(U, A) \oplus \mathbb{H}^i(V, A) \to \mathbb{H}^i(U \cap V, A) \to \mathbb{H}^{i+1}(X, A) \to \cdots$$

mirroring the Mayer-Vietoris sequence in topology.

We next discuss the locality of derived pushforward, and a fundamental computation which expresses pushforward in terms of cohomology. First, let $f\colon X \to Y$ be a map and $U \subset Y$ an open subset. Let $f|_{f^{-1}U}$ denote the restricted map $f^{-1}(U) \to U$. Then by definition we have an equality of functors

$$(-)|_U \circ f_* = (f|_{f^{-1}U})_* \circ (-)|_{f^{-1}U}.$$

Now restriction is exact, and we showed that it preserves injectives. It follows that we have a canonical identification

$$(Rf_*A)|_U = R(f|_{f^{-1}U})_*(A|_{f^{-1}U}),$$

and therefore

$$(R^p f_*\mathcal{F})|_U = R^p(f|_{f^{-1}U})_*(\mathcal{F}|_{f^{-1}U}) \qquad p \geq 0$$

for $\mathcal{F}$ an $\mathcal{O}_X$-module. This means that whenever we are checking a property which is target local, we may always check on an open cover. For example, if $U \subset Y$ we see that

$$R\Gamma(U, Rf_*A) = R\Gamma(U, R(f|_{f^{-1}U})_*A) = R\Gamma(f^{-1}(U), A),$$

viewed in the derived category of the ring $\mathcal{O}_X(f^{-1}(U))$ by restriction of scalars. Next let us consider a seemingly stupid composition of functors:

$$\mathrm{Mod}(\mathcal{O}_X) \xrightarrow{f_*} \mathrm{Mod}(\mathcal{O}_Y) \xrightarrow{i_Y} \mathrm{PMod}(\mathcal{O}_Y) \xrightarrow{\mathbf{a}} \mathrm{Mod}(\mathcal{O}_Y)$$

Sheafification $\mathbf{a}$ is exact, and the composite of the last two functors is the identity. Thus the whole composite equals f_*. Since everything is $\mathbf{a}$-acyclic, this implies that

$$Rf_* = \mathbf{a} \circ R(i_Y \circ f_*) = \mathbf{a} \circ Ri_Y \circ Rf_*,$$

where in the second equality we have used the fact that a flasque sheaf is i_Y-acyclic. That is the case because $R^p i_Y(\mathcal{G}) = \underline{H}^p(\mathcal{G}) = 0$ as $H^p(U, \mathcal{G}) = 0$ if $\mathcal{G}$ is flasque. Therefore, for $A \in D^+(\mathcal{O}_X)$, we have

$$\begin{aligned} H^p(Rf_*A) &= \mathbf{a}H^p(Ri_Y(Rf_*A)) = \mathbf{a}\underline{\mathbb{H}}^p(Rf_*A) \\ &= \mathbf{a}\big(U \mapsto \mathbb{H}^p(U, Rf_*A)\big) \\ &= \mathbf{a}\big(U \mapsto \mathbb{H}^p(f^{-1}(U), A)\big). \end{aligned}$$

Once again, we are implicitly viewing $U \mapsto \underline{\mathbb{H}}^p(f^{-1}(U), A)$ as a presheaf of $\mathcal{O}_Y$-modules via restriction of scalars.

Example 2.2.3. Take $A = \mathcal{F}[0]$ and $f = 1_X$. Then this says that $\mathcal{F} = \mathbf{a}(U \mapsto H^0(U, A))$, which we already know, and

$$0 = \mathbf{a}\big(U \mapsto H^p(U, \mathcal{F})\big) = \mathbf{a}\underline{H}^p(\mathcal{F})$$

for $p > 0$, which is new. In other words, each cohomology class in $H^p(X, \mathcal{F})$ restricts to zero in a nbhd of every point. This is a reincarnation of the fact that taking sections is locally exact, but globally not right exact.

(Perhaps reorganize to add something more apt here.)

We now discuss a useful criterion for derived pushforward and cohomology commuting with filtered colimits. First of all, exactness of a sequence of sheaves is checked on stalks, which is a certain colimit, and filtered colimits commute with colimits; therefore taking filtered colimits of $\mathcal{O}_X$-modules is exact. It follows that on each of the derived categories we obtain a functor

$$\varinjlim_I^D \colon D^*(\mathrm{Mod}(\mathcal{O}_X)^I) \to D^*(\mathcal{O}_X),$$

where we use the superscript D to distinguish this from some kind of colimit inside $D^*(\mathcal{O}_X)$. If we restrict to the bounded above (or below) case, we know that this can be computed by any preadapted class in $\mathrm{K}^+(\mathrm{Mod}(\mathcal{O}_X)^I)$, where I. We usually take this class to be the pointwise injectives. Every object of $\mathrm{Mod}(\mathcal{O}_X)^I$ admits a monomorphism into a pointwise injective object because we have functorial injective embeddings in $\mathrm{Mod}(\mathcal{O}_X)$.

This preadapted class has good properties. Even if its elements are not themselves injective objects of the functor category, we note that exactness in a functor category is checked pointwise. So, given a left exact functor $F\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathcal{A}$, the induced functor $F\colon \mathrm{Mod}(\mathcal{O}_X)^I \to \mathcal{A}^I$ is also left exact, and given an acyclic complex of pointwise injective objects A, $F(A)$ is acyclic, because $F(A)(i) = F(A(i))$ is acyclic as $A(i)$ is an acyclic complex of injectives. This means that we can define the right derived functor $RF\colon D^+(\mathrm{Mod}(\mathcal{O}_X)) \to D^+(\mathcal{A}^I)$.

Remark 2.2.4. More generally, here is what we have just said: If $\mathcal{A}$ is an abelian category and $\mathcal{R}$ is a preadapted class with *functorial embeddings*, then for any small category C, the class $\mathcal{R}^C$ forms a preadapted class in $\mathcal{A}^C$. Next, if $\mathcal{R}$ is adapted to a left exact functor $F\colon \mathcal{A} \to \mathcal{B}$, then $\mathcal{R}^C$ is adapted to $F^C\colon \mathcal{A}^C \to \mathcal{B}^C$ so that we can define the derived functor $RF^C\colon D^+(\mathcal{A}^C) \to D^+(\mathcal{B}^C)$. Lastly, suppose $A^\bullet \in \mathrm{Ch}^+(\mathcal{A}^C)$, and choose a resolution $A^\bullet \to X^\bullet$ with $X^\bullet(c) \in \mathrm{K}^+(\mathcal{R})$ for each $c \in C$. Then $RF^C(A) = F(X)$ pointwise and termwise, i.e. $RF^C(A)(c) = F(X(c)^\bullet)$. Since the evaluation functors are exact, $A^\bullet(c) \to X^\bullet(c)$ is also a quasi-isomorphism, and $RF(A^\bullet(c)) = F(X(c)^\bullet)$. Therefore, $RF(A)$ is represented by the object $c \mapsto RF(A(c))$, where if $c \to c'$ is a map in C, then the corresponding map $RF(A(c)) \to RF(A(c'))$ is given by functoriality of RF and the fact that $A(c) \to A(c')$ is a chain map. Another way to say this last part is that the derived functor RF^C factors through

$$D^+(\mathcal{A}^C) \to D^+(\mathcal{A})^C \dashrightarrow D^+(\mathcal{B}^I)$$

where the first map is induced by the exact evaluation functors.

We can combine these two observations above to discuss how filtered colimits interact with sheaf cohomology.

Proposition 2.2.5. *Let Y have a basis of qcqs opens, X have a basis of quasicompact opens, and $f\colon X \to Y$ be a qcqs morphism of ringed spaces. For any filtered category I, the diagram*

$$\begin{array}{ccc} D^+(\mathrm{Mod}(\mathcal{O}_X)^I) & \xrightarrow{\varinjlim_I^D} & D^+(\mathcal{O}_X) \\ {\scriptstyle Rf_*^I}\downarrow & & \downarrow{\scriptstyle Rf_*} \\ D^+(\mathrm{Mod}(\mathcal{O}_Y)^I) & \xrightarrow{\varinjlim_I^D} & D^+(\mathcal{O}_Y) \end{array}$$

commutes up to canonical isomorphism. In particular, if $\mathcal{F}(-)$ is a filtered diagram of $\mathcal{O}_X$-modules, then:

(a) With the assumptions on f as above, the natural map

$$\varinjlim_{i\in I} R^p f_* \mathcal{F}(i) \to R^p f_*\big(\varinjlim_{i\in I} \mathcal{F}(i)\big)$$

is an isomorphism for all $p \geq 0$.

(b) If X is qcqs and has a basis of quasicompact opens, then the natural map

$$\varinjlim_{i\in I} H^p(X, \mathcal{F}(i)) \to H^p\big(X, \varinjlim_{i\in I} \mathcal{F}(i)\big)$$

is an isomorphism for all $p \geq 0$.

Therefore, in the circumstances described above, the functors $R^p f_*$ and $H^p(X, -)$ preserve arbitrary coproducts for all $p \geq 0$.

Proof. Consequence (a) follows directly from the diagram and the fact that Rf_*^I is computed pointwise, which was discussed above. If we take $Y = (*, \mathcal{O}_X(X))$, then f qcqs means that X is qcqs, and we get (b). The fact about arbitrary coproducts holds because a coproduct is a filtered colimit of biproducts (or empty, in which case we already know the functors preserve 0).

Now the main result. First let us show that under the assumptions of the proposition, $\varinjlim_I f_*(\mathcal{F}(i)) \xrightarrow{\sim} f_*(\varinjlim_i \mathcal{F}(i))$. Since Y has a basis of qcqs opens and f is qcqs, we reduce immediately to showing that $\varinjlim_i H^0(X, \mathcal{F}(i)) \xrightarrow{\sim} H^0(X, \varinjlim_i \mathcal{F}(i))$ is an isomorphism whenever X is qcqs and has a basis of qc opens. So, consider each $\mathcal{F}(i)$ as a sheaf on the basis of all qc opens, which is closed under finite intersection by assumption.

We claim that the presheaf filtered colimit $U \mapsto \varinjlim_i \mathcal{F}(i)(U)$ (for U quasicompact) is already a sheaf on the basis, which completes the proof since X is quasicompact, and the sheaf colimit is the sheafification of the presheaf colimit. To demonstrate the sheaf condition for an open cover $\{U_j\}$ of some qc open U, it suffices to show the sheaf condition on any subcover, and hence we may assume $\{U_j\}$ is finite. Thus the sheaf condition for each $\mathcal{F}(i)$ is the exact sequence

$$0 \to \mathcal{F}(i)(U) \to \bigoplus_j \mathcal{F}(i)(U_j) \to \bigoplus_{j,k} \mathcal{F}(i)(U_j \cap U_k),$$

and taking filtered colimits gives the sheaf condition for $(\varinjlim_i \mathcal{F}(i))(U)$.

This shows that f_* commutes with filtered colimits, and so the underived version of the diagram in the proposition commutes. To conclude, we apply composition of derived functors. By the discussion above, it suffices to show that if $\mathcal{I}(-)$ is a filtered diagram of injective objects, then $\varinjlim_i \mathcal{I}(i)$ is f_*-acyclic. By the description of $R^p f_*$ as a sheafification, we may show that $H^p(f^{-1}(U), \varinjlim_i \mathcal{I}(i)) = 0$ for all $p > 0$ and $U \subset Y$ qcqs. Then $f^{-1}(U)$ is qcqs, so we reduce to showing that $H^p(X, \varinjlim_i \mathcal{I}(i)) = 0$ for $p > 0$ whenever X is qcqs and has a basis of quasicompact opens, and $\mathcal{I}(-)$ is a filtered diagram of injectives.

We show this using Čech cohomology. Let $\mathcal{U}$ be a finite open cover of a quasicompact open $U \subset X$ by qc opens. Since $U_{i_0 \cdots i_p}$ is still qcqs and has a basis of qc opens, we have

$$\varinjlim_{i \in I} H^0(U_{i_0 \cdots i_p}, \mathcal{F}(i)) \xrightarrow{\sim} H^0(U_{i_0 \cdots i_p}, \varinjlim_{i \in I} \mathcal{F}(i)),$$

so $\varinjlim_i \check{C}^p(\mathcal{U}, \mathcal{I}(i)|_U) \xrightarrow{\sim} \check{C}^p(\mathcal{U}, \varinjlim_i \mathcal{I}(i)|_U)$ by finiteness of $\mathcal{U}$. Since filtered colimits are exact, this statement holds on the level of Čech cohomology. Since each $\mathcal{I}(i)$ is injective, the cohomology on the LHS vanishes, which proves that

$$\check{H}^p(\mathcal{U}, \varinjlim_{i \in I} \mathcal{I}(i)|_U) = 0 \qquad p > 0$$

for every finite open cover $\mathcal{U}$ of U by qc opens. Since such coverings form a cofinal system, we conclude by Lemma 2.1.15 that higher cohomology of $\varinjlim_i \mathcal{I}(i)$ vanishes for all qc opens in X. □

Our next topic is $\mathbf{R}\mathcal{H}om$, which we will define in the exact same way as RHom. We will see, however, that to prove the relevant properties of $\mathbf{R}\mathcal{H}om$, many specific properties of sheaves are needed; namely, what follows is *not* just an obvious generalization of RHom.

Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ be (possibly unbounded) complexes of $\mathcal{O}_X$-modules, and define

$$\underline{\mathcal{H}om}^n_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \coloneqq \prod_{p \in \mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^p, \mathcal{G}^{n+p}),$$

with differential given by the same formula $df - (-1)^n fd$, where this formula makes sense on every open and hence as sheaves. So,

$$H^0(U, \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)) = \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{F}^\bullet|_U, \mathcal{G}^\bullet|_U)$$

as complexes of $\mathcal{O}_X(U)$-modules since products commute with sections, from which we immediately deduce e.g. that the definition above actually gives a complex. More generally, most facts that we prove about the sheaf hom complex will pass through the formula above. For example, we get for free that

$$\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet[k]) = \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet[-k], \mathcal{G}^\bullet) = \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)[k]$$

because it is true on every open.

Lemma 2.2.6. *Let X be a ringed space. In what follows,*

$$\mathcal{F}^\bullet, \mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet, \mathcal{G}^\bullet, \mathcal{G}_1^\bullet, \mathcal{G}_2^\bullet$$

are all possibly unbounded complexes of $\mathcal{O}_X$-modules.

(a) $\underline{\mathcal{H}om}$ naturally defines a functor

$$\mathrm{Ch}(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{Ch}(\mathcal{O}_X) \to \mathrm{Ch}(\mathcal{O}_X).$$

This restricts to $\mathrm{Ch}^-(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{Ch}^+(\mathcal{O}_X) \to \mathrm{Ch}^+(\mathcal{O}_X)$. Moreover, these are homotopy invariant, meaning that both restrict to functors

$$\mathrm{K}(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{K}(\mathcal{O}_X) \to \mathrm{K}(\mathcal{O}_X) \qquad \mathrm{K}^-(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{K}^+(\mathcal{O}_X) \to \mathrm{K}^+(\mathcal{O}_X)$$

on the homotopy categories.

(b) For any map $\phi\colon \mathcal{G}_1^\bullet \to \mathcal{G}_2^\bullet$, there is a canonical identification

$$\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathrm{Cone}(\phi)) = \mathrm{Cone}\left(\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \varphi)\right).$$

For any map $\phi\colon \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet$, there is a canonical identification

$$\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathrm{Cone}(\phi), \mathcal{G}^\bullet) = \mathrm{Cone}\left(\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\phi, \mathcal{G}^\bullet)\right)$$

as well.

(c) If $\mathcal{I}^\bullet \in \mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$ and $\phi\colon \mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet$ is a quasi-isomorphism, then

$$\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}_1^\bullet, \mathcal{I}^\bullet) \to \underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}_2^\bullet, \mathcal{I}^\bullet)$$

is a quasi-isomorphism.

(d) If $\mathcal{I}^\bullet \in \mathrm{K}(\mathcal{I}_{\mathcal{O}_X})$, then $\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ is a complex of flasque $\mathcal{O}_X$-modules. If in addition $\mathcal{F}^\bullet$ is a complex of flat $\mathcal{O}_X$-modules, then $\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ is a complex of injectives.

Proof. (a) The first part is from functoriality of sheaf hom and of products. Note that if the first argument is zero after degree q and the second is zero below degree r, then whenever $n < -q + r$, either $p > q$ or $n + p \leq n + q < r$, so $\mathcal{H}om(\mathcal{F}^p, \mathcal{G}^{n+p}) = 0$ either way. For homotopy invariance, note e.g. that a homotopy $\mathcal{G}_1^\bullet \to \mathcal{G}_2^{\bullet-1}$ induces a map $\mathcal{H}om(\mathcal{F}^p, \mathcal{G}_1^{n+p}) \to \mathcal{H}om(\mathcal{F}^p, \mathcal{G}_2^{n+p-1})$, and hence a map

$$\underline{\mathcal{H}om}^n_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}_1^\bullet) \to \underline{\mathcal{H}om}^{n-1}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}_2^\bullet) \qquad n \in \mathbb{Z}$$

which is a homotopy by functoriality. Likewise for the first argument.

(b) This is true after taking sections over every open set, and the identification is compatible with restriction. Here we are using the fact that $\mathcal{H}om$ preserves finite biproducts in either variable.

(c) It suffices by (b) to prove that if $\mathcal{F}^\bullet$ is acyclic, then $\underline{\mathcal{H}om}^\bullet_{\mathcal{O}_X}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)$ is acyclic. But to show exactness we may show exactness on sections over every open set, and indeed $\underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{F}^\bullet|_U, \mathcal{I}^\bullet|_U)$ is acyclic for all U because $\mathcal{I}^\bullet|_U$ is a bounded above complex of injectives.

(d) Note that a product of flasque sheaves is flasque, because products commute with sections, and products are exact for modules over a ring. Thus we may show that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I})$ is flasque for $\mathcal{I}$ an injective sheaf. Indeed, for an open inclusion $j\colon U \to X$, applying $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$ to the injective morphism $j_!\mathcal{F}|_U \hookrightarrow \mathcal{F}$ gives the exact sequence

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \to \mathrm{Hom}_{\mathcal{O}_X}(j_!\mathcal{F}|_U, \mathcal{I}) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{I}|_U) \to 0$$

by adjunction. This shows flasqueness.

For the second statement, we once again reduce to showing that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})$ is injective if $\mathcal{F}$ is flat and $\mathcal{I}$ is injective, because products preserve injectives. But by the tensor-hom adjunction,

$$\mathrm{Hom}_{\mathcal{O}_X}(-,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})) \cong \mathrm{Hom}_{\mathcal{O}_X}((-)\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{I}),$$

so the LHS is a composition of two exact functors. □

By part (a) of the lemma, the sheaf hom complex gives a functor $\mathrm{K}(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X}) \to \mathrm{K}(\mathcal{O}_X)$, which by (c) descends to a functor $D(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{K}^+(\mathcal{O}_X) \to D(\mathcal{O}_X)$. Via the equivalence $\mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X}) \to D^+(\mathcal{O}_X)$, we get a functor

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X} \colon D(\mathcal{O}_X)^{\mathrm{op}} \times D^+(\mathcal{O}_X) \to D(\mathcal{O}_X).$$

By part (c) of the lemma, the functors

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,-) \qquad \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(-,B)$$

are both triangulated for any $A \in D(\mathcal{O}_X)$ and $B \in D^+(\mathcal{O}_X)$. Note that by (a), if $A \in D^-(\mathcal{O}_X)$, then this lands in $D^+(\mathcal{O}_X)$. Lastly, the choice of an inverse equivalence $D^+(\mathcal{O}_X) \to \mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$ comes with a natural transformation which induces a natural morphism

$$\underline{\mathcal{H}om} \to \mathbf{R}\mathcal{H}om$$

of bifunctors, which on the bounded below complexes of injectives in the second variable is an isomorphism.

Proposition 2.2.7. *Let X be a ringed space.*

(a) If $U \subset X$, then

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B)|_U = \mathbf{R}\mathcal{H}om_{\mathcal{O}_U}(A|_U, B|_U)$$

for any $A \in D(\mathcal{O}_X)$ and $B \in D^+(\mathcal{O}_X)$.

(b) For $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$, there is a canonical identification

$$R\,\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},-) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}[0],-)$$

of the right derived functor with sheaf RHom.

(c) There is a canonical identification

$$R\Gamma\Big(X, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B)\Big) = \mathrm{RHom}_{\mathcal{O}_X}(A,B)$$

in $D^+(\mathcal{O}_X(X))$ for $A \in D^-(\mathcal{O}_X)$ and $B \in D^+(\mathcal{O}_X)$.

Proof. (a) Replace B by a bounded below complex of injectives, then we conclude by exactness of restriction and the fact that the restriction of a bounded below complex of injectives still has the same property.

(b) Since $\mathbf{R}\mathcal{H}om(\mathcal{F}[0],-)$ is triangulated, and due to the natural map from $\underline{\mathcal{H}om}(\mathcal{F}[0],-) = \mathcal{H}om(\mathcal{F},-)$, there is a natural map in the direction specified by the universal property of derived functors, making the obvious diagram commute. This is an isomorphism on $\mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$, and hence on $D^+(\mathcal{O}_X)$.

(c) Note that $\Gamma(X, \underline{\mathcal{H}om}(A,B)) = \underline{\mathrm{Hom}}(A,B)$, so we have some natural maps

$$\mathrm{RHom}(A,-) \leftarrow \underline{\mathrm{Hom}}(A,-) \to R\Gamma(\underline{\mathcal{H}om}(A,-)) \to R\Gamma(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,-))$$

in the derived category of $\mathcal{O}_X(X)$. Testing on $\mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$, the left- and right-hand maps are isomorphisms, and the remaining map is an isomorphism as well because $\underline{\mathcal{H}om}(A, \mathcal{I}^\bullet)$ is flasque, by Lemma 2.2.6. □

In light of (b) above, we write

$$\mathcal{E}xt^p_{\mathcal{O}_X}(-,-) \coloneqq H^p(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}((-)[0],(-)[0]))$$

for the classical right derived functors of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},-)$, so these form bifunctors

$$\mathrm{Mod}(\mathcal{O}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X).$$

Then by (c), or simply the fact that $\mathcal{H}om(\mathcal{F},\mathcal{I})$ is flasque if $\mathcal{I}$ is injective, we have the following spectral sequence:

$$H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})) \implies \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

This is called the *local-to-global* spectral sequence. More generally, instead of applying the Grothendieck spectral sequence, we can use the main ingredient in its proof, which gives a spectral sequence

$$H^p\Big(X, H^q(\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B))\Big) \implies \mathrm{Hom}_{D(\mathcal{O}_X)}(A, B[p+q])$$

of $\mathcal{O}_X(X)$-modules, for $A \in D^-(\mathcal{O}_X)$ and $B \in D^+(\mathcal{O}_X)$. The point of introducing $\mathbf{R}\mathcal{H}om$ is that it is in fact usually much easier to compute, and as we will see it behaves a lot more like classical RHom's for modules. For example, we know that $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X,-)$ is the identity functor while $\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{O}_X,-)$ is $R\Gamma$. Thus the spectral sequence above is a generalization of the hypercohomology spectral sequence.

Another reason that $\mathbf{R}\mathcal{H}om$ is more similar to the module case is because we can often compute it using resolutions in the first variable. Let us temporarily call an $\mathcal{O}_X$-module *internally projective* if $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},-)$ is exact. Thus in this case $\mathcal{E}xt^p(\mathcal{F},-) = 0$ for $p > 0$, and conversely if $\mathcal{E}xt^p(\mathcal{F},-) = 0$ for $p > 0$, then an exact sequence $0 \to \mathcal{G}_1 \to \mathcal{G}_2 \to \mathcal{G}_3 \to 0$ induces a long exact sequence

$$0 \to \mathcal{H}om(\mathcal{F},\mathcal{G}_1) \to \mathcal{H}om(\mathcal{F},\mathcal{G}_2) \to \mathcal{H}om(\mathcal{F},\mathcal{G}_3) \to \mathcal{E}xt^1(\mathcal{F},\mathcal{G}_1) = 0$$

by exactness of $\mathbf{R}\mathcal{H}om$. So being internally projective is equivalent to having $\mathcal{E}xt^p(\mathcal{F},-) = 0$ for $p > 0$. For example, a finite locally free sheaf is internally projective.

Example 2.2.8. If $\mathcal{F}$ is an internally projective module, then the local-to-global spectral sequence for Exts immediately gives a natural identification $H^p(X, \mathcal{H}om(\mathcal{F},-)) = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F},-)$. For instance if $\mathcal{F} = \mathcal{E}$ is finite locally free, then $H^p(X, \mathcal{E}^\vee \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{E},\mathcal{G})$.

Example 2.2.9. Just as in the case where the target is modules, $\underline{\mathcal{H}om}(\mathcal{F}^\bullet,\mathcal{G}^\bullet)$ is the product totalization of the bicomplex $C^{p,q} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^{-p},\mathcal{G}^q)$, so there is an associated spectral sequence. However this doesn't automatically converge to what we expect since products aren't exact; so, assume either that $\mathcal{F}^\bullet \in \mathrm{Ch}^b(\mathcal{O}_X)$

and $\mathcal{G}^\bullet$ is arbitrary, or $\mathcal{F}^\bullet \in \mathrm{Ch}^-(\mathcal{O}_X)$ and $\mathcal{G}^\bullet \in \mathrm{Ch}^+(\mathcal{O}_X)$. Both of these ensure that the products taken are finite, so we get spectral sequences

$$H_h^p(H_v^q(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet))) \implies H^{p+q}(\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet))$$

and

$$H_v^p(H_h^q(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet))) \implies H^{p+q}(\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)).$$

So, for example, if $\mathcal{F}^\bullet$ is a bounded above complex of internally projective sheaves, then for $\mathcal{G}^\bullet$ a bounded below acyclic complex, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^i, \mathcal{G}^\bullet)$ is acyclic. Thus the first spectral sequence degenerates to 0 and the total complex $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ is acyclic.

Lemma 2.2.10. *Let $\mathcal{F}^\bullet$ be a bounded above complex of internally projective modules. Then the natural map*

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, -) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, -)$$

is an isomorphism of functors $\mathrm{K}^+(\mathcal{O}_X) \to D^+(\mathcal{O}_X)$.

Proof. Let $\mathcal{G}^\bullet \in \mathrm{K}^+(\mathcal{O}_X)$ and $\mathcal{G}^\bullet \to \mathcal{I}^\bullet$ be a qis with $\mathcal{I}^\bullet \in \mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$. Then the diagram

$$\begin{array}{ccc} \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) & \longrightarrow & \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \\ \downarrow & & \downarrow \cong \\ \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) & \xrightarrow{\cong} & \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \end{array}$$

commutes by naturality in both variables, and we already know that the right and bottom maps are isomorphisms. By Lemma 2.2.6(b) and the fact that $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, -)$ preserves acyclics, the left map is also an isomorphism. □

Here's how this occurs in practice: we often care geometrically about computing $\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, e.g. where $\mathcal{F}$ is some coherent sheaf, and we have a finite resolution of $\mathcal{F}$ by vector bundles. So, we can first compute $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ which is just $\underline{\mathcal{H}om}$ applied to the resolution, and then compute cohomology and use the local-to-global spectral sequence. We will return to internally projective sheaves in the next section.

Now let's discuss Lf^* and the derived tensor product. First, the latter; we know that $\mathrm{Mod}(\mathcal{O}_X)$ is closed symmetric monoidal and has exact arbitrary sums, and enough flat objects, so the entire discussion about $(-) \otimes^{\mathbf{L}} (-)$ already applies. By convention, we denote the corresponding Tor sheaves as $\mathcal{T}or^n_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, instead of $\mathrm{Tor}^n_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{F}, \mathcal{G})$, though they mean the same thing.

Example 2.2.11. Let $A, B \in D^-(\mathcal{O}_X)$, and represent them by complexes $\mathcal{F}^\bullet \to A$, $\mathcal{G}^\bullet = B$ where $\mathcal{F}^\bullet$ is a bounded above complex of flats. Then we defined the complex $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ as the (sum) totalization of the natural bicomplex. Since this is a shifted third quadrant bicomplex, we have the associated spectral sequence

$$H_h^p H_v^q(\mathcal{F}^* \otimes \mathcal{G}^*) \implies H^{p+q}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet).$$

Since each $\mathcal{F}^i$ is flat, we have $H^q_v(\mathcal{F}^* \otimes \mathcal{G}^*) = \mathcal{F}^* \otimes H^q(\mathcal{G}^\bullet)$. Thus the LHS above is $H^p(\mathcal{F}^\bullet \otimes H^q(\mathcal{G}^\bullet))$. Therefore, this gives a bounded spectral sequence

$$H^p(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(B)) \implies H^{p+q}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B),$$

which is called the *hyper-Tor spectral sequence*. In particular, if $A = \mathcal{F}$ is represented by a single sheaf, then

$$\mathcal{T}or^p(\mathcal{F}, H^q(B)) \implies H^{p+q}(\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} B).$$

The general version is useful because it relates the cohomology of $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$, which isn't a derived functor of anything, to the cohomology of something which is a derived functor, since

$$A \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(B) = L\Big((-) \otimes_{\mathcal{O}_X} H^q(B)\Big)(A)$$

as $H^q(B)$ is just an $\mathcal{O}_X$-module.

Example 2.2.12. We can repeat the same trick as above for $\mathbf{R}\mathcal{H}om$, but we have to make sure the first argument is bounded above as explained earlier. So if $A \in D^-(\mathcal{O}_X)$ and $B \in D^+(\mathcal{O}_X)$, then by replacing B with a bounded below complex of injectives we obtain a spectral sequence

$$H^p(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(H^{-q}(A), B)) \implies H^{p+q}(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B)),$$

so in particular

$$\mathcal{E}xt^p_{\mathcal{O}_X}(H^{-q}(A), \mathcal{G}) \implies H^{p+q}\big(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{G})\big).$$

The same thing of course works for regular RHom, so we have

$$H^p(\mathrm{RHom}_{\mathcal{O}_X}(H^{-q}(A), B)) \implies H^{p+q}(\mathrm{RHom}_{\mathcal{O}_X}(A, B))$$

and likewise an Ext version as above.

The nice property of the derived tensor product is that it behaves well with respect to stalks. First of all, since the stalk functor $(-)_x \colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_{X,x})$ is exact, it induces a termwise map on the derived categories $(-)_x \colon D^*(\mathcal{O}_X) \to D^*(\mathcal{O}_{X,x})$. Then, here's a simple lemma about flat sheaves.

Lemma 2.2.13. *Let X be a ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is flat if and only if $\mathcal{F}_x$ is flat over $\mathcal{O}_{X,x}$.*

Proof. We already know the $\Leftarrow$ implication. For the other direction, we just need to know that for any $\mathcal{O}_{X,x}$ module M there is an $\mathcal{O}_X$-module with stalk M at x. So take the skyscraper sheaf corresponding to M at x. □

Remark 2.2.14. In the tensor product case, we want to (as before) assert the existence of a map

$$(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \to (-) \otimes_{\mathcal{O}_X} (-)$$

natural in both variables, of functors $\mathrm{K}^-(\mathcal{O}_X) \times \mathrm{K}^-(\mathcal{O}_X) \to D^-(\mathcal{O}_X)$. This is more delicate than the $\mathbf{R}\mathcal{H}om$ case because we don't have enough projectives (actually, that case was a bit delicate too, so I should go back and fix it); so, if

$\mathcal{R}$ is the class of flats, while we have a map $\mathrm{K}^-(\mathcal{O}_X) \to \mathrm{K}^-(\mathcal{R})[S_{\mathcal{R}}^{-1}]$, we don't automatically have a map $\mathrm{K}^-(\mathcal{O}_X) \to \mathrm{K}^-(\mathcal{R})$ coming from a choice of inverse.

So here's what we can do instead: By construction, for each $\mathcal{F}$, the functor $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)$ as $D^-(\mathcal{O}_X) \to D^-(\mathcal{O}_X)$ is a left derived functor of $\mathcal{F} \otimes (-)$ using an adapted class, so we have a natural map $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \to \mathcal{F} \otimes_{\mathcal{O}_X} (-)$ of functors $\mathrm{K}^-(\mathcal{O}_X) \to D^-(\mathcal{O}_X)$. Since a morphism $\mathcal{F} \to \mathcal{F}'$ gives a natural transformation $\mathcal{F} \otimes (-) \to \mathcal{F}' \otimes (-)$, there is an induced morphism of left derived functors making the diagram

$$\begin{array}{ccc} \mathcal{F} \otimes^{\mathbf{L}} (-) & \longrightarrow & \mathcal{F} \otimes (-) \\ \downarrow & & \downarrow \\ \mathcal{F}' \otimes^{\mathbf{L}} (-) & \longrightarrow & \mathcal{F}' \otimes (-) \end{array}$$

commute. We know that if $\mathcal{G}^\bullet \to B$ is a flat resolution, then the comparison map is induced by $\mathcal{F} \otimes^{\mathbf{L}} B = \mathcal{F} \otimes \mathcal{G}^\bullet \to \mathcal{F} \otimes B$; now if $\mathcal{F}^\bullet$ is any bounded above complex, the differentials induce commutative diagrams like the one above, and by taking direct sums we get a map of totalizations

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} B = \mathrm{Tot}(\mathcal{F}^* \otimes \mathcal{G}^\bullet) \to \mathcal{F}^\bullet \otimes B,$$

This is exactly the desired map of complexes. If B is a bounded above complex of flats, then this is obviously an isomorphism, and if $\mathcal{F}^\bullet$ is a bounded above complex of flats, then tensoring with it preserves quasi-isomorphisms.

Remark 2.2.15. Here is another thing we are using to construct natural maps when comparing derived functors. If $\mathcal{R}$ is a (say left) preadapted class, then we know that the natural map $Q\colon \mathrm{K}^-(\mathcal{R}) \to D^-(\mathcal{A})$ is a localization by the quasi-isomorphisms. This means that restriction along Q induces an equivalence

$$Q^*\colon \mathrm{Fun}(D^-(\mathcal{A}), \mathcal{C}) \to \mathrm{Fun}^S(\mathrm{K}^-(\mathcal{R}), \mathcal{C}),$$

where Fun^S denotes the full subcategory of functors which send quasi-isomorphisms in $\mathrm{K}^-(\mathcal{R})$ to isomorphisms. By currying/product-hom adjunction, this also induces an equivalence

$$\mathrm{Fun}(D^-(\mathcal{A}) \times \mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{S\times 1}(\mathrm{K}^-(\mathcal{R}) \times \mathcal{C}, \mathcal{D}),$$

where the RHS denotes the full subcategory of bifunctors G for which $G(-, C)$ sends quasi-isomorphisms to isomorphisms for every $C \in \mathcal{C}$. Therefore, to provide natural transformations between functors in the LHS, we can precompose with Q and then provide natural transformations there.

The natural map $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \to (-) \otimes_{\mathcal{O}_X} (-)$ gives rise to a zig-zag of maps

$$\Big((-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)\Big)_x \to (-)_x \otimes_{\mathcal{O}_{X,x}} (-)_x \leftarrow (-)_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} (-)_x$$

because regular tensor product commutes with stalks, all landing in $D^-(\mathcal{O}_{X,x})$. Now in the first variable, place a bounded above complex $\mathcal{F}^\bullet$ of flat sheaves. The left hand map becomes an isomorphism because the same was true before taking stalks. The RHS is an isomorphism because $\mathcal{F}^\bullet_x$ is a bounded above complex of flat $\mathcal{O}_{X,x}$-modules. Therefore, as functors $\mathrm{K}^-(\mathcal{R}) \times D^-(\mathcal{O}_X) \to D^-(\mathcal{O}_{X,x})$, the

LHS and RHS are isomorphic, so the same is true on $D^-(\mathcal{O}_X) \times D^-(\mathcal{O}_X)$ by the remark above. We have shown:

Lemma 2.2.16. *Let $A, B \in D^-(\mathcal{O}_X)$. For all $x \in X$, there is a canonical identification*

$$(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B)_x = A_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} B_x$$

in $D^-(\mathcal{O}_{X,x})$. In particular, for $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}(\mathcal{O}_X)$, the natural map

$$\mathcal{T}or_n^{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \to \mathrm{Tor}_n^{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

for every $x \in X$ and $n \in \mathbb{Z}$.

Now we discuss derived pullback. Let $f\colon X \to Y$ be a morphism of ringed spaces. Consider the class $\mathcal{R}$ of flat $\mathcal{O}_Y$-modules, which we already know is preadapted. We claim it is adapted to the pullback functor f^*. If $\mathcal{F}^\bullet$ is a bounded above acyclic complex of flat $\mathcal{O}_X$-modules, then apply f^*, and consider the resulting sequence on stalks at a point $x \in X$:

$$\cdots \to \mathcal{F}^{n-1}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \to \mathcal{F}^n_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \to \cdots$$

Since each $\mathcal{F}^n_{f(x)}$ is flat over $\mathcal{O}_{Y,f(x)}$ and the sequence is bounded above, the sequence above is in fact exact (by a result early on in the discussion of derived tensor product). Therefore $f^*\mathcal{F}^\bullet$ is an acyclic complex, and we deduce that $\mathcal{R}$ is an adapted class, so the derived functor $Lf^*\colon D^-(\mathcal{O}_Y) \to D^-(\mathcal{O}_X)$ exists. Here are the very basic properties:

Proposition 2.2.17. *Let $f\colon X \to Y$ be a morphism of ringed spaces.*

(a) If $g\colon Y \to Z$ is a morphism of ringed spaces, the natural map

$$L(g \circ f)^* \to Lf^* \circ Lg^*$$

is an isomorphism of functors $D^-(\mathcal{O}_Z) \to D^-(\mathcal{O}_X)$.

(b) Let $U \subset Y$ be an open subset, and $f|_{f^{-1}(U)}$ denote the restricted morphism $f^{-1}(U) \to U$. Then there is a canonical identification

$$Lf^*(A)|_{f^{-1}(U)} = L(f|_{f^{-1}(U)})^*(A|_U)$$

in $D^-(\mathcal{O}_X|_{f^{-1}(U)})$ for $A \in D^-(\mathcal{O}_Y)$.

(c) For any $A \in D^-(\mathcal{O}_Y)$ and $x \in X$, the natural map

$$Lf^*(A)_x \to A_{f(x)} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

is an isomorphism, where on the RHS we mean the derived functor of the base change by $\mathcal{O}_{X,x}$ defined previously.

(d) For any $A, B \in D^-(\mathcal{O}_Y)$, the natural map

$$Lf^*(A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B) \to Lf^*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(B)$$

is an isomorphism.

PROOF. (a) If $\mathcal{F}$ is flat over $\mathcal{O}_Y$, then $f^*\mathcal{F}$ is flat over $\mathcal{O}_X$, because its stalk at x is the base change $\mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ and flatness is stable under base change.

(b) This follows from the commutative diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f|_{f^{-1}(U)}} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

of ringed spaces, (a), and the fact that pulling back along an open inclusion is just restriction.

(c) First of all, recall that we justified the notation on the RHS because the restriction of scalars (which is exact) of the derived base change is indeed the derived tensor product. The statement is derived functors of the composite in the commutative square:

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{O}_Y) & \xrightarrow{(-)_{f(x)}} & \mathrm{Mod}(\mathcal{O}_{Y,f(x)}) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}} \\ \mathrm{Mod}(\mathcal{O}_X) & \xrightarrow{(-)_x} & \mathrm{Mod}(\mathcal{O}_{X,x}) \end{array}$$

Taking stalks is exact, so the only thing to do is find a preadapted class on $\mathrm{Mod}(\mathcal{O}_Y)$ whose stalks land in an adapted class for base change to $\mathcal{O}_{X,x}$, for which we can take the flat $\mathcal{O}_Y$-modules.

(d) We have maps

$$Lf^*(A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B) \to Lf^*(A \otimes_{\mathcal{O}_Y} B) \to f^*(A \otimes_{\mathcal{O}_Y} B)$$

and maps

$$Lf^*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(B) \to f^*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^*(B) \to f^*(A) \otimes_{\mathcal{O}_X} f^*(B),$$

all of which are natural in A and in B. The two RHS's are naturally isomorphic in A and B. If we take A and B to be bounded above complexes of flat sheaves, then the first maps in the bottom and top row are isomorphisms. The second in the top row is also an isomorphism because the tensor product of flat sheaves is flat, and the second in the bottom row is an isomorphism because the pullback of flat modules is flat. Thus we obtain a canonical isomorphism on $D^-(\mathcal{O}_Y) \times D^-(\mathcal{O}_Y)$. □

Remark 2.2.18. Let $A \to B$ be a morphism in $D(\mathcal{O}_X)$, and suppose that $A_x \to B_x$ is an isomorphism in $D(\mathcal{O}_{X,x})$ for every $x \in X$. Since taking stalks is exact and termwise, this implies that $H^n(A) \to H^n(B)$ is a stalkwise isomorphism, and hence an isomorphism of sheaves. Thus $A \to B$ is an isomorphism. Therefore we can often argue on stalks even in the derived category.

Remark 2.2.19. Let $A \to B$ be a morphism in $D^+(\mathcal{O}_X)$, and suppose that $R\Gamma(U, A) \to R\Gamma(U, B)$ is an isomorphism in $D^+(\mathcal{O}_X(X))$ (or $D^+(\mathcal{O}_X(U))$, or even $D^+(\mathbb{Z})$) for every open $U \subset X$. We claim $A \to B$ is an isomorphism. We may assume $A = \mathcal{I}_1^\bullet$ and $B = \mathcal{I}_2^\bullet$ are in $\mathrm{K}^+(\mathcal{I}_{\mathcal{O}_X})$, and then since $\mathcal{I}_2^\bullet$ is a bounded below complex of injectives, the map $\mathcal{I}_1^\bullet \to \mathcal{I}_2^\bullet$ comes from a chain map. Then for each n,

$\mathcal{I}_1^n(U) \to \mathcal{I}_2^n(U)$ is an isomorphism for every U, so $\mathcal{I}_1 \xrightarrow{\sim} \mathcal{I}_2$ in $\mathrm{K}^+(\mathcal{O}_X)$, which implies the result.

Note that for this result, we only need to check on all opens U in a basis for X, because that is all we used in the last step of the proof. Moreover, it is of course enough to show that $\mathbb{H}^n(U, A) \to \mathbb{H}^n(U, B)$ is an isomorphism for all n and all U in a basis, because that implies the condition in the statement.

Remark 2.2.20. Let $\varphi\colon \mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X, so that we get an associated morphism $f\colon (X, \mathcal{O}_2) \to (X, \mathcal{O}_1)$ of ringed spaces. We have already discussed the fact that f_* is exact in this case and is just restriction of scalars. Since $f^{-1} = \mathrm{Id}$, $f^*(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_1} \mathcal{O}_2$ is extension of scalars. Then we have a canonical identification $f_*(\mathcal{F} \otimes_{\mathcal{O}_1} \mathcal{O}_2) = \mathcal{F} \otimes_{\mathcal{O}_1} \mathcal{O}_2$, where on the RHS we mean as $\mathcal{O}_1$-modules. Since f_* is exact, this implies that $f_* L f^*(A) = A \otimes^{\mathbf{L}}_{\mathcal{O}_1} \mathcal{O}_2$, so we often write $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_1} \mathcal{O}_2$, for derived change of scalars.

Now let $f\colon X \to Y$ be a morphism, and observe that the functor f^{-1} is exact, so descends to the derived categories. We may write f as the composite

$$(X, \mathcal{O}_X) \to (X, f^{-1}(\mathcal{O}_Y)) \to (Y, \mathcal{O}_Y)$$

where the first map is induced by $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$, and the second is given on spaces by f and the morphism of sheaves $f^{-1}\mathcal{O}_Y \to f^{-1}\mathcal{O}_Y$ is the identity. The second map is flat, so we are justified in writing

$$L f^*(A) = f^{-1}(A) \otimes^{\mathbf{L}}_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

according to the convention above.

A simple and practical example of a left derived functor is the closed-embedding projection formula. Let $i\colon Z \to X$ be a morphism of ringed spaces which is a closed embedding on the underlying topological spaces. If $\mathcal{F}$ is a sheaf on Z and $\mathcal{G}$ is a sheaf on Y, then since $i^*(i_*\mathcal{F} \otimes \mathcal{G}) = i^* i_* \mathcal{F} \otimes i^* \mathcal{G}$, the counit map induces a morphism $i^*(i_*\mathcal{F} \otimes \mathcal{G}) \to \mathcal{F} \otimes i^*\mathcal{G}$, and thus a morphism $i_*\mathcal{F} \otimes \mathcal{G} \to i_*(\mathcal{F} \otimes i^*\mathcal{G})$ by adjunction.

Consider the stalk at a point $x \notin i(Z)$: the RHS is 0, and the LHS is $0 \otimes \mathcal{G} = 0$. On the other hand, if $x = i(z)$, then the stalk at x on either side is

$$\mathcal{F}_z \otimes_{\mathcal{O}_{X,i(z)}} \mathcal{G}_{i(z)} \to \mathcal{F}_z \otimes_{\mathcal{O}_{Z,z}} (\mathcal{G}_z \otimes_{\mathcal{O}_{X,i(z)}} \mathcal{O}_{Z,z}),$$

which is an isomorphism. Thus the natural projection map is an isomorphism. Since i_* is exact, this likewise induces natural morphisms

$$i_*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to i_* A \otimes_{\mathcal{O}_X} B \cong i_*(A \otimes_{\mathcal{O}_Z} i^* B) \leftarrow i_*(A \otimes^{\mathbf{L}}_{\mathcal{O}_Z} L i^* B).$$

If we replace B with a bounded above complex of flat modules, the left map is an isomorphism, and i^*B is a bounded above complex of flat $\mathcal{O}_Z$-modules, so the right-hand map is also an isomorphism. Thus we deduce the derived projection formula:

Lemma 2.2.21. *Let $i\colon Z \to X$ be a closed topological embedding, and $A \in D^-(\mathcal{O}_Z)$, $B \in D^-(\mathcal{O}_X)$. Then the natural map*

$$i_* A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to i_*(A \otimes^{\mathbf{L}}_{\mathcal{O}_Z} L i^* B)$$

is an isomorphism.

Because of this result, combined with the fact that hypercohomology is invariant under pushing forward by closed embeddings, we often drop the i_* and just implicitly view an object $A \in D^-(\mathcal{O}_Z)$ as an object of $D^-(\mathcal{O}_X)$. It is separately also true that $i_*\colon D^-(\mathcal{O}_Z) \to D^-(\mathcal{O}_X)$ is a fully faithful embedding, but we won't discuss this yet.

Example 2.2.22. Since i_* is exact, it commutes with cohomology (of complexes), and so

$$H^p(i_*A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B) = i_* H^p(A \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Li^*B).$$

In particular, for sheaves $\mathcal{F}$ and $\mathcal{G}$ we have

$$\mathcal{T}or_p^{\mathcal{O}_X}(i_*\mathcal{F}, \mathcal{G}) = i_* \mathcal{T}or_p^{\mathcal{O}_Z}(\mathcal{F}, Li^*\mathcal{G})$$

for all p. Now on the other hand, we have shown that i_* actually commutes with $R\Gamma$. So, suppose that $A \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Li^*B \in D^b(\mathcal{O}_Z)$. Thus when we apply i_* we land in $D^b(\mathcal{O}_X)$, and therefore

$$R\Gamma(X, i_*A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B) = R\Gamma(Z, A \otimes^{\mathbf{L}}_{\mathcal{O}_Z} Li^*B)$$

viewed in $D^+(\mathcal{O}_X(X))$.

Example 2.2.23. Suppose $i_1\colon Z_1 \to X$ and $i_2\colon Z_2 \to X$ are two closed embeddings, and consider the structure sheaves $i_{1,*}\mathcal{O}_{Z_1}$ and $i_{2,*}\mathcal{O}_{Z_2}$ as sheaves on X. In scheme theory, we often want to think of their derived tensor product as a derived intersection product, which generalizes the fact that their ordinary product is the structure sheaf of their scheme-theoretic intersection. Then the projection formula shows that:

$$\begin{aligned} i_{1,*}\mathcal{O}_{Z_1} \otimes^{\mathbf{L}}_{\mathcal{O}_X} i_{2,*}\mathcal{O}_{Z_2} &= i_{1,*}(\mathcal{O}_{Z_1} \otimes^{\mathbf{L}}_{\mathcal{O}_{Z_1}} Li_1^* i_{2,*}\mathcal{O}_{Z_2}) \\ &= i_{1,*}(Li_1^* i_{2,*}\mathcal{O}_{Z_2}) \\ &= i_{2,*}(Li_2^* i_{1,*}\mathcal{O}_{Z_1}) \end{aligned}$$

In particular, we can compute the derived intersection by pulling back $\mathcal{O}_{Z_2}$ to Z_1 or pulling back $\mathcal{O}_{Z_1}$ to Z_2, and we get the same answer.

For example, suppose Z_1 and Z_2 are effective divisors on a scheme X. Then we have two term resolutions

$$i_{1,*}\mathcal{O}_{Z_1} \simeq [\mathcal{O}_X(-Z_1) \hookrightarrow \mathcal{O}_X][1] \qquad i_{2,*}\mathcal{O}_{Z_2} \simeq [\mathcal{O}_X(-Z_2) \hookrightarrow \mathcal{O}_X][1]$$

by line bundles, which are in particular flat resolutions. So we have a few options for computing the derived intersection. If we replace both sides in the definition by resolutions, then we will obtain a three term complex

$$[\mathcal{O}_X(-Z_1 - Z_2) \to \mathcal{O}_X(Z_1) \oplus \mathcal{O}_X(Z_2) \to \mathcal{O}_X][2]$$

by definition of the tensor product. This is most ideal if we need to compute something like $\mathbf{R}\mathcal{H}om$ or further pullback, because it is a complex of locally frees. Alternatively, we can directly apply e.g. Li_1^* to this, which gives the complex

$$[\mathcal{O}_X(-Z_2)|_{Z_1} \to \mathcal{O}_{Z_1}][1],$$

which is best suited for directly computing cohomology.

The most interesting statements usually arise from relating left and right derived functors. This often happens through adjunctions. For example, let $f\colon X \to Y$ be a morphism, then we have the standard adjunction $f^* \dashv f_*$. Therefore, by our results on derived adjunctions, for $A \in D^-(\mathcal{O}_Y)$ and $B \in D^+(\mathcal{O}_X)$ we have a canonical isomorphism

$$\mathrm{RHom}_{\mathcal{O}_X}(Lf^*A, B) \cong \mathrm{RHom}_{\mathcal{O}_Y}(A, Rf_*B)$$

in $D^+(\mathcal{O}_Y(Y))$, where the LHS is viewed via restriction of scalars along $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$. We would like to generalize this statement to something on the level of sheaves.

To do this, first consider the sheaf-level adjunction; given a morphism $f\colon X \to Y$, applying the sheaf adjunction to $f^{-1}(U) \to U$ and recalling the behavior of pullback and pushforward with respect to this, we deduce a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{f^{-1}(U)}}(f^*\mathcal{F}|_{f^{-1}(U)}, \mathcal{G}|_{f^{-1}(U)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, f_*\mathcal{G}|_U).$$

This commutes with the restriction maps (i.e. is natural in U as well), so it induces an isomorphism

$$f_*\,\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}, f_*\mathcal{G})$$

of $\mathcal{O}_Y$-modules, by definition of the relevant objects, again natural in $\mathcal{F}$ and $\mathcal{G}$. By taking direct sums of both sides, this immediately generalizes to the case where $\mathcal{F}^\bullet$ is in $\mathrm{K}^-(\mathcal{O}_Y)$ and $\mathcal{G}^\bullet \in \mathrm{K}^+(\mathcal{O}_X)$. Now, we have natural morphisms

$$\begin{array}{ccc}
f_*\,\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) & \xrightarrow{\cong} & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^\bullet, f_*\mathcal{G}^\bullet) \\
\downarrow & & \downarrow \\
Rf_*\,\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) & & \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet) \\
\downarrow & & \downarrow \\
Rf_*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) & & \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F}^\bullet, Rf_*\mathcal{G}^\bullet) \\
\downarrow & & \\
Rf_*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*\mathcal{F}^\bullet, \mathcal{G}^\bullet) & &
\end{array}$$

as functors $\mathrm{K}^-(\mathcal{O}_Y) \times \mathrm{K}^+(\mathcal{O}_X) \to D^+(\mathcal{O}_Y)$. If we take $\mathcal{G}^\bullet$ to be a complex of injectives, then the two upper-most maps on the left are isomorphisms, the first one because $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{G}^\bullet)$ is flasque by Lemma 2.2.6. If we also take $\mathcal{F}^\bullet$ to be a bounded above complex of flats, then $Lf^*\mathcal{F}^\bullet \to f^*\mathcal{F}^\bullet$ is an isomorphism, so the bottom map on the left is an isomorphism. Therefore, this induces a natural map

$$(*) \qquad Rf_*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*A, B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, Rf_*B)$$

in $D^+(\mathcal{O}_Y)$, natural in $A \in D^-(\mathcal{O}_Y)$ and $B \in D^+(\mathcal{O}_X)$. We note at this stage that it does not obviously follow from the diagram that this is an isomorphism; a bounded above complex of flats in the first variable doesn't necessarily compute $\mathbf{R}\mathcal{H}om$, and it is not clear if the pushforward of a bounded below complex of

injectives (which is only a complex of flasques) computes $\mathbf{R}\mathcal{H}om$. Yet nonetheless we still can prove the following theorem by indirect means:

Proposition 2.2.24. *For $A \in D^-(\mathcal{O}_Y)$ and $B \in D^+(\mathcal{O}_X)$, the map $(*)$ is an isomorphism*

$$Rf_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*A, B) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, Rf_*B)$$

in $D^+(\mathcal{O}_Y)$, natural in A and B.

Proof. We are going to apply Remark 2.2.19, which allows us to check this after applying $R\Gamma(U, -)$ for every $U \subset Y$. But because of the properties we have proven regarding the relation of pushforward and pullback with restriction, doing this gives a map

$$\mathrm{RHom}_{\mathcal{O}_{f^{-1}(U)}}(Lg^*(A|_U), B|_{f^{-1}(U)}) \to \mathrm{RHom}_{\mathcal{O}_U}(A|_U, Rg_*(B|_{f^{-1}(U)}))$$

where g is the restricted map $f^{-1}(U) \to U$. This is the same map as the derived adjunction applied to g, because both were defined using the adjunction $g^* \dashv g_*$, as one can check easily. □

We note that this is a nice example of using derived categories, because its statement was not trivial after replacing with a complex of injectives or flats. We also note that, because we have ignored unbounded derived categories, the isomorphism is not induced by any unit or counit map, but instead we describe it as the unique extension of the underived formula upon restricting to injectives and flats.

Example 2.2.25. Let $A = \mathcal{F}$ be a sheaf, and take the map f to be the inclusion $x \hookrightarrow X$ of a point where the domain has structure sheaf $\mathcal{O}_{X,x}$. Then f^* is the functor $(-)_x$ which is exact, and f_* is the skyscraper sheaf functor, which is also exact. So, both commute with cohomology, and since sheaf ext on a point is just module ext, we get

$$f_* \operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M) = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, f_*M)$$

for any $\mathcal{O}_{X,x}$-module M, so $\operatorname{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x, M) = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, f_*M)_x$. In particular, if $\mathcal{F}$ is internally projective, then $\mathcal{F}_x$ is projective for every $x \in X$. E.g. if $\mathcal{F}$ is also of finite presentation and X is a locally ringed space, then $\mathcal{F}$ is a locally free sheaf of finite rank. So, being internally projective is usually equivalent to being a vector bundle.

Example 2.2.26. Let us record the fact from the example above that if we denote the skyscraper sheaf functor by Sk_x, which descends to the derived category, then

$$\mathrm{Sk}_x \mathrm{RHom}_{\mathcal{O}_{X,x}}(A_x, B) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, \mathrm{Sk}_x B)$$

for $B \in D^+(\mathcal{O}_{X,x})$ and $A \in D^-(\mathcal{O}_X)$. In particular, suppose A is itself of the form $\mathrm{Sk}_x A$ for $A \in D^-(\mathcal{O}_{X,x})$, then

$$\mathrm{Sk}_x \mathrm{RHom}_{\mathcal{O}_{X,x}}(A, B) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathrm{Sk}_x A, \mathrm{Sk}_x B).$$

As an aside, we note that the usual derived adjunction formula tells us that $\mathrm{RHom}_{\mathcal{O}_{X,x}}(A_x, B) = \mathrm{RHom}_{\mathcal{O}_X}(A, \mathrm{Sk}_x B)$, which is an incarnation of the fact that skyscraper sheaves are (clearly) flasque.

Remark 2.2.27. Let A be a bounded above complex of flats, and B a bounded below complex of injectives. Now we can verify that

$$\underline{\mathcal{H}om}_{\mathcal{O}_Y}(A, f_*B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, Rf_*B)$$

is in fact an isomorphism. Indeed by Prop. 2.2.24, it suffices to check that

$$\underline{\mathcal{H}om}_{\mathcal{O}_Y}(f^*A, B) \to Rf_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(Lf^*A, B)$$

is an isomorphism, which we already showed in this case.

For the last result in this section, let us show the easier proposition that the tensor-hom adjunction derives well. Recall that from the adjunction, one immediately proves

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2, \mathcal{F}_3) \cong \mathcal{H}om_{\mathcal{O}_X}\Big(\mathcal{F}_1, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2, \mathcal{F}_3)\Big)$$

Now suppose $\mathcal{F}_1^\bullet, \mathcal{F}_2^\bullet$, and $\mathcal{F}_3^\bullet$ are (arbitrary) complexes of $\mathcal{O}_X$-modules. Then

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}\Big(\bigoplus_{p+q=n} \mathcal{F}_1^p \otimes_{\mathcal{O}_X} \mathcal{F}_2^q, \mathcal{F}_3^r\Big) &\cong \prod_{p+q=n} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_1^p \otimes_{\mathcal{O}_X} \mathcal{F}_2^q, \mathcal{F}_3^r) \\ &\cong \prod_{p+q=n} \mathcal{H}om_{\mathcal{O}_X}\Big(\mathcal{F}_1^p, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2^q, \mathcal{F}_3^r)\Big).\end{aligned}$$

Now take the product over all n, and set $r = n + m$; thus we will get

$$\begin{aligned}\underline{\mathcal{H}om}^m_{\mathcal{O}_X}(\mathcal{F}_1^\bullet \otimes_{\mathcal{O}_X} \mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet) &\cong \prod_{p,q} \mathcal{H}om_{\mathcal{O}_X}\Big(\mathcal{F}_1^p, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}_2^q, \mathcal{F}_3^{p+q+m})\Big) \\ &\cong \prod_p \mathcal{H}om_{\mathcal{O}_X}\Big(\mathcal{F}_1^p, \underline{\mathcal{H}om}^{p+m}_{\mathcal{O}_X}(\mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet)\Big) \\ &\cong \underline{\mathcal{H}om}^m_{\mathcal{O}_X}\Big(\mathcal{F}_1^\bullet, \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}_2^\bullet, \mathcal{F}_3^\bullet)\Big),\end{aligned}$$

which gives a tensor-hom adjunction in the category $\mathrm{Ch}(\mathcal{O}_X)$. Now we bring this to the derived category.

Proposition 2.2.28. *Let X be a ringed space. There is a canonical isomorphism*

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\Big(A, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C)\Big)$$

in $D^+(\mathcal{O}_X)$ natural in $A, B \in D^-(\mathcal{O}_X)$ and $C \in D^+(\mathcal{O}_X)$.

Proof. We will do this more quickly than the previous results. Choosing representatives, the natural maps $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to A \otimes_{\mathcal{O}_X} B$ and the natural maps from $\underline{\mathcal{H}om}$ to $\mathbf{R}\mathcal{H}om$, induce maps

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(A \otimes_{\mathcal{O}_X} B, C) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C)$$

and

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}\Big(A, \underline{\mathcal{H}om}_{\mathcal{O}_X}(B, C)\Big) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\Big(A, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C)\Big).$$

By the result in $\mathrm{Ch}(\mathcal{O}_X)$, the two LHS's are identified. Now, take B to be a bounded above complex of flat $\mathcal{O}_X$-modules, and C to be a bounded below complex of injectives. Then the top map is an isomorphism. Moreover, by Lemma 2.2.6(d), each $\underline{\mathcal{H}om}_{\mathcal{O}_X}(B,C)$ is a bounded below complex of injectives, so the bottom map is an isomorphism too. □

In particular, under the same assumptions for A, B, C, we have

$$\mathrm{RHom}_{\mathcal{O}_X}\left(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C\right) \cong \mathrm{RHom}_{\mathcal{O}_X}\Big(A, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B,C)\Big).$$

This is a result that we already knew if $B \in \mathrm{Mod}(\mathcal{O}_X)$, so this is a direct generalization of that.

2.2.2. Appendix: Cohomology of a Noetherian space. The goal of this short section is to prove the following theorem:

THEOREM 2.2.29. *Let X be a ringed space whose underlying space is Noetherian, and suppose $\dim(X) = d$. Then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_X)$.*

Corollary 2.2.30. *Let X be a Noetherian ringed space of dimension d.*

(a) For $A \in D^{[a,b]}(\mathcal{O}_X)$, $R\Gamma(X, A) \in D^{[a,b+d]}(\mathcal{O}_X(X))$. In particular $R\Gamma$ restricts to a functor $D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_X(X))$.

(b) Suppose $f\colon X \to Y$ is a morphism. For $A \in D^{[a,b]}(\mathcal{O}_X)$, $Rf_*A \in D^{[a,b+d]}(\mathcal{O}_X)$. In particular Rf_* restricts to a functor $D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_Y)$.

PROOF. (a) is a special case of (b). For each $\mathcal{F}$, $R^p f_*(\mathcal{F})$ is the sheafification of $U \mapsto H^p(f^{-1}(U), \mathcal{F})$, which for $p > d$ is zero since $f^{-1}(U)$ is a Noetherian space of dimension $\leq d$. Thus the result follows from Example 1.2.29. □

The proof of the main result is by some trickery. First, to simplify notation, we may assume that $\mathcal{O}_X = \underline{\mathbb{Z}}$, so that all sheaves considered are abelian (because cohomology commutes with change of ring). Thus all sheaves and ringed spaces in this section are considered abelian. Also, the number in the theorem (namely $\dim(X)$) is definitely achieved, as e.g. the example of $\mathbb{P}^n$ will show.

For this section, we will use the following special notation: for an open set $U \subset X$, we write $\underline{\mathbb{Z}}_U \coloneqq j_!\underline{\mathbb{Z}}$ where $j\colon U \hookrightarrow X$. If $\mathcal{F}$ is an abelian sheaf and $\{s_i\}$ is a collection of local sections with $s_i \in \mathcal{F}(U_i)$, we say that $\mathcal{F}$ is generated by $\{s_i\}$ if the induced map

$$\bigoplus_i \underline{\mathbb{Z}}_{U_i} \to \mathcal{F} \to 0$$

is surjective. Warning: a sheaf generated by finitely many local sections need not be of finite type (e.g. $\underline{\mathbb{Z}}_U$ itself has support U, so on a connected space it can't be of finite type).

But, what is true is that every sheaf $\mathcal{F}$ is a filtered colimit of sheaves generated by finitely many sections. Indeed, let $\{\mathcal{F}_\alpha\}$ be the collection of all subsheaves generated by finitely many sections of $\mathcal{F}$, considered as a filtered system with inclusion maps. Then the natural map $\varinjlim_\alpha \mathcal{F}_\alpha \to \mathcal{F}$ is an isomorphism. Indeed,

it's injective, and surjectivity can be checked on stalks; but this is clear because a local section $s \in \mathcal{F}(U)$ generates a subsheaf of $\mathcal{F}$ which is generated by finitely many sections.

The main lemma which actually does real 'work' is somehow the following simple one. We will explain how to prove the theorem after stating the lemma, and then prove the lemma at the end.

Lemma 2.2.31. *Let X be an irreducible Noetherian space and $\mathcal{F} \subset \underline{\mathbb{Z}}$ an abelian subsheaf generated by finitely many local sections. Then there is a filtration*

$$0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$$

such that for $0 < i \leq n$, there are open sets $U, V \subset X$ and an exact sequence of the form $0 \to \underline{\mathbb{Z}}_U \to \underline{\mathbb{Z}}_V \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$.

Intuitively, this lemma says that we can finitely filter the sheaf $\mathcal{F}$ so that the filtration quotients are generated by one local section with one local relation. As mentioned, let's prove the main theorem. The main trick we'll apply aside from the lemma and the previous remark is that for a Noetherian space, as we showed, cohomology over all opens commutes with filtered colimits of sheaves.

Proof (Thm. 2.2.29). The proof is a sequence of a bunch of reductions. We prove by induction on d that if $\dim(X) \leq d$, then all sheaves $\mathcal{F}$ satisfy $H^p(X, \mathcal{F}) = 0$ for $p > d$.

1. We may assume X is irreducible. Indeed, induct on the (finite) number of components and assume the base case is done. Let $X_1 \cup \cdots \cup X_k = X$ be the components and write $U = X \setminus X_1$, then we have the extension-by-zero sequence

   $$0 \to j_U, !\mathcal{F}|_U \to \mathcal{F} \to i_{X_1,*}\mathcal{F}|_{X_1} \to 0.$$

   Then $j_{U,!}\mathcal{F}|_U$ is supported in $U \subset X_2 \cup \cdots \cup X_k$ and $i_{X_1,*}\mathcal{F}|_Z$ is supported in X_1, so both are pushed forward from a Noetherian space of lower or equal dimension, and with strictly fewer irreducible components.
2. Now if $\dim(X) = 0$, then since X is irreducible it is the only nonempty closed set: if there were another one Z, then it would contain some nonempty irreducible component Z_1, and then $Z_1 \subset X$ would contradict dimension 0 unless $Z = \varnothing$. But then X has only one nonempty open, so every sheaf on X is flasque.
3. For the remainder of the proof, assume X is irreducible and $d > 0$. Since cohomology commutes with filtered colimits, we may assume $\mathcal{F}$ is generated by finitely many local sections, say $\{s_1, \ldots, s_k\}$. If we let $\mathcal{G} \subset \mathcal{F}$ be the subsheaf generated by $s_1, \ldots, s_{k-1}$, then in the exact sequence $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{F}/\mathcal{G} \to 0$ the left and right sheaves are generated by strictly fewer than k local sections. Thus inductively we may assume $\mathcal{F}$ is generated by one local section.
4. We claim that we can actually assume that $\mathcal{F}$ is of the form $\underline{\mathbb{Z}}_V$. Indeed, by (3) there is an exact sequence $0 \to \mathcal{G} \to \underline{\mathbb{Z}}_U \to \mathcal{F} \to 0$ for some open set $U \subset X$, so under such an assumption $H^p(X, \mathcal{F}) \xrightarrow{\sim} H^{p+1}(X, \mathcal{G})$ for

$p > d$, and hence by replacing $\mathcal{F}$ with $\mathcal{G}$ we may assume $\mathcal{F} \subset \underline{\mathbb{Z}}_U \subset \underline{\mathbb{Z}}$. Once again by writing $\mathcal{F}$ as a filtered colimit of its subsheaves generated by finitely many sections, we may assume $\mathcal{F} \subset \mathbb{Z}$ is generated by finitely many sections. Thus we have a filtration $0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = \mathcal{F}$ where the lemma and claimed assumption show that $H^p(X, \mathcal{F}_i/\mathcal{F}_{i-1}) = 0$ for $p > d$. Thus the sequence $0 \to \mathcal{F}_{i-1} \to \mathcal{F}_i \to \mathcal{F}_i/\mathcal{F}_{i-1} \to 0$ shows inductively that $H^p(X, \mathcal{F}_i,) = 0$ for $p > d$, which includes the result.

5. Thus, let $V \subset X$ be a nonempty open and consider the extension-by-zero sequence $0 \to \underline{\mathbb{Z}}_V \to \underline{\mathbb{Z}} \to \mathcal{G} \to 0$ where $\mathcal{G}$ is supported on the closed subset $X \setminus V$. Since X is irreducible, and V is nonempty, $\dim(X \setminus V) \leq d - 1$, and hence $H^p(X, \mathcal{G}) = 0$ for $p \geq d$. Thus, $H^p(X, \underline{\mathbb{Z}}_V) \xrightarrow{\sim} H^p(X, \underline{\mathbb{Z}}) = 0$ for $p > d$, however $\underline{\mathbb{Z}}$ is flasque because every open set of X is irreducible and hence connected.

This proves the induction step as desired. □

We will see a few arguments of this form in the sequel, where we argue by various inductions and sub-inductions, and keep reducing cohomology statements via exact sequences until we get to something trivial. Finally, the lemma:

Proof (of Lemma 2.2.31). Since X is irreducible, $\mathcal{F}(U) \subset \mathbb{Z}$ for all U, so $\mathcal{F}$ is generated by $(U_1, n_1), \ldots, (U_k, n_k)$, where we may assume $n_i > 0$ for all i. Note that for any subcollection $\{(U_i, n_i) : i \in I\}$ of these generators, $\mathcal{F}(\bigcap_i U_i)$ contains the ideal $(n_i : i \in I) = (d)$ where $d = \gcd(n_i : i \in I)$; so by adding all such opens and gcds over all $I \subset \{1, \ldots, k\}$, we may assume that for each $x \in X$, $\gcd(n_i : i \in \{i : x \in U_i\})$ is attained by some (U_i, n_i) containing x.

Let $\mathcal{F}_0 = 0$ and for each n let $\mathcal{F}_n$ be generated by $\{(U_i, n_i) : n_i \leq n\}$. Thus we get an ascending filtration as in the lemma statement which ends at $\mathcal{F}$. Write $U_{\leq n} = \bigcup_{i, n_i \leq n} U_i$, and note that

$$(\mathcal{F}_n)_x = (n_i : \{i \leq n : x \in U_i\}) = \gcd(n_i : \{i \leq n : x \in U_i\})\mathbb{Z}.$$

$(\mathcal{F}_n)_x = (\mathcal{F}_{n-1})_x = 0$ if $x \notin U_{\leq n}$, and $(\mathcal{F}_n)_x = (\mathcal{F}_{n-1})_x$ if $x \in U_{\leq n-1}$, because in the latter case the gcd already occurs as a generator at x with value $\leq n - 1$. In the remaining case $(\mathcal{F}_n)_x = n\mathbb{Z}$ but $(\mathcal{F}_{n-1})_x = 0$. Thus, the section n over $U_{\leq n}$ generates $\mathcal{F}_n/\mathcal{F}_{n-1}$, and the kernel of $\underline{\mathbb{Z}}_{U_{\leq n}} \to \mathcal{F}_n/\mathcal{F}_{n-1} \to 0$ is exactly the subsheaf $\underline{\mathbb{Z}}_{U_{\leq n-1}}$. □

2.2.3. Pseudocoherent and perfect complexes. In this section, we start to focus on more advanced notions regarding the derived category $D(\mathcal{O}_X)$ where X is a ringed space. Pseudocoherent objects of the derived category will be a generalization of finitely presented $\mathcal{O}_X$-modules, and perfect objects will be a generalization of finite locally free $\mathcal{O}_X$-modules. In practice, pseudocoherence will not be very different from a finite type assumption because we mostly care about Noetherian schemes. Perfect complexes will not be very different from bounded complexes of vector bundles.

Before moving forward, let us also make the following convention: so far D^+, D^- and D^b have referred to the localizations of the respective homotopy categories. We now will use these to refer to the essential images of these subcategories in $D(\mathcal{O}_X)$. In other words, these are the subcategories with cohomology bounded below, above, or in both directions. Indeed, one direction is clear, and for the other observe that the map from/to the smart truncation τ will be a quasi-isomorphism. We may also use the notation $D^{[a,b]}(\mathcal{O}_X)$ for objects whose cohomology is concentrated in the interval $[a, b]$. We want to start to view the derived category $D(\mathcal{O}_X)$ more abstractly, and think of its objects as more than just complexes. So, we use letters $A, B \in D(\mathcal{O}_X)$ when talking abstractly about its objects, and $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{K}(\mathcal{O}_X)$ when thinking about chain complexes of $\mathcal{O}_X$-modules.

Here are the major definitions of this section.

Definition 2.2.32. An $\mathcal{O}_X$-module $\mathcal{E}$ is *strictly perfect* if it is a direct summand of a finite free $\mathcal{O}_X$-module. An object $\mathcal{E}^\bullet \in \mathrm{Ch}(\mathcal{O}_X)$ is *strictly perfect* if it is bounded, and $\mathcal{E}^n$ is strictly perfect for all $n \in \mathbb{Z}$.

Definition 2.2.33. Let X be a ringed space.

1. A complex $\mathcal{F}^\bullet \in \mathrm{K}(\mathcal{O}_X)$ is *perfect* if there is an open cover $\{U_i\}$ of X such that for each i, there exists a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}|_{U_i}$ in $\mathrm{Ch}(\mathcal{O}_{U_i})$ such that $\mathcal{E}$ is strictly perfect.
2. An object $A \in D(\mathcal{O}_X)$ is *perfect* if it is represented by a perfect complex. We denote by $\mathrm{Perf}^*(\mathcal{O}_X) \subset D^*(\mathcal{O}_X)$ the full subcategory of perfect objects.

Definition 2.2.34. Let X be a ringed space and $n \in \mathbb{Z}$.

1. A complex $\mathcal{F}^\bullet \in \mathrm{K}(\mathcal{O}_X)$ is *n-pseudocoherent* if there is an open cover $\{U_i\}$ of X such that for each i, there exists a morphism $\mathcal{E}^\bullet \to \mathcal{F}|_{U_i}$ in $\mathrm{Ch}(\mathcal{O}_X)$ with $\mathcal{E}$ strictly perfect, such that $H^p(\mathcal{E}^\bullet) \to H^p(\mathcal{F}^\bullet|_{U_i})$ is an isomorphism for $j > m$ and an epimorphism for $j = m$.
2. A complex $\mathcal{F}^\bullet \in \mathrm{K}(\mathcal{O}_X)$ is $(-\infty)$-pseudocoherent, or just pseudocoherent, if it is n-pseudocoherent for all $n \in \mathbb{Z}$.
3. An object $A \in D(\mathcal{O}_X)$ is *n-pseudocoherent* if it is represented by an n-pseudocoherent complex for $n \in [-\infty, \infty)$.

Remark 2.2.35. Here are some basic observations about these definitions. First, a strictly perfect sheaf is clearly finitely presented, and thus over a locally ringed space it is finite locally free. In that case, it doesn't change anything if we define perfectness and pseudocoherence using finite free sheaves, or finite locally free

sheaves. Second, if X is quasicompact, then a perfect object A is in $D^b(\mathcal{O}_X)$, because there is a finite open cover $\{U_i\}$ such that $H^p(A)|_{U_i} = 0$ for $p \gg 0$ and $p \ll 0$. Similarly, if X is quasicompact then a pseudocoherent object A is in $D^-(\mathcal{O}_X)$.

Here are the basic properties of strictly perfect complexes.

Lemma 2.2.36. *Let X be a ringed space.*

1. *If $\phi\colon \mathcal{E}_1^\bullet \to \mathcal{E}_2^\bullet$ is a morphism of strictly perfect complexes, then* $\mathrm{Cone}(\phi)$ *is a strictly perfect complex.*
2. *If $\mathcal{E}_1^\bullet, \mathcal{E}_2^\bullet$ are strictly perfect complexes, then $\mathcal{E}_1^\bullet \otimes \mathcal{E}_2^\bullet$ is strictly perfect.*
3. *If $f\colon X \to Y$ is a morphism of ringed spaces and $\mathcal{E}^\bullet$ is strictly perfect on Y, then $f^*\mathcal{E}^\bullet$ is strictly perfect on X. In particular if $\mathcal{E}^\bullet$ is strictly perfect, then $\mathcal{E}^\bullet|_U$ is strictly perfect for any $U \subset X$.*
4. *A strictly perfect complex is a bounded complex of internally projective sheaves. In particular, if $\mathcal{E}^\bullet$ is perfect, then the natural map $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, -) \to \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{E}^\bullet, -)$ is an isomorphism in $D(\mathcal{O}_X)$.*

Proof. It is clear from the definition that the direct sum of two strictly perfect $\mathcal{O}_X$-modules is strictly perfect. Therefore (1) is trivial. This also reduces (2) to showing that if $\mathcal{F}, \mathcal{G}$ are strictly perfect then $\mathcal{F} \otimes \mathcal{G}$ is. Indeed if $\mathcal{F} \oplus \mathcal{F}' = \mathcal{O}_X^n$ and $\mathcal{G} \oplus \mathcal{G}' = \mathcal{O}_X^m$, then taking the tensor product of both sides and distributing has one summand as $\mathcal{F} \otimes \mathcal{G}$, and on the RHS we have $\mathcal{O}_X^{mn}$. (3) is clear because f^* takes $\mathcal{O}_Y^n$ to $\mathcal{O}_X^n$, and preserves direct sums. For (4), note that if $\mathcal{F} \oplus \mathcal{F}'$ is finite free, then $\mathcal{E}xt^p(\mathcal{F}, -) \oplus \mathcal{E}xt^p(\mathcal{F}', -) = 0$ for $p > 0$. □

Next we have some results that we aim to apply to the strictly perfect case, but we can prove in more generality about internally projective objects.

Lemma 2.2.37. *Let $\phi\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes with $\mathcal{E}^\bullet$ a bounded above complex of internally projective objects. Assume we're in one of the following situations:*

(a) $a \in \mathbb{Z}$ is such that $\mathcal{E}^p = 0$ for $p < a$, and $H^p(\mathcal{F}^\bullet) = 0$ for $p \geq a$.
(b) $\mathcal{E}^\bullet$ is bounded, and $\mathcal{F}^\bullet$ is acyclic.
(c) $\mathcal{F}^\bullet$ is acyclic and bounded below.

Then there is an open cover $\{U_i\}$ of X such that $\phi|_{U_i}$ is nullhomotopic.

Proof. We claim that in any of the cases, $H^0\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = 0$. For cases (b) and (c), we showed this in Example 2.2.9. For (a), just use the same spectral sequence of Example 2.2.9 slightly more carefully, which converges as expected because $\mathcal{E}^\bullet$ is bounded.

Now, observe that $H^0\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ is the sheafification of the presheaf

$$U \mapsto H^0 \underline{\mathrm{Hom}}_{\mathcal{O}_U}(\mathcal{E}^\bullet|_U, \mathcal{F}^\bullet|_U) = \mathrm{Hom}_{K(\mathcal{O}_U)}(\mathcal{E}^\bullet|_U, \mathcal{F}^\bullet|_U),$$

so the fact that this is zero means that every global section of this presheaf is zero on an open cover. □

Corollary 2.2.38. *Let $\mathcal{E}^\bullet$ be a bounded complex of internally projective objects and $\mathcal{F}^\bullet$ a complex.*

1. *Suppose $\mathcal{E}^\bullet \in \mathrm{K}^{[a,\infty)}(\mathcal{O}_X)$, and let $\phi\colon \mathcal{G}^\bullet \to \mathcal{F}^\bullet$ be a morphism of complexes for which $H^p(\phi)$ is an isomorphism for $p > a$, and an epimorphism for $p = a$. Then for any map $\alpha\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$, there is an open cover $\{U_i\}$ of X and maps $\beta_i\colon \mathcal{E}^\bullet|_U \to \mathcal{G}^\bullet|_U$ such that $\phi|_U\beta_i = \alpha$ in $\mathrm{K}(\mathcal{O}_U)$.*
2. *For any $\alpha \in \mathrm{Hom}_{D(\mathcal{O}_X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$, there exists an open cover $\{U_i\}$ of X such that $\alpha|_{U_i}$ is given by a morphism of complexes $\mathcal{E}^\bullet|_{U_i} \to \mathcal{F}^\bullet|_{U_i}$.*
3. *If $\alpha\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ is a morphism of complexes which becomes 0 in $D(\mathcal{O}_X)$, then it is locally nullhomotopic.*

Proof. (1) The distinguished triangle $\mathcal{G}^\bullet \to \mathcal{F}^\bullet \to \mathrm{Cone}(\phi) \to \mathcal{G}^\bullet[1]$ shows that $H^p(\mathrm{Cone}(\phi)) = 0$ for $p \geq a$. This also induces an exact sequence

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{O}_X)}(\mathcal{E}^\bullet|_U, \mathcal{G}^\bullet|_U) \to \mathrm{Hom}_{\mathrm{K}}(\mathcal{E}^\bullet|_U, \mathcal{F}^\bullet|_U) \to \mathrm{Hom}_{\mathrm{K}}(\mathcal{E}^\bullet|_U, \mathrm{Cone}(\phi)|_U)$$

because restriction is exact. By the lemma part (a), the composite map $\mathcal{E}^\bullet \to \mathrm{Cone}(\phi)$ is locally 0, so on any such open cover, the exact sequence says that we can lift it to a map $\mathcal{E}^\bullet|_U \to \mathcal{G}^\bullet|_U$.

(2) Write α as a roof $\mathcal{E}^\bullet \to \mathcal{G}^\bullet \xleftarrow{q} \mathcal{F}^\bullet$ where q is a qis. Applying (1) to q, we can locally lift against q to get $\mathcal{E}^\bullet|_U \to \mathcal{F}^\bullet|_U$ making the diagram commute, which is therefore a representative for α.

(3) A map becomes zero iff there is a roof $\mathcal{E}^\bullet \to \mathcal{G}^\bullet \xleftarrow{q} \mathcal{F}^\bullet$ with the first map being zero, commuting with α. Again by (a) we can locally lift against q, which means that α has to locally be 0. □

Now we are going to apply these results to m-pseudocoherent and perfect complexes. We'll first focus on m-pseudocoherence. First of all, as we have already explained, the condition in the definition of m-pseudocoherent on the map $\mathcal{E}^\bullet \to \mathcal{F}|_U$ is equivalent to the cone having no cohomology in degrees $\geq m$. We'll use this all the time.

Lemma 2.2.39. *Let X be a ringed space and $E \in D(\mathcal{O}_X)$.*

(a) Suppose that for each U in an open cover of X, there is a strictly perfect complex $\mathcal{E}^\bullet \in \mathrm{K}(\mathcal{O}_U)$ and a morphism $\mathcal{E}^\bullet \to E|_U$ whose cone is in $D^{<m}(\mathcal{O}_U)$. Then E is m-pseudocoherent. Moreover, any complex representing E is m-pseudocoherent.

(b) Suppose that for each U in an open cover of X, there is a strictly perfect complex $\mathcal{E}^\bullet \in \mathrm{K}(\mathcal{O}_U)$ such that $\mathcal{E}|_U$ is isomorphic in $D(\mathcal{O}_U)$ to $\mathcal{E}^\bullet$. Then E is perfect. Moreover, any complex representing E is perfect.

Proof. In either case, choose a representative $\mathcal{F}^\bullet$ for E, and consider the triangle $\mathcal{E}^\bullet \to \mathcal{F}^\bullet|_U$ in $D(\mathcal{O}_U)$. It suffices to show that $\mathcal{F}^\bullet$ is m-pseudocoherent or perfect. By Cor. 2.2.38, we may shrink U and assume $\mathcal{E}^\bullet \to \mathcal{F}^\bullet|_U$ is represented by a morphism of complexes, with the same hypothesis that either the cone is in $D^{<m}(\mathcal{O}_U)$ or it is 0. This is the definition of m-pseudocoherent or perfect, and the representative was arbitrary. □

Next, there are some obvious properties of pseudocoherence which we write down.

Lemma 2.2.40. *Let X be a ringed space and $E \in D(\mathcal{O}_X)$.*

(a) If E is m-pseudocoherent, then it is k-pseudocoherent for all $k \geq m$.
(b) If E is m-pseudocoherent, then $E[n]$ is $(m-n)$-pseudocoherent.
(c) If $E \in D^{<m}(\mathcal{O}_X)$, then E is m-pseudocoherent.
(d) If E is perfect, then it is pseudocoherent.

Proof. (a) and (d) are by definition. (b) Locally $\mathcal{E}^\bullet[n] \to E|_U[n] \to C[n] \to$ has $C[n] \in D^{<m-n}$ since $C \in D^{<m}$. (c) The triangle $0 \to E \to E \to 0$ has $E \in D^{<m}$. □

Remark 2.2.41. In the following discussion, we will use the following facts (that we have essentially already discussed) often. First, a left derived functor LF (in the presence of an adapted class) sends $D^{\leq m}(\mathcal{A}) \to D^{\leq m}(\mathcal{B})$. The proof is that if we choose a representative in $\mathrm{K}^{\leq m}(\mathcal{A})$, then we can choose the resolution in the adapted class to be in $\mathrm{K}^{\leq m}$ as well. Next, specializing to $\mathrm{Mod}(\mathcal{O}_X)$, the derived tensor product and $\mathbf{R}\mathcal{H}om$ preserve the following categories.

$$(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \colon D^{\leq a}(\mathcal{O}_X) \times D^{\leq b}(\mathcal{O}_X) \to D^{\leq a+b}(\mathcal{O}_X).$$

Indeed, we can choose a bounded above complex of flats in $\mathrm{K}^{\leq a}(\mathcal{O}_X)$ for the first variable, and a representative in $\mathrm{K}^{\leq b}$ for the second, then the tensor product lands in $\mathrm{K}^{\leq a+b}$. Similarly,

$$\mathbf{R}\mathcal{H}om \colon D^{\leq a}(\mathcal{O}_X)^{\mathrm{op}} \times D^{\geq b}(\mathcal{O}_X) \to D^{-a+b}(\mathcal{O}_X),$$

because choosing a bounded below complex of injectives in $\mathrm{K}^{\geq b}(\mathcal{O}_X)$ for the second variable and a representative in $\mathrm{K}^{\leq a}$ for the first, any $m > b - a$ satisfies either $p > a$ or $p < m + b$.

Lemma 2.2.42. *Let $A \to B \to C \to A[1]$ be a distinguished triangle in $D(\mathcal{O}_X)$.*

(a) If A is $(m+1)$-pseudocoherent and B is m-pseudocoherent, then C is m-pseudocoherent.
(b) If A and C are m-pseudocoherent, then B is m-pseudocoherent.
(c) If B is m-pseudocoherent and C is $(m-1)$-pseudocoherent, then A is m-pseudocoherent.

In particular, if any two of A, B, and C are pseudocoherent, then the third is pseudocoherent as well.

Proof. The conclusion follows immediately from the three statements, and observe that (b) and (c) follow from (a) by rotation and the fact that $A[1]$ is m-pseudocoherent iff A is $(m+1)$-pseudocoherent.

For (a), fix $x \in X$ so that on a nbhd U we have maps as in the solid part of the following diagram:

$$\begin{array}{ccccccc} \mathcal{F}^\bullet & \dashrightarrow & \mathcal{G}^\bullet & \dashrightarrow & \mathcal{H}^\bullet & \dashrightarrow & \mathcal{F}^\bullet[1] \\ \downarrow & & \downarrow & & \vdots & & \vdots \\ A|_U & \longrightarrow & B|_U & \longrightarrow & C|_U & \longrightarrow & A|_U[1] \end{array}$$

So, here $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ are strictly perfect, and the maps to $A|_U$ and $B|_U$ preserve cohomology in degrees above $(m+1)$ and m respectively, and are surjections on those degrees. By throwing out lower degree terms we may assume $\mathcal{F}^\bullet \in \mathrm{K}^{\geq m+1}$. Now represent B by a complex (but we use the same letter).

By further shrinking U, we may assume that the composite $\mathcal{F}^\bullet \to B|_U$ is represented by a morphism of complexes (Cor. 2.2.38(2)). Likewise, we may assume that $\mathcal{G}^\bullet \to B|_U$ is represented by a morphism of complexes by shrinking again. By the condition on $\mathcal{G}^\bullet \to B|_U$, the composite $\mathcal{F}^\bullet \to B|_U$ lifts to a map $\mathcal{F} \dashrightarrow \mathcal{G}$ by again shrinking U, by Cor. 2.2.38(1).

By the triangulated structure, this data can be completed to a morphism of triangles as shown in the diagram, where $\mathcal{H}^\bullet$ is strictly perfect as a cone of morphisms of strictly perfect complexes. Now consider the diagram of long exact sequences: the epimorphism 4-lemma shows that $H^m(\mathcal{H}^\bullet) \to H^m(C|_U)$ is surjective, and the 5-lemma shows that $H^p(\mathcal{H}^\bullet) \to H^p(C|_U)$ is an isomorphism for $p > m$. □

This is a useful lemma because often we will try to prove things by inducting along smart truncation triangles. Also, note that in particular, the pseudocoherent complexes form a full triangulated subcategory of $D(\mathcal{O}_X)$. The same is true for $\mathrm{Perf}^*(\mathcal{O}_X)$, though this is much easier; we reduce immediately to the case of strictly perfect complexes, and we already know they are closed under cones. Here is one application:

Lemma 2.2.43. *For $m \leq 0$, an $\mathcal{O}_X$-module $\mathcal{F}$ is m-pseudocoherent if and only if for each point there is a nbhd U and an exact sequence*

$$\mathcal{O}_U^{n_m} \to \mathcal{O}_U^{n_{m+1}} \to \cdots \to \mathcal{O}_U^{n-1} \to \mathcal{O}_U^{n_0} \to \mathcal{F}|_U \to 0$$

of finite free modules.

Proof. Since a finite free module is strictly perfect, such a local resolution implies m-pseudocoherence. We remark that the only reason this isn't trivial is because we may not be in the locally ringed case. Conversely, note that if $\mathcal{F}$ is 0-pseudocoherent then it admits a surjection from a strictly perfect (so finite type) module, hence is finite type. So 0-pseudocoherent is equivalent to finite type.

To complete the proof, suppose that there is a resolution as in the lemma statement of length m, and assume $\mathcal{F}$ is $(m-1)$-pseudocoherent. Split up into short exact sequences

$$0 \to \mathcal{G}^{i-1} \to \mathcal{O}_U^{n_i} \to \mathcal{G}^i \to 0$$

where $\mathcal{G}^0 = \mathcal{F}|_U$. It suffices to show that $\mathcal{G}^{m-1}$ is finite type, i.e. 0-pseudocoherent. Each sequence corresponds to a triangle in $D(\mathcal{O}_X)$, each $\mathcal{O}_U^{n_i}$ is $(-\infty)$-pseudocoherent, and $\mathcal{G}^0$ is $(m-1)$-pseudocoherent. Thus $\mathcal{G}^1$ is m-pseudocoherent, $\mathcal{G}^2$ is $(m+1)$-pseudocoherent, and so on, so $\mathcal{G}^{m-1}$ is 0-pseudocoherent. □

So, in particular, for modules 0-pseudocoherent is finite type, and (-1)-pseudocoherent is finite presentation.

Example 2.2.44. Let $A \in D^{\leq a}(\mathcal{O}_X)$ be m-pseudocoherent with $m \leq a$. Then in fact $H^a(A)$ is of finite type. Indeed, by assumption $\tau_{\leq a}A = A$, so we have an exact triangle

$$\tau_{<a}A \to A \to H^a(A)[-a] \to \tau_{<a}A[1].$$

Since $m \leq a$, A is a-pseudocoherent, and so is $\tau_{<a}A$, thus $H^a(A)[-a]$ is a-pseudocoherent, so $H^a(A)$ is 0-pseudocoherent.

Proposition 2.2.45. *Let $A \in D^{\leq a}(\mathcal{O}_X)$ and $B \in D^{\leq b}(\mathcal{O}_X)$.*

(a) If A and B are perfect, then $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$ is perfect.

(b) If A is m-pseudocoherent and B is n-pseudocoherent, then $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$ is $\max(a+n, b+m)$-pseudocoherent.

(c) Now let $f\colon X \to Y$ be a morphism of ringed spaces and $A \in D^-(\mathcal{O}_Y)$. If A is m-pseudocoherent (resp. perfect), then $Lf^*(A)$ is m-pseudocoherent (resp. perfect).

Proof. For (a), we may choose U in an open cover so that $A|_U$ and $B|_U$ are isomorphic in $D(\mathcal{O}_X)$ to strictly perfect complexes. Then by locality of tensor product, flatness of strictly perfect complexes, and the fact that strictly perfect complexes are preserved under tensor product (Lemma 2.2.36), we are done. For (c) in the perfect case, the same thing works as f^* preserves strictly perfect complexes.

For (c) in the m-pseudocoherent case, let $\mathcal{F}^\bullet$ be strictly perfect on U so that we have a triangle $\mathcal{F}^\bullet \to A|_U \to C \to \mathcal{F}^\bullet[1]$ with $C \in D^{<m}(\mathcal{O}_U)$. Applying Lf^* gives the distinguished triangle

$$f^*\mathcal{F}^\bullet \to Lf^*(A)|_U \to Lf^*C \to$$

and we conclude because $f^*\mathcal{F}^\bullet$ is strictly perfect on $f^{-1}(U)$, $Lf^*(C) \in D^{<m}(\mathcal{O}_{f^{-1}(U)})$.

Now do (b): we'll suppress open sets. We may also assume that $n \leq b$; if not, then $a+n > a+b$, and since $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D^{\leq a+b}$, it is automatically $(a+n)$ (hence $\max(a+n, b+m)$)-pseudocoherent. Locally on X, we have distinguished triangles $\mathcal{E}_1^\bullet \to B \to C_1 \to$ and $\mathcal{E}_2^\bullet \to A \to C_2 \to$ with $\mathcal{E}_1^\bullet$ and $\mathcal{E}_2^\bullet$ strictly perfect, and $C_1 \in D^{<m}$, $C_2 \in D^{<n}$. Since $n \leq b$, $C_2 \in D^{<b}$, so $\mathcal{E}_2^\bullet \in D^{\leq b}$.

By exactness of $\otimes^{\mathbf{L}}$, we get three distinguished triangles:

$$\mathcal{E}_1^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \to C_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \to$$
$$A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} C_2 \to$$
$$\mathcal{E}_1^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to C_3 \to$$

Here, $C_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \in D^{<m+b}$ and $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} C_2 \in D^{<a+n}$, so both are in $D^{<\max(a+n,m+b)}$. By the octahedral axiom, there is a distinguished triangle

$$C_1 \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet \to C_3 \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} C_2 \to,$$

so $C_3 \in D^{<\max(a+n,m+b)}$. But $\mathcal{E}_1^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{E}_2^\bullet = \mathcal{E}_1^\bullet \otimes \mathcal{E}_2^\bullet$ is strictly perfect by Lemma 2.2.36, so we are done. □

Now let us prove a lemma which will be the key method in which we actually use pseudocoherent and perfect objects of the derived category.

Lemma 2.2.46. *Let $\mathcal{F}^\bullet, \mathcal{G}^\bullet, \mathcal{H}^\bullet \in \mathrm{Ch}(\mathcal{O}_Y)$, and let $f\colon X \to Y$ be a morphism of ringed spaces.*

(a) There are canonical natural maps

$$\mathrm{ev}\colon \underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes \mathcal{F}^\bullet \to \mathcal{G}^\bullet$$
$$\underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes \mathcal{H}^\bullet \to \underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet \otimes \mathcal{H}^\bullet)$$
$$\mathcal{F}^\bullet \otimes \mathcal{H}^\bullet \to \underline{\mathcal{H}om}(\underline{\mathcal{H}om}(\mathcal{F}, \mathcal{O}_Y), \mathcal{H})$$
$$f^* \underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}(f^*\mathcal{F}^\bullet, f^*\mathcal{G}^\bullet)$$

all induced by the relevant adjunctions.

(b) If $\mathcal{G}^\bullet_{(-)}$ is a complex of filtered diagrams of sheaves, then there is a canonical map

$$\varinjlim_{i \in I} \underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}_i^\bullet) \to \underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \varinjlim_{i \in I} \mathcal{G}_i^\bullet)$$

where on either side we mean the termwise filtered colimit.

(c) Assume that $\mathcal{F}^\bullet$ is a bounded complex of finitely presented modules. Then the map of (b) is an isomorphism. If $\mathcal{H}^\bullet$ is a complex of flat sheaves, then second map of (a) is an isomorphism. If f is a flat morphism, then the fourth map of (a) is an isomorphism. In particular, the canonical map

$$\underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)_y \to \underline{\mathrm{Hom}}_{\mathcal{O}_{Y,y}}(\mathcal{F}_y^\bullet, \mathcal{G}_y^\bullet)$$

is an isomorphism for all $y \in Y$.

(d) If $\mathcal{F}^\bullet$ is strictly perfect, the second, third, and fourth maps of (a) are isomorphisms. In particular, $\mathcal{F}^\bullet$ is dualizable, meaning that the natural map

$$\mathcal{F}^\bullet \to \underline{\mathcal{H}om}(\underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{O}_Y), \mathcal{O}_Y)$$

is an isomorphism.

(e) If $\mathcal{F}^\bullet$ is bounded, and $\mathcal{H}^\bullet$ is strictly perfect, then the second map of (a) is an isomorphism.

Proof. (a) Recall that we have already shown the internal tensor hom adjunction for complexes. The external version of this (i.e. the actual adjunction) is obtained by taking zero-cycles. The counit of this adjunction is the first map (and it is also natural in $\mathcal{F}^\bullet$). Now take the first map and tensor both sides with $\mathcal{H}^\bullet$.

Then the tensor-hom adjunction produces the second map. For the third map, we have a natural map

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, f_* f^* \mathcal{G}^\bullet) \cong f_* \underline{\mathcal{H}om}_{\mathcal{O}_X}(f^*\mathcal{F}^\bullet, f^*\mathcal{G}^\bullet)$$

where the first one is induced by the unit map $\mathcal{G}^\bullet \to f_* f^* \mathcal{G}^\bullet$ of the standard pull-push adjunction on complexes. Therefore by the same adjunction this corresponds to the desired map. (b) just follows from the universal property applied termwise.

(c-e) Let us first prove all of these statements in the case of single sheaves, and then we'll generalize this to complexes. Throughout, note that if we have a natural transformation α of additive functors and $\alpha_{\mathcal{F}\oplus\mathcal{G}}$ is an isomorphism, then $\alpha_{\mathcal{F}}$ is an isomorphism. This is because e.g. we can consider the kernel and cokernel. Therefore, to prove the strictly perfect case for sheaves we usually only need to consider the finite free case.

First, consider the natural map

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} \to \mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{H}).$$

If $\mathcal{F} \cong \mathcal{O}_Y^n$, then this reduces to $\mathcal{G}^n \otimes \mathcal{H} \to (\mathcal{G} \otimes \mathcal{H})^n$ which is an isomorphism. Hence this works if $\mathcal{F}$ is strictly perfect too. Likewise, if $\mathcal{H} \cong \mathcal{O}_Y^n$, then the map reduces to $\mathcal{H}om(\mathcal{F}, \mathcal{G})^n \operatorname{Tor} \mathcal{H}om(\mathcal{F}, \mathcal{G}^n)$ which is also an isomorphism. So we're done if $\mathcal{H}$ is strictly perfect as well. Finally, let $\mathcal{F}$ be finitely presented and $\mathcal{H}$ be flat. Since the question is local, we may assume there is a presentation $\mathcal{O}_X^n \to \mathcal{O}_X^m \to \mathcal{F} \to 0$, then we obtain a commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{H}om(\mathcal{F}, \mathcal{G}) \otimes \mathcal{H} & \longrightarrow & \mathcal{G}^m \otimes \mathcal{H} & \longrightarrow & \mathcal{G}^n \otimes \mathcal{H} \\
 & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
0 & \longrightarrow & \mathcal{H}om(\mathcal{F}, \mathcal{G} \otimes \mathcal{H}) & \longrightarrow & (\mathcal{G} \otimes \mathcal{H})^m & \longrightarrow & (\mathcal{G} \otimes \mathcal{H})^n
\end{array}$$

The two right-hand maps are isomorphisms because tensor products commute with sums, so the left-hand one is.

Next consider the single sheaf case of the fourth map of (a). If $\mathcal{F} \cong \mathcal{O}_Y^n$, then this becomes the map $f^*(\mathcal{G}^n) \to (f^*\mathcal{G})^n$. This proves the statement when $\mathcal{F}$ is strictly perfect. Now assume f is flat and $\mathcal{F}$ is finitely presented. Just as above, we may assume that we have $\mathcal{O}_Y^n \to \mathcal{O}_Y^m \to \mathcal{F} \to 0$, and then applying either $f^* \mathcal{H}om(-, \mathcal{G}^\bullet)$ or $\mathcal{H}om(f^*(-), f^*\mathcal{G})$ preserves exactness by flatness and gives the result by the finite free case. In particular, if we take $f = \{y\} \hookrightarrow Y$, which is flat because taking stalks is exact, we get that $\mathcal{H}om(\mathcal{F}, \mathcal{G})_y = \mathcal{H}om(\mathcal{F}_x, \mathcal{G}_x)$ when $\mathcal{F}$ is finitely presented.

This last fact proves that (b) is an isomorphism in the case of sheaves with $\mathcal{F}$ finitely presented, because stalks commute with filtered colimits of sheaves, so we are reduced to the case of rings. But we have shown previously that finitely presented modules over a ring are compact. Note that this implies that $\mathcal{H}om(\mathcal{F}, -)$ commutes with arbitrary direct sums if $\mathcal{F}$ is finitely presented.

Finally, consider the third map of (a) in the sheaf case and assume $\mathcal{F}$ is finite free. Then it is just the map

$$\mathcal{H}^n \to \mathcal{H}om(\mathcal{O}_Y^n, \mathcal{H}) = \mathcal{H}om(\mathcal{O}_Y, \mathcal{H})^n$$

which is an isomorphism. Thus the isomorphism holds in the strictly perfect case, which completes all proofs in the case of individual sheaves.

It remains to reduce all claims to the sheaf case. Let's start with the second map of (a), and note that in all cases we are assuming $\mathcal{F}^\bullet$ is bounded, so the products in the hom complexes become sums. Consider the chain of isomorphisms:

$$\begin{aligned}
(\underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes \mathcal{H}^\bullet)^n &\cong \bigoplus_p \underline{\mathcal{H}om}^p_{\mathcal{O}_Y}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes \mathcal{H}^{n-p} \\
&\cong \bigoplus_{r,p} \mathcal{H}om(\mathcal{F}^r, \mathcal{G}^{r+p}) \otimes \mathcal{H}^{n-p} \\
&\cong \bigoplus_{r,p} \mathcal{H}om(\mathcal{F}^r, \mathcal{G}^{r+p} \otimes \mathcal{H}^{n-p}) \\
&\cong \bigoplus_r \mathcal{H}om(\mathcal{F}^r, \bigoplus_p \mathcal{G}^{r+p} \otimes \mathcal{H}^{n-p}) \\
&= \bigoplus_r \mathcal{H}om(\mathcal{F}^r, (\mathcal{G}^\bullet \otimes \mathcal{H}^\bullet)^{r+n}) \\
&= \underline{\mathcal{H}om}^n_{\mathcal{O}_Y}(\mathcal{F}^\bullet, G^\bullet \otimes \mathcal{H}^\bullet).
\end{aligned}$$

Here, in any of the three cases we are considering, the third step follows from the individual sheaf case. Now the fourth step: in two of the cases, $\mathcal{F}^r$ is finitely presented, so $\mathcal{H}om(\mathcal{F}^r, -)$ preserves arbitrary sums as explained. In the remaining case, $\mathcal{H}^\bullet$ is bounded, so there are only finitely many p for which $\mathcal{G}^{r+p} \otimes \mathcal{H}^{n-p}$ is nonzero.

Next consider the fourth map of (a). Again in the cases we consider, $\mathcal{F}^\bullet$ is bounded so the products turn into sums, and f^* commutes with sums. So termwise, we have

$$\begin{aligned}
f^* \underline{\mathcal{H}om}^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) &= \bigoplus_p f^* \mathcal{H}om(\mathcal{F}^p, \mathcal{G}^{n+p}) \\
&\cong \bigoplus_p \mathcal{H}om(f^*\mathcal{F}^p, f^*\mathcal{G}^{n+p}) \\
&\cong \underline{\mathcal{H}om}^n(f^*\mathcal{F}^\bullet, f^*\mathcal{G}^\bullet),
\end{aligned}$$

as desired, by the sheaf cases. The stalkwise consequence follows by taking f to be the inclusion of a point, as before.

For the map of (b), note that we reduce immediately to the sheaf case because finite direct sums commute with filtered colimits, so the Hom complex doesn't affect anything. All that remains is the third map of (a) when $\mathcal{F}^\bullet$ is strictly perfect.

Indeed,

$$\begin{aligned}(\mathcal{F}^\bullet \otimes \mathcal{H}^\bullet)^n &= \bigoplus_p \mathcal{F}^p \otimes \mathcal{H}^{n-p} \\ &\cong \bigoplus_p \mathcal{H}om(\mathcal{H}om(\mathcal{F}^p, \mathcal{O}_Y), \mathcal{H}^{n-p}) \\ &= \bigoplus_p \mathcal{H}om(\underline{\mathcal{H}om}^{-p}(\mathcal{F}^\bullet, \mathcal{O}_Y), \mathcal{H}^{n-p}) \\ &= \bigoplus_p \mathcal{H}om(\underline{\mathcal{H}om}^{p}(\mathcal{F}^\bullet, \mathcal{O}_Y), \mathcal{H}^{n+p}) \\ &= \underline{\mathcal{H}om}^n(\underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{O}_Y), \mathcal{H}^\bullet),\end{aligned}$$

as desired, where in the fourth step we just replaced p with $-p$. □

It is a good moment to quickly discuss the relationship between some of the conditions on sheaves we have been discussing. The upshot is that, in the algebro-geometric setting, considering objects which are locally isomorphic in the derived category to bounded complexes of finite locally free sheaves is really the most general nice situation we can get for computing $\mathbf{R}\mathcal{H}om$ in the first variable. Thus, hereafter, we will forget about the notion of internally projective sheaves.

Proposition 2.2.47. *Let X be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. We have the following implications: $\mathcal{F}$ locally strictly perfect $\implies$ $\mathcal{F}$ internally projective $\implies$ $\mathcal{F}$ stalkwise projective $\implies$ $\mathcal{F}$ flat. Moreover:*

(a) If $\mathcal{F}$ is of finite type, then the first implication may be reversed.
(b) If $\mathcal{F}$ is of finite presentation, then the second and third implications may be reversed as well.

Now suppose X is a locally ringed space. Then $\mathcal{F}$ finite locally free $\iff$ $\mathcal{F}$ locally strictly perfect. Moreover:

(a) If $\mathcal{F}$ is of finite type, then $\mathcal{F}$ stalkwise projective implies $\mathcal{F}$ stalkwise free.
(b) If $\mathcal{F}$ is of finite presentation, then $\mathcal{F}$ flat implies $\mathcal{F}$ locally free, i.e. all assertions are equivalent.

Proof. In the first part, we have proven all of the starting implications. If $\mathcal{F}$ is finite type and internally projective, then we can find a surjection $\mathcal{O}_U^n \to \mathcal{F} \to 0$ and thus an exact sequence

$$\mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{O}_U^n) \to \mathcal{H}om_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{F}|_U) \to 0.$$

Thus by further restricting U we can lift the identity section, which gives a local splitting. Next, suppose $\mathcal{F}$ is of finite presentation. Then since $\mathcal{H}om(\mathcal{F}, (-))_x \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, (-)_x)$, the RHS being an exact functor for all x implies that $\mathcal{H}om(\mathcal{F}, -)$ is an exact functor. If $\mathcal{F}$ is flat and finitely presented, then the same is true of $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$ for each x, which implies that $\mathcal{F}_x$ is projective.

Now let X be locally ringed. If $\mathcal{F}$ is finite locally free, then clearly it is locally strictly perfect. Since being locally strictly perfect implies being of finite presentation, proving (b) will prove the other direction. Note that (a) holds

because $\mathcal{F}_x$ is a finite projective module over a local ring for each x. For (b), we may assume $\mathcal{F}$ is stalkwise free.

Fix $x \in X$; choose a basis for $\mathcal{F}_x$ over $\mathcal{O}_{X,x}$, then by finiteness we can lift this to a map $\mathcal{O}_U^n \to \mathcal{F}|_U$ whose stalk at x is an isomorphism. Considering the cokernel $\mathcal{O}_U^n \to \mathcal{F}|_U \to \mathcal{G} \to 0$, $\mathcal{G}$ is of finite type so its support is closed, and $\mathcal{G}_x = 0$, so by shrinking U we may assume $\mathcal{O}_U^n \to \mathcal{F}|_U \to 0$ is surjective. Consider the kernel $\mathcal{K}$, which is of finite type since $\mathcal{F}$ is of finite presentation, hence its support is closed. Again $\mathcal{K}_x = 0$, so by shrinking U again $\mathcal{O}_U^n \to \mathcal{F}|_U$ becomes an isomorphism. □

Now we come to some applications of pseudocoherence and perfectness to derived notions. We will not do things in full generality right now, and instead will prefer to discuss perfectness in more detail until after we have talked about tor-dimension.

Proposition 2.2.48. *Let Y be a ringed space with $A \in D^-(\mathcal{O}_Y)$ and $B \in D^{\geq b}(\mathcal{O}_Y)$.*

(a) Assume that $\mathcal{F}^\bullet \in \mathrm{Ch}^{\geq a}(\mathcal{O}_Y)$ is a bounded complex of flat sheaves. There is a morphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, B) \otimes \mathcal{F}^\bullet \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, B \otimes_{\mathcal{O}_Y} \mathcal{F}^\bullet)$$

in $D(\mathcal{O}_Y)$ natural in A and B.

(b) Assume that $f\colon X \to Y$ is a flat morphism. Then there is a morphism

$$f^* \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(f^*A, f^*B)$$

natural in A and B.

(c) Let I be a filtered category. Then there is a morphism

$$\varinjlim^D_{i \in I} \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, B_i) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(A, \varinjlim^D_I B_\bullet)$$

natural in A and $B_\bullet \in D^+(\mathrm{Mod}(\mathcal{O}_Y)^I)$.

(d) Let A be m-pseudocoherent. Then the morphism of (a) induces an isomorphism on H^p for $p > m + a + b$ and a monomorphism on H^{m+a+b}. The morphism of (b) induces an isomorphism on H^p for $p > m + b$ and a monomorphism on H^{m+b}. In particular, if A is pseudocoherent both are isomorphisms.

Remark 2.2.49. We are going to ignore (c) for right now, but it's easy to prove with the same arguments as in (a) and (b). It is interesting to note that it implies, in the case e.g. of a qcqs scheme, that $\mathrm{RHom}_{\mathcal{O}_Y}(A, -)$ commutes with filtered colimits for A pseudocoherent. So for example in that case A is 'derived-compact'.

Proof. (a-b) Note that since $\mathcal{F}^\bullet$ is a bounded complex of flat sheaves, $(-)\otimes\mathcal{F}^\bullet$ preserves arbitrary quasi-isomorphisms. Likewise, the same holds for f^* as it is exact. We have a natural map

$$\mathbf{R}\mathcal{H}om(A, B) \otimes \mathcal{F}^\bullet \leftarrow \underline{\mathcal{H}om}(A, B) \otimes \mathcal{F}^\bullet \to \mathbf{R}\mathcal{H}om(A, B \otimes \mathcal{F}^\bullet)$$

where the second map factors through $\underline{\mathcal{H}om}(A, B \otimes \mathcal{F}^\bullet)$. When B is a complex of injectives the left map is an isomorphism, so we obtain the desired map. Similarly

we have

$$f^* \mathbf{R}\mathcal{H}om(A, B) \leftarrow f^*\underline{\mathcal{H}om}(A, B) \to \underline{\mathcal{H}om}(f^*A, f^*B) \to \mathbf{R}\mathcal{H}om(f^*A, f^*B),$$

and the left map is an isomorphism when $B \in \mathrm{K}^+(\mathcal{O}_Y)$

(d) Let's do e.g. (b), and the proof of (a) will be the same. The statement is local on Y, so by replacing f with $f^{-1}(U) \to U$, we may assume there is a triangle $\mathcal{E}^\bullet \to A \to C \to \mathcal{E}^\bullet[1]$ with $C \in D^{<m}(\mathcal{O}_Y)$ and $\mathcal{E}^\bullet$ is strictly perfect. By naturality, we get a morphism of distinguished triangles:

$$\begin{array}{ccccccc}
\longrightarrow & f^* \mathbf{R}\mathcal{H}om(C, B) & \longrightarrow & f^* \mathbf{R}\mathcal{H}om(A, B) & \longrightarrow & f^*\underline{\mathcal{H}om}(\mathcal{E}^\bullet, B) \\
& \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & \mathbf{R}\mathcal{H}om(f^*C, f^*B) & \longrightarrow & \mathbf{R}\mathcal{H}om(f^*A, f^*B) & \longrightarrow & \underline{\mathcal{H}om}(f^*\mathcal{E}^\bullet, f^*B)
\end{array}$$

Here we are using the fact that the pullback of strictly perfect complex is strictly perfect, and that a strictly perfect complex computes $\mathbf{R}\mathcal{H}om$ in the first variable. The right-hand map is an isomorphism by Lemma 2.2.46 for two separate reasons, and both of the objects in the left-hand map lie in $D^{<b+m}$. Therefore the 5-lemma and the monomorphism 4-lemma provide the result regarding the middle map. □

We record the important consequence that if A is m-pseudocoherent and $B \in D^{\geq b}$, then we get a natural isomorphism

$$H^p(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B))_x \xrightarrow{\sim} H^p(\mathrm{RHom}_{\mathcal{O}_{X,x}}(A_x, B_x))$$

for $p > m + b$. So $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B)_x = \mathrm{RHom}_{\mathcal{O}_{X,x}}(A_x, B_x)$ in the pseudocoherent case, which is important because on $\mathcal{O}_{X,x}$ we have enough projectives, so we can use alternative techniques to compute things there. Note in particular that for individual sheaves $\mathcal{F}, \mathcal{G}$, we have

$$\mathrm{Ext}^p_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})_x \xrightarrow{\sim} \mathrm{Ext}^p_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x)$$

for $p > m$, where $\mathcal{F}$ is m-pseudocoherent.

Before moving to tor-dimension, let us prove a few more results which allow us to relate pseudocoherence to having pseudocoherent terms or pseudocoherent cohomology.

Lemma 2.2.50. *A direct summand of an m-pseudocoherent object of $D(\mathcal{O}_X)$ is m-pseudocoherent.*

Remark 2.2.51. Here we will use that the direct sum of exact triangles is exact. This is true in any triangulated category, but in $\mathrm{K}^*(\mathcal{A})$ or $D^*(\mathcal{A})$ for an abelian category $\mathcal{A}$, this is especially easy because cones commute with direct sums of morphisms.

Proof. Suppose $A \oplus B = C$ is m-pseudocoherent. We have distinguished triangles $A \to A \to 0 \to A[1]$ and $B \xrightarrow{0} B \to B \oplus B[1] \to B[1]$, which add up to the triangle:

$$C \to C \to B \oplus B[1] \to C[1].$$

Thus $B \oplus B[1]$ is m-pseudocoherent. Applying this same result inductively to $B \oplus B[1]$ shows that $B[n] \oplus B[n+1]$ is m-pseudocoherent. Now consider the triangles

$$B[n] \to B[n] \oplus B[n-1] \to B[n-1] \to B[n+1]$$

coming from the corresponding split short exact sequence. Since this is local we may assume B is bounded above (since C is). Thus for $n \gg 0$, $B[n]$ is in $D^{<m}$ and hence is m-pseudocoherent. Thus $B[n-1]$ is m-pseudocoherent, and so inductively B is m-pseudocoherent. □

Lemma 2.2.52. *Let X be a ringed space.*

(a) If $\mathcal{F}^\bullet \in \mathrm{K}^-(\mathcal{O}_X)$ and $\mathcal{F}^i$ is $(m-i)$-pseudocoherent for all i, then $\mathcal{F}^\bullet$ is m-pseudocoherent.

(b) If $A \in D^-(\mathcal{O}_X)$ and $H^i(A)$ is $(m-i)$-pseudocoherent for all i, then A is m-pseudocoherent.

Proof. We prove (b), and (a) follows from the same trick but using stupid truncations instead of smart ones. For every $p \in \mathbb{Z}$, use the canonical truncation triangle

$$\tau_{\leq p-1}A \to \tau_{\leq p}A \to H^p(A)[-p] \to \tau_{\leq p-1}A[1].$$

By assumption, $H^p(A)[-p]$ is m-pseudocoherent for all p. Also, for $p \ll 0$, $\tau_{\leq p-1}A$ will also be m-pseudocoherent (e.g. for $p < m$). Thus at that point $\tau_{\leq p}A$ is m-pseudocoherent, and $\tau_{\leq p+1}A$ is as well, and so on. Now for $p \gg 0$, $A = \tau_{\leq p}A$ so we're done. □

Now we will discuss a nice condition on complexes which is that of having finite tor-dimension or having bounded tor amplitude. An object $A \in D(\mathcal{O}_X)$ has *tor amplitude in* $[a,b]$ if it is bounded above and $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} \in D^{[a,b]}(\mathcal{O}_X)$ for each $\mathcal{O}_X$-module $\mathcal{F}$. In particular, it must in fact be bounded. An object A has *finite tor dimension* if it has tor amplitude in $[a,b]$ for some a, b.

The good behavior of tor dimension comes from the following result.

Proposition 2.2.53. *Let X be a ringed space.*

(a) Let $\mathcal{F}^\bullet$ be a bounded above complex of flat $\mathcal{O}_X$-modules. If $\mathcal{F}^\bullet$ has tor-amplitude in $[a,b]$, then $\operatorname{cok}\left(\mathcal{F}^{a-1} \to \mathcal{F}^a\right)$ is flat. In particular, $\mathcal{F}^\bullet \to \tau_{\geq a}\mathcal{F}^\bullet$ is a quasi-isomorphism and $\tau_{\geq a}\mathcal{F}^\bullet$ is a bounded complex of flat modules.

(b) An object $E \in D(\mathcal{O}_X)$ has tor amplitude in $[a,b]$ if and only if it is represented by a complex of flat $\mathcal{O}_X$-modules in $\mathrm{Ch}^{[a,b]}(\mathcal{O}_X)$.

(c) Let $\mathcal{R}_{flat} \subset \mathrm{Mod}(\mathcal{O}_X)$ denote the preadapted class of flat $\mathcal{O}_X$-modules. Let $D^{[a,b]}_{\mathrm{tor}}(\mathcal{O}_X)$ and $D^b_{\mathrm{tor}}(\mathcal{O}_X)$ be the full subcategories of $D(\mathcal{O}_X)$ of objects with tor amplitude in $[a,b]$, and finite tor dimension respectively. Then the natural maps

$$\mathrm{K}^{[a,b]}(\mathcal{R}_{flat}) \to D^{[a,b]}_{\mathrm{tor}}(\mathcal{O}_X) \qquad \mathrm{K}^b(\mathcal{R}_{flat}) \to D^b_{\mathrm{tor}}(\mathcal{O}_X)$$

are triangulated localizations by the quasi-isomorphisms.

PROOF. By assumption, $\mathcal{F}^\bullet$ is exact in degrees $< a$. Therefore

$$\cdots \to \mathcal{F}^{a-2} \to \mathcal{F}^{a-1} \to \mathcal{F}^a \to \operatorname{cok}(d^{a-1}) \to 0$$

is a flat resolution of the cokernel, so $\mathcal{T}or^{-1}_{\mathcal{O}_X}(\operatorname{cok}(d^{a-1}), \mathcal{G})$ is isomorphic to the cohomology of

$$\mathcal{F}^{a-2} \otimes \mathcal{G} \to \mathcal{F}^{a-1} \otimes \mathcal{G} \to \mathcal{F}^a \otimes \mathcal{G}.$$

in the middle (for any $\mathcal{G}$). But

$$H^{a-1}(\mathcal{F}^\bullet \otimes \mathcal{G}) = H^{a-1}(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}) = 0$$

since $\mathcal{F}^\bullet$ has tor amplitude in $[a, b]$, so $\mathcal{T}or^{-1}_{\mathcal{O}_X}(\operatorname{cok}(d^{a-1}), \mathcal{G}) = 0$ for every $\mathcal{G}$. For the second statement, the map is a quasi-isomorphism because $\mathcal{F}^\bullet$ only has cohomology in degrees $\geq a$.

(b) One direction is obvious because a bounded complex of flats computes $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$. For the other direction, by (a) we may choose a complex of flats $\mathcal{F}^\bullet$ concentrated in $[a, \infty)$. Since it can't have cohomology in degrees $\geq b$, the sequence

$$0 \to \ker d^b \to \mathcal{F}^b \to \mathcal{F}^{b+1} \to 0$$

is exact, and thus $\ker d^b$ is flat. Thus $\tau_{\leq b}\mathcal{F}^\bullet \to \mathcal{F}^\bullet$ is a quasi-isomorphism and $\tau_{\leq b}\mathcal{F}$ has the desired properties.

(c) Let us prove the bounded case, which is the one that we'll actually use. Consider the functor $\mathrm{K}^b(\mathcal{R}_{\mathrm{flat}}) \to D^-(\mathcal{O}_X)$, which has essential image D^b_{tor} by (b). Thus we have to show it is fully faithful. Since $\mathcal{R}$ is preadapted, we already know that quasi-isomorphisms form a localizing class (compatible with the triangulation) in the source category, so we apply the criterion of Prop. 1.1.21.

Namely, we want to show that for any quasi-isomorphism $\mathcal{F}_1^\bullet \to \mathcal{G}^\bullet$ in $\mathrm{K}^-(\mathcal{R}_{\mathrm{flat}})$ with $\mathcal{F}_1^\bullet$ in $\mathrm{K}^b(\mathcal{R}_{\mathrm{flat}})$, there is a map $\mathcal{G}^\bullet \to \mathcal{F}_2^\bullet$ with $\mathcal{F}_2^\bullet \in \mathrm{K}^b(\mathcal{R}_{\mathrm{flat}})$ such that the composite is a quasi-isomorphism. Assume $\mathcal{F}_1^\bullet$ is concentrated in degrees $\geq a$, then the map $\mathcal{G}^\bullet \to \tau_{\geq a}\mathcal{G}^\bullet$ satisfies these hypotheses by (a). □

Remark 2.2.54 (experimental)**.** Take the unbounded tensor product functor $\mathrm{K}(\mathcal{O}_X) \times \mathrm{K}(\mathcal{O}_X) \to \mathrm{K}(\mathcal{O}_X)$, and restrict it to the two subcategories

$$\mathrm{K}(\mathcal{O}_X) \times \mathrm{K}^b(\mathcal{R}_{flat}) \to \mathrm{K}(\mathcal{O}_X) \qquad \mathrm{K}^b(\mathcal{R}_{flat}) \times \mathrm{K}(\mathcal{O}_X) \to \mathrm{K}(\mathcal{O}_X).$$

Now if $\mathcal{F}^\bullet \in \mathrm{K}^b(\mathcal{R}_{flat})$, then we know that $(-) \otimes \mathcal{F}^\bullet$ preserves quasi-isomorphisms of unbounded complexes, because in fact we showed this for all bounded above complexes of flat $\mathcal{O}_X$-modules. But now suppose $\mathcal{G}^\bullet$ is an unbounded complex, then we claim $\mathcal{G}^\bullet \otimes (-)$ preserves quasi-isomorphisms in $\mathrm{K}^b(\mathcal{R}_{flat})$. Note that it is closed under cones, so we may show it preserves acyclics. But both spectral sequences for a totalization apply here, because the corresponding bicomplex looks like either a vertical or horizontal strip. Thus taking cohomology of $\mathcal{G}^\bullet \otimes \mathcal{F}^n$ first shows that the totalization is acyclic. Therefore this descends to a bifunctor

$$(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-) \colon D(\mathcal{O}_X) \times D^b_{\mathrm{tor}}(\mathcal{O}_X) \to D(\mathcal{O}_X),$$

and likewise for the one in the other direction.

Thus, we can take derived tensor products when one of the factors has finite tor dimension; but we should be very careful using this because we haven't proven

many properties of it. However, in general, almost everything carries over with similar proofs as long as two unbounded complexes are not interacting with each other. For instance, it is easy to see from the corresponding properties that $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \cong B \otimes^{\mathbf{L}}_{\mathcal{O}_X} A$ if A has finite tor dimension (in the sense of the two functors above). Similarly,

$$(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \cong A \otimes^{\mathbf{L}}_{\mathcal{O}_X} (B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C)$$

if e.g. both B and C have finite tor dimension, so that both sides above make sense.

Remark 2.2.55. We note that in the real world (i.e. under Noetherianity hypotheses), pseudocoherence is usually not a real condition beyond finite-type/coherence properties. But having finite tor dimension *is* a subtler condition. For example, let $X = \operatorname{Spec} k[\epsilon]/(\epsilon)^2$ and $Z = V(\epsilon) \subset X$. We essentially showed earlier that $\mathcal{O}_Z$ has infinite tor dimension on X; let us demonstrate this again. Indeed, we have an infinite resolution

$$\cdots \to \mathcal{O}_X \xrightarrow{\epsilon} \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

where each map corresponds to multiplication by ϵ and the final map is the canonical quotient. The section ϵ is zero on Z, so

$$\begin{aligned} \mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_X &\simeq [\cdots \to \mathcal{O}_Z \xrightarrow{0} \mathcal{O}_Z \xrightarrow{0} \mathcal{O}_Z \to 0] \\ &= \bigoplus_{n \geq 0} \mathcal{O}_Z[n], \end{aligned}$$

i.e. $\mathcal{T}or^{-n}_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_Z) = \mathcal{O}_Z$ for all $n \geq 0$. Here, the point is that infinite tor dimension is a detector of singularities. We will see later that sheaves supported in the regular locus of a scheme tend to have finite tor dimension, and those whose support intersects the singular locus have infinite tor dimension. Thus, in general, we cannot expect sheaves to automatically have finite tor dimension unless we are working over a regular scheme.

Lemma 2.2.56. *Let $A \in D(\mathcal{O}_Y)$ have tor-amplitude in $[a, b]$.*

(a) For any $f\colon X \to Y$, $Lf^*(A)$ has tor-amplitude in $[a, b]$.

(b) For any $B \in D(\mathcal{O}_Y)$ with tor-amplitude in $[c, d]$, $A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B$ has tor-amplitude in $[a + c, b + d]$.

Proof. Represent A by a complex $\mathcal{E}^\bullet \in \mathrm{Ch}^{[a,b]}$ of flats. (a) Then $Lf^*(A)$ is represented by $f^*\mathcal{E}^\bullet$, which is still a complex of flats in $[a, b]$. (b) Represent B by a complex of flats $\mathcal{F}^\bullet$ in $[c, d]$, then $\mathcal{E}^\bullet \otimes \mathcal{F}^\bullet$ represents the derived tensor product, the totalization is concentrated in $[a + c, b + d]$ by definition, and the tensor product and direct sum of flat sheaves is flat. □

Let $f\colon X \to Y$ be a morphism of ringed spaces. We say f has tor-dimension $\leq n$ if $Lf^*(\mathcal{F}) \in D^{[-n,0]}(\mathcal{O}_X)$ for every $\mathcal{F} \in \mathrm{Mod}(\mathcal{O}_Y)$.

Lemma 2.2.57. *Let $A \in D(\mathcal{O}_X)$.*

(a) A has tor-amplitude in $[a, b]$ if and only if $A_x \in D(\mathcal{O}_{X,x})$ has tor-amplitude in $[a, b]$ for all x.

(b) If $f\colon X \to Y$ is a morphism, then f has tor-dimension $\leq n$ if and only if $\mathcal{O}_X$ has tor-amplitude in $[-n, 0]$ as an object of $D(f^{-1}(\mathcal{O}_Y))$, if and only if $\mathcal{O}_{X,x}$ has tor-amplitude in $[-n, 0]$ as an object of $D(\mathcal{O}_{Y,f(x)})$ for x.

PROOF. (a) One direction follows from (a) of the previous proposition. For the other, note that an object being in $D^{[a,b]}(\mathcal{O}_X)$ can be checked on stalks, and $\otimes^{\mathbf{L}}_{\mathcal{O}_X}$ commutes with taking stalks.

(b) The same proof as in (a) shows that f has tor dimension n iff $(Lf^*\mathcal{F})_x = \mathcal{F}_{f(x)} \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ is in $D^{[-n,0]}$ for every x and $\mathcal{F}$. The other condition in (b) says that $M \otimes^{\mathbf{L}}_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$ is in $D^{[-n,0]}$ for every x and $\mathcal{O}_{Y,f(x)}$-module M. Thus the question is whether or not every $\mathcal{O}_{Y,f(x)}$-module M is of the form $\mathcal{F}_{f(x)}$ for some sheaf $\mathcal{F}$. This is true because we can take the skyscraper sheaf of M at $f(x)$ and apply f^{-1}. □

Example 2.2.58. Let $i\colon Z \to X$ be a closed embedding on the underlying spaces. By the projection formula, $i_*\mathcal{O}_Z \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G} \cong i_*Li^*\mathcal{G}$ for every $\mathcal{G}$ on X. Thus $i_*\mathcal{O}_Z$ has tor amplitude in $[-n, 0]$ iff $i_*Li^*\mathcal{G}$ is in $D^{[-n,0]}(\mathcal{O}_X)$ for every $\mathcal{G}$, iff $Li^*\mathcal{G} \in D^{[n,0]}(\mathcal{O}_Z)$ for every $\mathcal{G}$. Thus the tor dimension of $i_*\mathcal{O}_Z$ coincides with the tor-dimension of i.

Lemma 2.2.59. *Let $A \to B \to C \to A[1]$ be a distinguished triangle in $D(\mathcal{O}_X)$*

(a) If A has tor-amplitude in $[a+1, b+1]$ and C has tor-amplitude in $[a, b]$, then B has tor-amplitude in $[a, b]$.
(b) If A and C have tor-amplitude in $[a, b]$, then B has tor-amplitude in $[a, b]$.
(c) If B has tor amplitude in $[a, b]$ and C has tor-amplitude in $[a-1, b-1]$, then A has tor-amplitude in $[a, b]$.

PROOF. Note that $(-)[1]$ takes $D^{[a,b]}$ to $D^{[a-1,b-1]}$. Thus it suffices by rotation to prove only one of the statements, and we'll do (b). For each sheaf $\mathcal{F}$,

$$A \otimes^{\mathbf{L}} \mathcal{F} \to B \otimes^{\mathbf{L}} \mathcal{F} \to C \otimes^{\mathbf{L}} \mathcal{F} \to A[1] \otimes^{\mathbf{L}} \mathcal{F}$$

is exact, and the first and third objects lie in $D^{[a,b]}$, so the middle one does too, completing the proof. □

Next we prove the main structure result relating perfect complexes to finite tor dimension. The main element is the following trivial observation: if $A \to B$ is a map in $D(\mathcal{A})$ and $B \in D^{\geq n}(\mathcal{A})$, then the map factors as $A \to \tau_{\geq n}A \to B$. Proof: Consider the canonical triangle $\tau_{\leq n-1}A \to A \to \tau_{\geq n}A \to$ and apply the cohomological functor $\mathrm{Hom}(-, B)$ to get an exact sequence

$$\mathrm{Hom}(\tau_{\geq n}A, B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(\tau_{\leq n-1}A, B) = 0$$

where the 0 is because $\tau_{\leq n-1}A \in D^{\leq n-1}$ and $B \in D^{\geq n}$.

Lemma 2.2.60. *Let $E \in D(\mathcal{O}_X)$ have tor amplitude in $[a, b]$. Then the following are equivalent:*

i. E is $(a-1)$-pseudocoherent.
ii. E is perfect.

iii. E is pseudocoherent.
iv. E is locally isomorphic to a strictly perfect complex concentrated in $[a, b]$.

PROOF. The obvious claims are $(iv) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$, so assume (i) and we prove (iv). The claim is local so assume there is a map $\mathcal{E}^\bullet \to E$ inducing isomorphisms on H^p for $p \geq a$ with $\mathcal{E}^\bullet$ strictly perfect and $\mathcal{E}^p = 0$ for $p < a$. Since $E \in D^{>a}$, this map factors as $\mathcal{E}^\bullet \to \tau_{\geq a}\mathcal{E}^\bullet \to E$ so the second map is an isomorphism. Moreover, by Prop. 2.2.53 $\tau_{\geq a}\mathcal{E}^\bullet$ is a complex of flats. By definition we have an exact sequence

$$\mathcal{E}^{a-1} \to \mathcal{E}^a \to (\tau_{\geq a}\mathcal{E}^\bullet)^a \to 0$$

which shows that $(\tau_{\geq a}\mathcal{E}^\bullet)^a$ is (-1)-pseudocoherent, i.e. finitely presented. Thus $(\tau_{\geq a}\mathcal{E}^\bullet)^a$ is locally strictly perfect (Prop. 2.2.47), so by restricting further $\tau_{\geq a}\mathcal{E}^\bullet$ is strictly perfect.

Now we know that E can be locally represented by a strictly perfect complex $\mathcal{E}^\bullet$ concentrated in $[a, \infty)$. In particular it is exact in degrees $> b$, so we have an exact sequence $0 \to (\tau_{\leq b}\mathcal{E}^\bullet)^b \to \mathcal{E}^b \to \mathcal{E}^{b+1} \to 0$, and hence $(\tau_{\leq b}\mathcal{E}^\bullet)^b$ is flat. Since $\mathcal{E}^b$ and $\mathcal{E}^{b+1}$ are pseudocoherent, it is also pseudocoherent, hence finitely presented, hence locally strictly perfect. So by restricting, $E \cong \tau_{\leq b}\mathcal{E}^\bullet$ becomes an isomorphism with a strictly perfect complex concentrated in $[a, b]$. □

Corollary 2.2.61. *An object $E \in D(\mathcal{O}_X)$ is perfect if and only if E is pseudocoherent and locally has finite tor dimension.*

PROOF. ($\Rightarrow$) Clear because E is locally isomorphic to a strictly perfect complex. ($\Leftarrow$) The question is local on X, and thus the claim follows from Lemma 2.2.60. □

Definition 2.2.62. If X is a ringed space, we define $\mathrm{Perf}^b(\mathcal{O}_X) \coloneqq \mathrm{Perf}(\mathcal{O}_X) \cap D^b_{\mathrm{tor}}(\mathcal{O}_X)$ to be the full subcategory of $D(\mathcal{O}_X)$ of perfect objects with finite tor dimension.

In particular, this is *not* the category of bounded perfect objects. However, in practice this is not a big restriction because e.g. if X is quasicompact, then $\mathrm{Perf}^b(\mathcal{O}_X) = \mathrm{Perf}(\mathcal{O}_X)$ by the above corollary.

Example 2.2.63. Perfect objects have the 2/3 property for distinguished triangles. Indeed, we showed pseudocoherent objects do and it holds for finite tor dimension too. Another property is that a direct summand of a perfect object is perfect, because we showed this for pseudocoherence, and it's obvious for finite tor dimension.

Now let's use this formalism to prove stuff. We showed earlier that if $C \in D^b_{\mathrm{tor}}(\mathcal{O}_X)$, then there is a natural map

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C),$$

and that if A is pseudocoherent then it is an isomorphism.

Now assume that $A \in \mathrm{Perf}^b(\mathcal{O}_X)$, so it has tor amplitude in $[a, b]$, and hence is locally a strictly perfect complex in $[a, b]$. It follows that $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X)$ is also

locally a strictly perfect complex in $[-b,-a]$, i.e. $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,\mathcal{O}_X) \in \mathrm{Perf}^b(\mathcal{O}_X)$. If (and only if) $A \in \mathrm{Perf}^b(\mathcal{O}_X)$, we define

$$A^\vee \coloneqq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,\mathcal{O}_X) \in \mathrm{Perf}^b(\mathcal{O}_X)$$

to be the *dual object of* A. Also, perfect objects (or just objects of $D^b_{\mathrm{tor}}(\mathcal{O}_X)$) have the property that tensoring with them sends $D^b(\mathcal{O}_X) \to D^b(\mathcal{O}_X)$, which we will also use. First, by the tensor-hom adjunction,

$$\mathrm{Hom}_{D(\mathcal{O}_X)}(A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} A, \mathcal{O}_X) \cong \mathrm{Hom}_{D(\mathcal{O}_X)}(A^\vee, A^\vee)$$

if $A \in \mathrm{Perf}^b(\mathcal{O}_X)$, which makes sense because of what we have just said. Thus we obtain a natural map $\mathrm{ev}\colon A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} A \to \mathcal{O}_X$ called the evaluation map. Now, for $B \in D^-(\mathcal{O}_X)$, this gives a natural map

$$A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \otimes^{\mathbf{L}}_{\mathcal{O}_X} A \to B$$

by associativity. If we assume $B \in D^b(\mathcal{O}_X)$, then the tensor-hom adjunction makes sense and gives a natural map

$$A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B).$$

We claim this is an isomorphism; indeed, on an open set where $A^\vee$ is strictly perfect, this reduces to Lemma 2.2.46(d). Now, if $C \in D^-(\mathcal{O}_X)$, consider the associativity isomorphism

$$(A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \xrightarrow{\sim} A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C).$$

Thus if we assume $B, C, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \in D^b(\mathcal{O}_X)$ to be safe, we obtain a natural isomorphism

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C).$$

E.g. if we assume $C \in D^b_{\mathrm{tor}}(\mathcal{O}_X)$ then this is a special case of what we showed before, but otherwise this is more general.

Now return to the evaluation map. Since $A^\vee$ is in $\mathrm{Perf}^b(\mathcal{O}_X)$, $(A^\vee)^\vee$ is also in $\mathrm{Perf}^b(\mathcal{O}_X)$, so the tensor-hom adjunction also gives the natural double-dual map

$$A \to (A^\vee)^\vee$$

corresponding to $A^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} A \to \mathcal{O}_X$. This is also an isomorphism: on an open set where A is a strictly perfect complex, the question reduces to the corresponding statement in Lemma 2.2.46(d). Thus perfect complexes are *dualizable* in $D(\mathcal{O}_X)$. Therefore, we have a natural isomorphism

$$A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \xrightarrow{\sim} (A^\vee)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A^\vee, B)$$

For any $B \in D^+(\mathcal{O}_X)$.

Let's play around with this some more. Under our hypotheses, as long as A and C are bounded, the natural map

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C)$$

is an isomorphism if $C \in \mathrm{Perf}^b(\mathcal{O}_X)$ (meaning we don't need to assume A is pseudocoherent). To see this, recall that we showed this in Lemma 2.2.46(e) for

C strictly perfect and underived, so the only issue is that $B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C$ may not be a bounded above complex of injectives even if C is perfect and B is a bounded above complex of injectives. The trick is that $B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \cong \mathbf{R}\mathcal{H}om(C^\vee, B)$, and $C^\vee \in \mathrm{Perf}^b$, so locally on X it restricts to $\mathcal{H}om(\mathcal{E}^\bullet, B|_U)$ where $\mathcal{E}^\bullet$ is strictly perfect. But since $\mathcal{E}^\bullet$ is a complex of flats, this is a complex of injectives. Therefore in fact $B \otimes^{\mathbf{L}}_{\mathcal{O}_X} C$ does compute $\mathbf{R}\mathcal{H}om$ after restricting to a small enough open set.

Now let's add in the sheafy derived tensor-hom adjunction. Assume that $A, B \in D^b(\mathcal{O}_X)$ and $E \in \mathrm{Perf}^b(\mathcal{O}_X)$. Then:

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee, B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} E$$

We sometimes like to write the first isomorphism in more perverse ways, like an inverse tensor-hom adjunction:

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, A), B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} B)$$

Next we investigate the consequences of the derived pull-push adjunction under the assumption of perfectness. First of all, let $f\colon X \to Y$ be a morphism and suppose $A \in D^b(\mathcal{O}_Y)$ is such that $Lf^*(A) \in D^b(\mathcal{O}_X)$ (for whatever reason, e.g. A or f have finite tor-dimension). Then by adjunction there is a canonical unit map $A \to Rf_*Lf^*(A)$ corresponding to the identity $Lf^*A \to Lf^*A$. In the special case that $A = \mathcal{O}_Y$, this gives a map $\mathcal{O}_Y \to Rf_*\mathcal{O}_X$, so

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E, \mathcal{O}_Y) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E, Rf_*\mathcal{O}_X) \cong Rf_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*(E), \mathcal{O}_Y)$$

for any $E \in D^-(\mathcal{O}_Y)$. Now suppose $E \in \mathrm{Perf}^b(\mathcal{O}_Y)$, so that the $\mathbf{R}\mathcal{H}om$'s on both sides are finite, since Lf^*E is also in $\mathrm{Perf}^b(\mathcal{O}_Y)$. Then this by adjunction induces a map

$$Lf^*(E^\vee) \to (Lf^*(E))^\vee$$

which we claim is an isomorphism. Indeed, look on an open nbhd where E is strictly perfect; then $Lf^*(E)$ and $\mathbf{R}\mathcal{H}om(E, \mathcal{O}_Y)$ are strictly perfect, so they compute $\mathbf{R}\mathcal{H}om$ and Lf^* respectively. Therefore this reduces to Lemma 2.2.46(d). Now it's easy to generalize this isomorphism: For any $B \in D^b(\mathcal{O}_Y)$ for which $Lf^*B \in D^b(\mathcal{O}_X)$, this means

$$\begin{aligned} Lf^* \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E, B) &\cong Lf^*(E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B) \\ &\cong Lf^*(E^\vee) \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(B) \\ &\cong Lf^*(E)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(B) \\ &\cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*(E), Lf^*(B)), \end{aligned}$$

extending the already-known case when f is flat.

Now, we finish this section with a discussion of the projection formula. Let $f\colon X \to Y$ be a morphism, and let $A \in D^b(\mathcal{O}_X)$ be an object such that $Rf_*(A) \in D^b(\mathcal{O}_Y)$. Then by adjunction there is a morphism $Lf^*Rf_*A \to A$ in $D(\mathcal{O}_X)$, so we get a morphism

$$Lf^*(Rf_*A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B) \xrightarrow{\sim} Lf^*Rf_*A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Lf^*B \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*B$$

for any $B \in D^-(\mathcal{O}_Y)$. Now assume $B = E \in \mathrm{Perf}^b(\mathcal{O}_Y)$, so that $Rf_*A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} E$ and $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*E$ are bounded. Thus we get by adjunction a morphism

$$Rf_*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} E \to Rf_*(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(E))$$

called the projection map. We claim it is an isomorphism. The proof is easy:

$$\begin{aligned} Rf_*A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} E &\cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E^\vee, Rf_*A) \\ &\cong Rf_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*E^\vee, A) \\ &\cong Rf_*(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*E) \end{aligned}$$

by the previous result. As a concrete application,

$$R\Gamma(X, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*E) = R\Gamma(Y, Rf_*A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} E)$$

on the level of sheaf cohomology.

Remark 2.2.64. The remark about tor-dimension also applies to derived pullback along morphisms $f\colon X \to Y$: f having finite tor dimension, and $Lf^*(B)$ being bounded for some B, is a delicate question relating to singularities. For example, if $p \in X$ is a point of a scheme, then $\operatorname{Spec} \kappa(p) \to X$ has finite tor-dimension essentially exactly when p is singular. However, asking for Rf_*A to be bounded is not a real condition; e.g. it's satisfied for any bounded A with X Noetherian, or any A with quasicoherent cohomology and f a qcqs morphism of schemes.

To conclude this section, we will summarize all of the results that we just proved for perfect objects.

Proposition 2.2.65. *Let $E \in \mathrm{Perf}^b(\mathcal{O}_X)$ and $A, B \in D^b(\mathcal{O}_X)$. Then we have the following canonical isomorphisms as described above:*

(a) $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E^\vee, B)$.
(b) $E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, B)$.
(c) $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, A) \otimes^{\mathbf{L}}_{\mathcal{O}_X} C \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B)$ *assuming* $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D^b(\mathcal{O}_X)$.
(d) $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} E \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} E)$.
(e) $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee, B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} B)$.

Assume that $f\colon X \to Y$ is a morphism, $E \in \mathrm{Perf}^b(\mathcal{O}_Y)$, $B \in D^b(\mathcal{O}_Y)$, $A \in D^b(\mathcal{O}_X)$. Then:

(a) $Lf^* \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E, B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*E, Lf^*B)$, *assuming* $Lf^*B \in D^b(\mathcal{O}_X)$.
(b) $Rf_*A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} E \cong Rf_*(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*E)$, *assuming* $Rf_*A \in D^b(\mathcal{O}_Y)$.

2.3. The classical theory

In this section, we will specialize our discussions above to the case where the ringed space X is a scheme. In the first section we will prove the foundational results which allow us to work properly with the derived category of a scheme. In the following sections, we will begin to do some basic algebraic geometry, however the most interesting results and applications of derived categories will need to wait until after an in-depth section on commutative algebra.

Let X be a scheme. In this case, we give the privileged notation $D^*(X) \coloneqq D^*(\mathcal{O}_X)$ to differentiate it from the case of a ringed space. We write $D^*_{\mathrm{qc}}(X) \subset D^*(X)$ to mean the full subcategory of objects A such that $H^p(A) \in \mathrm{QCoh}(X)$ for all $p \in \mathbb{Z}$. Likewise, $D^*_{\mathrm{coh}}(X) \subset D^*(X)$ means the full subcategory of $D^*(X)$ of objects A such that $H^p(A) \in \mathrm{Coh}(X)$ for all $p \in \mathbb{Z}$. We will have to work a bit to relate this to $D^* \mathrm{QCoh}(X)$ and $D^b \mathrm{Coh}(X)$, so we will care more about D_{qc} and D_{coh} for now.

At the very least, however, by the embedding of $\mathrm{Mod}(\mathcal{O}_X) \subset D(X)$ as H^0-objects, we do have a canonical embedding $\mathrm{QCoh}(X) \subset D^*_{\mathrm{qc}}(X)$ and $\mathrm{Coh}(X) \subset D^*_{\mathrm{coh}}(X)$. More generally, since the kernel and image of a map of quasicoherent sheaves is quasicoherent, a complex $\mathcal{F}^\bullet$ of quasicoherent sheaves has $\mathcal{F}^\bullet \in D_{\mathrm{qc}}(X)$. The inclusion $\mathrm{QCoh}(X) \to \mathrm{Mod}(\mathcal{O}_X)$ is also exact, so we do have a natural map $D^*(\mathrm{QCoh}(X)) \to D^*_{\mathrm{qc}}(X)$.

The categories $D^*_{\mathrm{qc}}(X) \subset D(X)$ form triangulated replete subcategories. Replete is clear, and closed under shifts is clear. If $A \to B \to C \to A[1]$ is an exact triangle with $A, B \in D^*_{qc}(X)$, then consider the exact sequence

$$H^n(A) \to H^n(B) \to H^n(C) \to H^{n+1}(A) \to H^{n+1}(B)$$

which we can split up into:

$$H^n(A) \to H^n(B) \to \mathcal{F} \to 0 \qquad 0 \to \mathcal{G} \to H^{n+1}(A) \to H^{n+1}(B) \to \mathcal{G} \to 0$$

and

$$0 \to \mathcal{F} \to H^n(C) \to \mathcal{G} \to 0.$$

The first sequence shows that $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent, and thus the second sequence shows that $H^n(C)$ is quasicoherent.

2.3.1. Cohomology of quasicoherent sheaves.

THEOREM 2.3.1. *Let X be an affine scheme and $\mathcal{F}$ a quasicoherent sheaf.*

(a) For an arbitrary cover $\mathcal{U}$ of X by quasicompact opens, $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

(b) $H^p(X, \mathcal{F}) = 0$ for all $p > 0$. More generally, if $\{\mathcal{F}_\alpha\}$ is a family of quasicoherent sheaves on X, then $H^p(X, \prod_\alpha \mathcal{F}_\alpha) = 0$ for all $p > 0$.

PROOF. (a) First assume that the cover $\mathcal{U}$ by quasicompact opens is finite. Then in the Čech resolution of Lemma 2.1.16

$$0 \to \mathcal{F} \to \prod_{\ell(I)=0} j_{I,*}\mathcal{F}|_{U_I} \to \prod_{\ell(I)=1} j_{I,*}\mathcal{F}|_{U_I} \to \cdots,$$

we can write each product as a direct sum. All opens in an affine scheme are separated, so for each I the inclusion $j_I\colon U \to X$ is in particular a qcqs map, so $j_{I,*}\mathcal{F}|_{U_I}$ is quasicoherent. Thus each term in the exact sequence above is quasicoherent, so taking global sections shows that $\mathcal{F}(X)[0] \to C^\bullet(\mathcal{U},\mathcal{F})$ is a quasi-isomorphism. We'll come back to the case that $\mathcal{U}$ is infinite momentarily.

(b) If $\{\mathcal{F}_\alpha\}$ is a family of quasicoherent sheaves on an affine scheme, then $\check{H}^p(\mathcal{U},\prod_\alpha \mathcal{F}_\alpha) = \prod_\alpha \check{H}^p(\mathcal{U},\mathcal{F}_\alpha) = 0$ for every (e.g.) finite affine open cover $\mathcal{U}$, by (a). Since X is separated, the affine opens form a basis of X which is closed under finite intersection. Therefore by Lemma 2.1.15 we conclude $H^p(X,\prod_\alpha \mathcal{F}_\alpha) = 0$ for all $p > 0$.

(a) Now we do the infinite case. Consider the same Čech resolution as written above; even though the terms aren't all quasicoherent, it is a resolution of $\mathcal{F}$ by products of quasicoherent sheaves, namely a Γ-acyclic resolution. Thus $H^*(X,\mathcal{F})$ coincides with the (complex) cohomology of $\Gamma(X,\check{\mathcal{C}}(\mathcal{U},\mathcal{F})) = \check{C}(\mathcal{U},\mathcal{F})$, i.e. $\check{H}^*(\mathcal{U},\mathcal{F})$, so we are done by (b). □

We say that a scheme X is *semiseparated* if the intersection of any two affine opens is affine, i.e. if the diagonal map $X \to X \times X$ is affine. Note that:

$$\text{separated} \implies \text{semiseparated} \implies \text{quasiseparated}$$

The first implication is clear, and the second follows from the fact that an affine morphism is quasicompact. We don't really care about the difference between semiseparated and separated schemes, but instead the terminology helps us remember how things are proved. We also sometimes say that a morphism of schemes is semiseparated if $f^{-1}(U)$ is semiseparated for every affine open U. In particular a separated morphism is semiseparated.

Corollary 2.3.2. *Let X be a scheme and $\mathcal{F}^\bullet$ a bounded below complex of quasicoherent sheaves.*

(a) If X is affine then the canonical map $\Gamma(X,\mathcal{F}^\bullet) \to R\Gamma(X,\mathcal{F}^\bullet)$ is an isomorphism in $D(\mathcal{O}_X(X))$.

(b) If $f\colon X \to Y$ is an affine morphism, then the natural map $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is an isomorphism in $D(Y)$. In particular, the natural map

$$R\Gamma(Y,f_*\mathcal{F}^\bullet) \to R\Gamma(Y,Rf_*\mathcal{F}^\bullet) = R\Gamma(X,\mathcal{F}^\bullet)$$

is an isomorphism in $D(\mathcal{O}_Y(Y))$.

(c) Suppose $\{U_i\}$ is an affine open cover of X such that $U_{i_0\cdots i_p}$ is affine for all indices $i_0,\dots,i_p$. If $\mathcal{F}$ is a quasicoherent sheaf, the natural map $\check{C}^\bullet(\{U_i\},\mathcal{F}) \to R\Gamma(X,\mathcal{F})$ is an isomorphism, so the natural maps $\check{H}^p(\{U_i\},\mathcal{F}) \to H^p(\{U_i\},\mathcal{F})$ are isomorphisms for all $p \geq 0$. More generally, this holds if $\mathcal{F}$ is replaced by an arbitrary product of quasicoherent sheaves.

Proof. (a) Then $\mathcal{F}^\bullet$ is a bounded below complex of Γ-acyclic sheaves. (b) To prove both claims it suffices to show that $R^pf_*\mathcal{F} = 0$ for $\mathcal{F}$ quasicoherent and

$p > 0$. Indeed, $R^p f_* \mathcal{F}$ is the sheafification of the presheaf

$$U \mapsto H^p(f^{-1}(U), \mathcal{F})$$

on Y, yet on the basis of affine opens this presheaf is zero because $f^{-1}(U)$ is still affine. (c) This follows from Thm. 2.1.11, because if $\prod_\alpha \mathcal{F}_\alpha$ is a product of quasicoherent sheaves, and U is affine, then $(\prod_\alpha \mathcal{F}_\alpha)|_U = \prod_\alpha \mathcal{F}_\alpha|_U$, so $H^p(U, \prod_\alpha \mathcal{F}_\alpha) = 0$ for all $p > 0$. □

Corollary 2.3.3. *Let X be a scheme and $A \in D^+(X)$.*

(a) There are natural maps $\mathbb{H}^p(X, A) \to H^0(X, H^p(A))$. If X is affine and $A \in D^+_{\mathrm{qc}}(X)$, then these are isomorphisms.

(b) If $f\colon X \to Y$ is a morphism of schemes, then there are natural maps $H^p(Rf_* A) \to f_* H^p(A)$. If f is affine and $A \in D^+_{\mathrm{qc}}(X)$, then these are isomorphisms.

Proof. Let $\mathcal{I}^\bullet$ be a bounded below complex of injectives. Then there is the natural sheafification unit map given by $H^p_{\mathrm{PSh}}(\mathcal{I}^\bullet) \to H^p(\mathcal{I}^\bullet)$, and hence $H^p(\mathcal{I}^\bullet(X)) \to H^p(\mathcal{I}^\bullet)(X)$. This induces the desired map. Likewise for (b), this construction gives a map $H^p(\mathcal{I}^\bullet(f^{-1}(U))) \to H^p(\mathcal{I}^\bullet)(f^{-1}(U))$ for all $U \subset Y$, and thus

$$H^p(f_* \mathcal{I}^\bullet) \to f_* H^p(\mathcal{I}^\bullet),$$

so this gives the desired map. To prove the isomorphisms, consider the two spectral sequences:

$$E_2^{p,q} = H^p(X, H^q(A)) \implies \mathbb{H}^{p+q}(X, A)$$
$$E_2^{p,q} = R^p f_*(H^q(A)) \implies H^{p+q} R f_*(A).$$

By cohomology vanishing for quasicoherent sheaves and the fact that the constructed maps are the edge maps of the spectral sequences we conclude. □

Note that in (c), if X is a semiseparated scheme, then every affine open cover satisfies this property. Thus for such schemes we may compute sheaf cohomology of quasicoherent sheaves using Cech cohomology with respect to an affine open cover. A useful observation is that an open $U \subset X$ of an affine scheme is, even if not affine, separated. Therefore we still have control over the higher terms in the Čech spectral sequence.

Lemma 2.3.4. *Let X be a scheme and $\{\mathcal{F}_\alpha\}$ a family of $\mathcal{O}_X$-modules.*

(a) If X is semiseparated and each $\mathcal{F}_\alpha$ is quasicoherent, then the natural map

$$H^p(X, \prod_\alpha \mathcal{F}_\alpha) \to \prod_\alpha H^p(X, \mathcal{F}_\alpha)$$

is an isomorphism for all p.

(b) Suppose $f\colon X \to Y$ is a quasicompact semiseparated morphism of schemes. If each $\mathcal{F}_\alpha$ is quasicoherent, then the natural map

$$R^p f_*(\prod_\alpha \mathcal{F}_\alpha) \to \prod_\alpha R^p f_*(\mathcal{F}_\alpha)$$

is an isomorphism for all p.

(c) If X is qcqs, then the natural map

$$\bigoplus_\alpha H^p(\mathcal{F}_\alpha) \to H^p(\bigoplus_\alpha \mathcal{F}_\alpha)$$

is an isomorphism for all p.

(d) If $f\colon X \to Y$ is a qcqs morphism of schemes, then the natural map

$$\bigoplus_\alpha R^p f_*(\mathcal{F}_\alpha) \to R^p f_*(\bigoplus_\alpha \mathcal{F}_\alpha)$$

is an isomorphism for all p.

Proof. (c), (d) We know these results because schemes have a basis of qcqs opens, and because of our result on cohomology and filtered colimits (Prop. 2.2.5).

(a) By Cor. 2.3.2(c), by choosing an affine open cover we may replace both sides by Čech cohomology. But then we are done because Čech cohomology commutes with arbitrary products.

For (b), note the extra quasicompactness. We have:

$$\begin{aligned} R^p f_*(\prod_\alpha \mathcal{F}_\alpha) &\cong \mathbf{a}(U \mapsto H^p(f^{-1}(U), \prod_\alpha \mathcal{F}_\alpha)) \\ &\cong \mathbf{a}(U \mapsto \prod_\alpha H^p(f^{-1}(U), \mathcal{F}_\alpha)) \\ &= \mathbf{a}(U \mapsto \prod_\alpha H^0(U, R^p f_* \mathcal{F}_\alpha)) \\ &= \mathbf{a}(U \mapsto H^0(U, \prod_\alpha R^p f_* \mathcal{F}_\alpha)) \\ &= \prod_\alpha R^p f_*(\mathcal{F}_\alpha) \end{aligned}$$

where we are considering sheafification over the basis of affine opens of Y. Here, the second step is by part (a) and the fact that $f^{-1}(U)$ is semiseparated, and the third step is by quasicoherence of higher direct images, which will be proven in Prop. 2.3.9. □

Remark 2.3.5. Recall that for (c) and (d), we actually know that cohomology and derived pushforward (in the qcqs cases) commute with all filtered colimits.

Let X be a scheme. Observe from the definition that X is semiseparated iff for each affine open U, the map $U \hookrightarrow X$ is an affine morphism. So, let $\mathcal{F}$ be a quasicoherent sheaf and consider the Čech resolution:

$$0 \to \mathcal{F} \to \prod_i j_{i,*} j_i^* \mathcal{F} \to \prod_{|I|=1} j_{I,*} j_I^* \mathcal{F} \to \cdots$$

If X is semiseparated, then $H^p(X, j_{I,*} j_i^* \mathcal{F}) = H^p(U_I, \mathcal{F}|_{U_I}) = 0$, so by our results above, the complex above is a resolution of $\mathcal{F}$ by Γ-acyclic objects. Now if X is quasicompact and semiseparated, then we may even assume that the open cover above is finite, so each term above after $\mathcal{F}$ is a quasicoherent Γ-acyclic sheaf. Thus this gives a functorial resolution by Γ-acyclic sheaves.

Lemma 2.3.6. *Let X be a quasicompact semiseparated scheme. Denote by Γ_{qc} the sections functor $\mathrm{QCoh}(X) \to \mathrm{Mod}(\mathcal{O}_X(X))$. Then the collection of Γ-acyclic quasicoherent sheaves is a right adapted class for Γ_{qc}, and hence the derived functor*

$$R\Gamma_{\mathrm{qc}} \colon D^+ \mathrm{QCoh}(X) \to D^+(\mathcal{O}_X(X))$$

exists and makes the diagram

$$\begin{array}{ccc} D^+ \mathrm{QCoh}(X) & \longrightarrow & D^+(X) \\ & \searrow^{R\Gamma_{\mathrm{qc}}} & \downarrow{\scriptstyle R\Gamma} \\ & & D^+(\mathcal{O}_X(X)) \end{array}$$

commutes.

Proof. We just showed above that upon choosing a finite affine cover, the first map in the Čech resolution gives an embedding of any quasicoherent sheaf into a Γ-acyclic quasicoherent sheaf. Both the Γ-acyclics and quasicoherent sheaves are closed under finite direct sums. If $\mathcal{F}^\bullet$ is a bounded below complex of quasicoherent Γ-acyclics, then of course applying Γ_{qc} gives an acyclic because $\Gamma_{\mathrm{qc}} = \Gamma$. This proves the first claim. Finally, since the inclusion functor $D^+ \mathrm{QCoh}(X) \to D^+(X)$ sends the adapted class to Γ-acyclic sheaves we are done. □

Next we discuss boundedness of cohomology, i.e. the fact that for the schemes we care about, all quasicoherent sheaves have cohomology vanishing in large enough degree.

Lemma 2.3.7. *Let X be a scheme.*

- *(a) If X is Noetherian of dimension d, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and all $\mathcal{O}_X$-modules $\mathcal{F}$.*
- *(b) If X is quasicompact semiseparated, then $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and quasicoherent sheaves $\mathcal{F}$, where $d+1$ is the minimal size of an affine open cover.*
- *(c) If X is qcqs, then there exists d such that $H^p(X, \mathcal{F}) = 0$ for all $p > d$ and all quasicoherent sheaves $\mathcal{F}$.*

Assuming the number d as in all three cases above, $R\Gamma$ maps $D_{\mathrm{qc}}^{[a,b]}(X)$ to $D^{[a,b+d]}(\mathcal{O}_X(X))$.

Proof. (a) is known from Thm. 2.2.29. (b) Note that if we use the ordered Čech complex for a cover of size $d+1$, then $\check{C}^p_{\mathrm{ord}}(\mathcal{U}, \mathcal{F}) = 0$ for $p \geq d+1$, so the result follows from Čech cohomology. (c) Consider the Čech spectral sequence

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \implies H^{p+q}(X, \mathcal{F})$$

for a finite affine cover $\mathcal{U}$. For $p \gg 0$ and all q, $E_2^{p,q} = 0$ (e.g. once p exceeds the size of the cover). For each $i_0, \ldots, i_p$, $U_{i_0 \cdots i_p}$ is a quasicompact subscheme in an affine scheme, hence quasicompact separated. Hence by (b), for fixed p, the presheaf $\underline{H}^q(\mathcal{F})$ for $q \gg 0$ gives zero when evaluated on all $U_{i_0 \cdots i_p}$. This means that $E_2^{p,q}$ is contained inside some large rectangle, and so the spectral sequence above gives the result.

For the final statement, if $A \in D_{\mathrm{qc}}^{[a,b]}(X)$, then consider the hypercohomology spectral sequence

$$H^p(X, H^q(A)) \implies \mathbb{H}^{p+q}(X, A).$$

Then p is concentrated in $[0, d]$, and q in $[a, b]$, so the resulting diagonals have nonzero range in $[a, b + d]$. □

Now we come to the result on higher pushforwards. The proof is simple, and to simplify it even more one can use the following observation:

Remark 2.3.8. Suppose $E_{r_0}^{p,q} \implies H^{p+q}$ is a bounded spectral sequence of sheaves converging to a bounded filtration on an object H^{p+q}. Assume that $E_{r_0}^{p,q}$ is quasicoherent for all p, q. Then in fact $E_r^{p,q}$ is quasicoherent for all $r \geq r_0$, and H^n is quasicoherent for all n. Indeed, the first claim is inductively clear: the kernel and image of a morphism of quasicoherent sheaves is quasicoherent, and quasicoherent sheaves have the 2-out-of-3 property. Now, we have a finite filtration $H^n = F^a H^n \supset F^{a+1} H^n \cdots \supset F^b H^n = 0$ such that

$$0 \to F^{p+1} H^n \to F^p H^n \to E_\infty^{p,n-p} \to 0$$

is exact. By boundedness, $E_\infty^{p,n-p}$ is quasicoherent, so if $F^{p+1} H^n$ is quasicoherent then $F^p H^n$ is as well. By induction, $F^a H^n = H^n$ is quasicoherent.

Proposition 2.3.9. *Let $f \colon X \to Y$ be a qcqs morphism of schemes. For $A \in D_{\mathrm{qc}}^+(X)$, we have $Rf_*(A) \in D_{\mathrm{qc}}^+(Y)$. In particular, for each affine open $U \subset Y$, the natural map*

$$\mathbb{H}^p(f^{-1}(U), A) \to H^0(U, H^p(Rf_* A))$$

is an isomorphism of $\mathcal{O}_Y(U)$-modules for all p.

Proof. We may assume $A = \mathcal{F}$ is a quasicoherent sheaf, because in general this case and the hypercohomology spectral sequence for pushforwards, i.e.

$$R^p f_*(H^q(A)) \implies H^{p+q} Rf_*(A),$$

along with Remark 2.3.8 imply that $Rf_* A$ has quasicoherent cohomology. We will apply the spectral sequence of Prop. 2.1.17 associated to a finite cover $\{U_i\}$ on X:

$$E_1^{p,q} = \bigoplus_{|I|=p} R^q f_{I,*} \mathcal{F}|_{U_I} \implies R^{p+q} f_* \mathcal{F}$$

Since the claim is local on Y, we may assume Y is affine, and hence X is qcqs, so we can take an affine open cover and consider the spectral sequence.

Now if X was semiseparated, then each morphism f_I would be a map of affine schemes, so $R^q f_{I,*} \mathcal{F}|_{U_I} = 0$ for $q > 0$ and $R^0 f_{I,*} \mathcal{F}|_{U_I} = f_{I,*} \mathcal{F}$. Thus in this case it follows from Remark 2.3.8 that $R^p f_* \mathcal{F}$ is quasicoherent for all p. Now, if X is only qcqs, then each map $f_I \colon U_I \to Y$ is qcqs out of a quasicompact semiseparated scheme, and thus $R^q f_{I,*} \mathcal{F}|_{U_I}$ is quasicoherent by the first case. Thus we conclude again by Remark 2.3.8.

Now to establish the second claim, we apply Cor. 2.3.3: Since U is affine and $Rf_* A|_U = Rf_*(A|_{f^{-1}(U)}) \in D_{\mathrm{qc}}^+(U)$,

$$\mathbb{H}^p(f^{-1}(U), A) = \mathbb{H}^p(U, Rf_* A) \to H^0(U, H^p(Rf_* A))$$

is an isomorphism. □

Example 2.3.10. Let X be a qcqs A-scheme and $p\colon X \to \operatorname{Spec} A$ be the structure map. If $\mathcal{F}$ is a quasicoherent sheaf, we can consider $M = H^n(X, \mathcal{F})$ as an A-module. Then the above result says that $R^n p_* \mathcal{F} = \widetilde{M}$. We will give a more functorial way to view this shortly.

Here is another example of this technique.

Lemma 2.3.11. *Let $f\colon X \to Y$ be a qcqs morphism. If Y is quasicompact, then there exists $d \geq 0$ such that $R^p f_*(\mathcal{F}) = 0$ for all quasicoherent $\mathcal{F}$ and $p > d$.*

Proof. By replacing Y with elements of a finite affine open cover we may assume Y is affine. We again use the spectral sequence

$$E_1^{p,q} = \bigoplus_{|I|=p} R^q f_{I,*} \mathcal{F}|_{U_I} \implies R^{p+q} f_* \mathcal{F}$$

associated to some finite affine open cover $\{U_i\}$ of X, but this time its alternating version, so that $E_1^{p,q} = 0$ once p exceeds the size of the cover. If X is semiseparated, then each $f_I\colon U_I \to Y$ is a morphism of affine schemes hence affine, so terms with $q > 0$ are zero, proving the result in this case. In the general case, note that $f_I\colon U_I \to Y$ is a morphism between quasicompact separated schemes, so for each of the finitely many columns with nonzero entries, vertical entries are zero for q large enough, independent of $\mathcal{F}$. Therefore the spectral sequence is contained in a rectangle with size independent of $\mathcal{F}$ and we are done. □

In the remainder of this section, we want to show that the derived category constructions of the previous section are affine local, meaning that for an affine open they just give the corresponding construction in the derived category of the ring. At the very end, we will show that this holds for the actual derived category itself, with some difficulty.

Example 2.3.12. Note that if $Y = \operatorname{Spec} A$ is affine, then an A-module M is flat if and only if $\widetilde{M}$ is a flat sheaf over $\mathcal{O}_Y$. This is because flatness is checked stalkwise in both cases, and both have the same stalks as $A_{\mathfrak{p}}$-modules. Therefore, given a module M, we can choose a flat resolution of M by quasicoherent sheaves; so if $f\colon X \to Y$ is any morphism, $Lf^*(\widetilde{M})$ can be represented by a complex of quasicoherent sheaves, and hence is in $D_{\mathrm{qc}}^-(X)$. So, if $f\colon X \to Y$ is an arbitrary morphism of schemes, and $\mathcal{F}$ is a quasicoherent sheaf on Y, then by locality of derived pullback, $Lf^*(\mathcal{F}) \in D_{\mathrm{qc}}^-(X)$.

Next suppose $A \in D_{\mathrm{qc}}^b(Y)$. We could use the canonical truncations

$$\tau_{\leq n-1} A \to \tau_{\leq n} A \to H^n(A)[-n] \to \tau_{\leq n} A[1]$$

and induct starting at the bottom nonzero cohomology group of A: then since $D_{\mathrm{qc}}(X)$ is triangulated, induction will show that $Lf^*(\tau_{\leq n} A) \in D_{\mathrm{qc}}(X)$ for all n, and hence $Lf^* A \in D_{\mathrm{qc}}(X)$.

Finally, assume $A \in D^-_{\mathrm{qc}}(Y)$ is possibly unbounded. For each $n \in \mathbb{Z}$ we have the canonical truncations

$$\tau_{\leq n-1}A \to A \to \tau_{\geq n}A \to \tau_{\leq n-1}A[1],$$

so applying Lf^* gives a new triangle. Note that $Lf^*(\tau_{\leq n-1}A) \in D^{\leq n-1}(X)$, so $H^p(Lf^*(A)) \xrightarrow{\sim} H^p(Lf^*(\tau_{\geq n}A))$ for $p \geq n$. But $\tau_{\geq n}A$ is in $D^b_{\mathrm{qc}}(Y)$, so $H^p(Lf^*A)$ is quasicoherent for $p \geq n$. Since n was arbitrary we are done. Note that in this argument, we couldn't have naively applied a spectral sequence here because we don't have enough projectives.

Example 2.3.13. We *can* get away with using spectral sequences in the case of derived tensor products. First, the same argument as before by using a flat resolution on affine opens, shows that if $\mathcal{F}$ and $\mathcal{G}$ are quasicoherent, then $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}$ has quasicoherent cohomology, i.e. $\mathcal{T}or^{-n}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasicoherent for all n. Now recall the hyper-tor spectral sequence (Example 2.2.11):

$$H^p(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(B)) \implies H^{p+q}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B)$$

First, if $A = \mathcal{F}$ is quasicoherent and $B \in D^-_{\mathrm{qc}}(X)$, then this shows that $\mathcal{F} \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D^-_{\mathrm{qc}}(X)$. By commutativity, we can reverse the logic: if $A, B \in D^-_{\mathrm{qc}}(X)$, then $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} H^q(B) \in D^-_{\mathrm{qc}}(X)$ by the first case, so $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D^-_{\mathrm{qc}}(X)$ by the spectral sequence. We remark that everything still works if we take one of the arguments to have finite tor dimension and the other one to be unbounded.

So, in the last two examples we have shown:

Lemma 2.3.14. *Let $A, B \in D^-_{\mathrm{qc}}(Y)$, and let $f\colon X \to Y$ be a morphism. Then $A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B \in D^-_{\mathrm{qc}}(Y)$, and $Lf^*(A) \in D^-_{\mathrm{qc}}(X)$.*

Next we will give another example, which involves pseudocoherence. Observe that if $A \in D(X)$ is m-pseudocoherent, then $H^p(A)$ is locally isomorphic to $H^p(\mathcal{E}^\bullet)$ for $\mathcal{E}^\bullet$ a strictly perfect complex for $p > m$. Hence $H^p(A)$ is quasicoherent for $p > m$, so if A is pseudocoherent then $A \in D_{\mathrm{qc}}(X)$.

Lemma 2.3.15. *Let $A \in D^-(X)$ be pseudocoherent and $B \in D^+_{\mathrm{qc}}(X)$. Then $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \in D^+_{\mathrm{qc}}(X)$.*

Proof. First consider the case $A = \mathcal{E}^\bullet$ of a strictly perfect complex, which is in particular bounded with each term finite locally free. Since the claim is local, we may assume that each term of $\mathcal{E}^\bullet$ is finite free; choose $B = \mathcal{F}^\bullet$, then we claim $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$ has quasicoherent cohomology. Since $\mathcal{E}^\bullet$ is bounded, all products in the totalization are sums, and we can consider the corresponding spectral sequence, which gives

$$E_1^{p,q} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-p}, H^q(\mathcal{F}^\bullet)) \implies H^{p+q}(\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet))$$

since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^{-p}, -)$ is exact. Then the LHS is of the form $H^q(B)^n$ since $\mathcal{E}^{-p}$ is finite free, so $E_1^{p,q}$ is quasicoherent for all p, q. Thus the RHS is quasicoherent.

Now we do the general case, so again since the claim is local, consider an exact triangle of the form $\mathcal{E}^\bullet \to A \to C \to \mathcal{E}^\bullet[1]$ with $C \in D^{<m}$ and $\mathcal{E}^\bullet$ strictly perfect. Then

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet,B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(C,B)[1] \to$$

is exact. If $B \in D^{\geq b}$, then $\mathbf{R}\mathcal{H}om(C,B)[1] \in D^{>b-m-1}$. Thus right-hand map above is an isomorphism on H^p for $p \leq b-m-1$. In particular, $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B)$ has quasicoherent cohomology in degrees $\leq b-m-1$. Since this holds for every m, we are done. □

Remark 2.3.16. We can extract a slightly more specific statement from the proof of the lemma. Namely, let $A \in D^-(X)$ be m-pseudocoherent and $B \in D_{\mathrm{qc}}^{\geq b}(X)$, then what we really showed is that $H^p(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B))$ is quasicoherent for all $p \leq b-m-1$, i.e. $\tau_{\leq b-m-1}\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A,B) \in D_{\mathrm{qc}}(X)$. So, e.g., if $\mathcal{F}$ is $(-d)$-pseudocoherent and $\mathcal{G}$ is quasicoherent, then

$$\mathcal{E}xt^0_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}),\ldots,\mathcal{E}xt^{d-1}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

are all quasicoherent sheaves.

Now we come to an important result, which is the fact that the bounded below derived category is affine local. Let $X = \operatorname{Spec} R$, and recall that we have an adjunction

$$\widetilde{(-)}\operatorname{Mod}(R) \rightleftarrows \operatorname{Mod}(\mathcal{O}_X) : \Gamma(X,-)$$

where $\widetilde{(-)}$ is an exact functor. Therefore, $\widetilde{(-)}$ extends to $D^*(R) \to D^*(X)$, and by Remark 1.3.21 we obtain a derived adjunction

$$(*) \qquad \widetilde{(-)}\colon D^+(R) \rightleftarrows D^+(X) : R\Gamma(X,-)$$

which comes with a unit and counit map. Since the original map factors through $\operatorname{QCoh}(X)$, the left-hand functor factors through $D^+_{\mathrm{qc}}(X)$.

Proposition 2.3.17. *Let R be a ring and $X = \operatorname{Spec} R$. The adjunction $(*)$ restricts to an equivalence*

$$\widetilde{(-)}\colon D^+(R) \rightleftarrows D^+_{\mathrm{qc}}(X) : R\Gamma(X,-)$$

of triangulated categories.

Proof. First consider the unit of $(*)$, which is the map $A \to R\Gamma(X,\widetilde{A})$. We claim this is an isomorphism; indeed, writing $A = M^\bullet$ this map is given by

$$M^\bullet \to \Gamma(X,\widetilde{M^\bullet}) \to R\Gamma(X,\widetilde{M^\bullet}),$$

and the first map is an isomorphism since $\operatorname{Mod}(R) \to \operatorname{Mod}(\mathcal{O}_X)$ is a fully faithful adjoint, and the second map is an isomorphism since $\widetilde{M^\bullet}$ is a bounded below complex of quasicoherent sheaves. Now consider the counit

$$\widetilde{R\Gamma(X,B)} \to B \qquad B \in D^+(\mathcal{O}_X)$$

and restrict B to lie in $D^+_{\mathrm{qc}}(X)$. We may show it is an isomorphism after applying H^p, giving the map

$$\widetilde{\mathbb{H}^p(X,B)} \to H^p(B).$$

Since both sides are quasicoherent, this reduces to showing $\mathbb{H}^p(X, B) \to H^0(X, H^0(B))$ is an isomorphism, but this is exactly the isomorphism of Cor. 2.3.3(a). □

It is in fact true that even the unbounded version $(\widetilde{-})\colon D(R) \to D_{\mathrm{qc}}(X)$ is an equivalence, but its proof is much harder since we haven't set up unbounded derived category theory. We will anyway give a self-contained proof of this in an appendix to this section by introducing some ideas.

But, we anyway use this result to convert local questions about the derived functors to commutative algebra. For example, this implies that if $p\colon X \to \operatorname{Spec} A$ is the structure map of a qcqs A-scheme, then for $A \in D^+_{\mathrm{qc}}(X)$, Rp_*A is just $(\widetilde{-})$ applied to $R\Gamma(X, A)$. Another example is the following: If R is a ring, $A \in D^-(R)$, $B \in D^+(R)$, and suppose $\widetilde{A}$ is pseudocoherent. Then the natural map

$$\widetilde{\mathrm{RHom}_R(A, B)} \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\widetilde{A}, \widetilde{B})$$

is an isomorphism, where $X = \operatorname{Spec} R$. Indeed, since we know the RHS is in $D^+(X)$, we may show this after applying $R\Gamma$, and then after applying H^i, from which we obtain

$$\mathrm{Hom}_{D(R)}(A, B[i]) \to \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\widetilde{A}, \widetilde{B}[i]).$$

This follows from our result if A is bounded, and will follow from the unbounded result once we prove it. We are also interested in the corresponding results for tensor product and derived pullback. To get this, we will want to apply the functor from the proof of the unbounded case, so we will prove this case later.

2.3.2. Unbounded right derived functors on schemes. The point of this section is to extend the result of the previous section to show that $D(R) = D_{\mathrm{qc}}(\operatorname{Spec} R)$. The idea behind the proof is to try to construct an inverse by extending $R\Gamma$ to all of $D_{\mathrm{qc}}(\operatorname{Spec} R)$, which we will do by taking a derived limit of the smart truncations $\tau_{\geq -n}A$ of an object as $n \to \infty$. It will therefore require a short detour into the theory of derived limits of inverse systems.

Let $\mathcal{D}$ be a triangulated category and consider an *inverse system* (K_n) of objects, which means a countable sequence

$$K_1 \xleftarrow{f_1} K_2 \xleftarrow{f_2} K_3 \to \cdots$$

of objects and morphisms as above. Suppose that the product $\prod_i K_i$ exists in $\mathcal{D}$; then this system induces a canonical map $s\colon \prod_i K_i \to \prod_i K_i$ so that the nth component $\prod_i K_i \to K_n$ is given by $\prod_i K_i \to K_{n+1} \xrightarrow{f_n} K_n$. The *derived limit of the system* is an object K of $\mathcal{D}$, equipped with an exact triangle

$$K \to \prod_i K_i \xrightarrow{1-s} \prod_i K_i \to K[1]$$

in $\mathcal{D}$. In this case, we write $K = \mathbf{R}\varprojlim_i K_i$, and we view K to be equipped with the maps $K \to \prod_i K_i \to K_i$.

Remark 2.3.18. Here's why we make this definition. Suppose (K_n) is an inverse system in $\mathrm{Mod}(R)$, then it is easy to check from the universal property that $\varprojlim_i K_i$ is canonically the kernel of the corresponding map $\prod_i K_i \xrightarrow{1-s} \prod_i K_i$ as defined above. When we're in the case of a module category, we write $R^1 \varprojlim_i K_i$ to denote the cokernel of the map $1-s$, so that we have a sequence

$$0 \to \varprojlim_i K_i \to \prod_i K_i \xrightarrow{1-s} \prod_i K_i \to R^1 \varprojlim_i K_i \to 0$$

that is exact. We could, but at the moment will not consider this as a derived functor of anything, and instead just view this as notation.

This construction in a triangulated category has an extremely weak kind of universal property. Suppose $C \in \mathcal{D}$ and that we have maps $g_n \colon C \to K_n$ compatible with the (f_i). It follows that the induced map $g \colon C \to \prod_i K_i$ satisfies $gs = g$, i.e. $g(1-s) = 0$. Now applying the homological functor $\mathrm{Hom}(C, -)$ to our distinguished triangle, we obtain the exact sequence:

$$\mathrm{Hom}_{\mathcal{D}}(C, K) \to \prod_i \mathrm{Hom}_{\mathcal{D}}(C, K_i) \xrightarrow{(-)\circ(1-s)} \prod_i \mathrm{Hom}_{\mathcal{D}}(C, K_i)$$

In particular, such a system (C, g_n) is induced by a morphism $C \to K$, but this is certainly not unique.

Note that for each C, $\mathrm{Hom}(C, K_i)$ is its own inverse system, and the kernel of the middle morphism above is $\varprojlim_i \mathrm{Hom}(C, K_i)$ in the category **Ab**. Therefore we have an exact sequence

$$0 \to R^1 \varprojlim_i \mathrm{Hom}(C, K_i[-1]) \to \mathrm{Hom}(C, K) \to \varprojlim_i \mathrm{Hom}(C, K_i) \to 0$$

by splitting into short exact sequences. Note e.g. that if K were the categorical limit then the right-hand map above would be an isomorphism, but here it's only surjective; the ambiguity is controlled by $R^1 \varprojlim_i$. We call this the *Milnor sequence*.

Example 2.3.19. Consider the case $\mathcal{D} = D(R)$ for a ring R. For $L \in D(R)$, we know that $H^i(L) = \mathrm{Hom}_{D(R)}(R[-i], L)$. Thus given a system (K_i, f_i) which has a derived limit, we can take $C = R[-i]$ in the Milnor sequence to obtain an exact sequence

$$0 \to R^1 \varprojlim_n H^{i-1}(K_n) \to H^i(K) \to \varprojlim_n H^i(K_n) \to 0$$

in $\mathrm{Mod}(R)$ in cohomology.

Now let us study the basic properties of this operation. We do this by reducing to the case of modules or abelian groups. For an inverse system (A_n) in a category, it is called *eventually constant* if there exists k such that the maps f_n are isomorphisms for $n > k$.

Lemma 2.3.20. *Let (A_n) be an inverse system in* $\mathrm{Mod}(R)$.

(a) Removing finitely many terms doesn't change $\varprojlim$ or $R^1 \varprojlim$. More precisely, for any k the natural maps

$$\varprojlim_{i \geq 1} A_i \to \varprojlim_{i \geq k} A_i \qquad R^1 \varprojlim_{i \geq 1} A_i \to R^1 \varprojlim_{i \geq k} A_i$$

are isomorphisms.

(b) Suppose (A_n) is eventually constant. Then $R^1 \varprojlim_i A_i = 0$, and if k is such that f_n is an isomorphism for $n \geq k$, then $\varprojlim_i A_i \to A_k$ is an isomorphism.

PROOF. (a) We can just do the case where we go from $i \geq k$ to $i \geq k+1$. There is a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_k & \longrightarrow & \prod_{i\geq k} A_i & \longrightarrow & \prod_{i\geq k+1} A_i & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow 1-s & & \downarrow 1-s & & \\ 0 & \longrightarrow & A_k & \longrightarrow & \prod_{i\geq k} A_i & \longrightarrow & \prod_{i\geq k+1} A_i & \longrightarrow & 0 \end{array}$$

The kernel and cokernel of the left-hand map are zero, so by the snake lemma the induced maps on kernels and cokernels of $1-s$ are isomorphisms.

(b) By (a), we may assume all maps in A_n are isomorphisms, and thus identities, so that $A_n = A$ for all n. The claim that the limit is A is clear in any category since this is a connected diagram of identities. For the claim about $R^1 \varprojlim$, we must show that

$$\prod_i A_i \xrightarrow{1-s} \prod_i A_i \qquad (a_i)_i \mapsto (a_i - a_{i+1})_i$$

is surjective. Indeed if $(b_i) \in \prod_i A_i$, then let $a_1 = 0$, and inductively $a_{i+1} = a_i - b_i$ for all $i \geq 1$. □

In the next result, we use the following construction: suppose (K_n) and (L_n) are inverse systems in a triangulated category, then a morphism of such systems is a system of maps $g_i \colon K_{a(i)} \to L_i$ where $a(1) < a(2) < \cdots$ is an increasing sequence of numbers, so that these commute with the defining morphisms. This data defines a morphism $g \colon \prod_i K_i \to \prod_i L_i$ by first projecting to $\prod_i K_{a(i)}$ and then taking the product morphism. On the other hand, we can define another map g' with nth component

$$g'_n \colon \prod_i K_i \to L_n \qquad g_n \circ \Big(\sum_{p=a(n)}^{a(n+1)-1} \prod_i K_i \to K_p \to K_{a(n)} \Big).$$

This is a telescoping sum and one easily checks that it satisfies $g'_n \circ (1 - s_K) = (1 - s_L) \circ g$, so we obtain a diagram

$$\begin{array}{ccccccc} \mathbf{R}\varprojlim_i K_i & \longrightarrow & \prod_i K_i & \longrightarrow & \prod_i K_i & \longrightarrow & \mathbf{R}\varprojlim_i K_i[1] \\ \downarrow \widetilde{g} & & \downarrow g & & \downarrow g' & & \downarrow \\ \mathbf{R}\varprojlim_i L_i & \longrightarrow & \prod_i L_i & \longrightarrow & \prod_i L_i & \longrightarrow & \mathbf{R}\varprojlim_i L_i[1] \end{array}$$

of triangles. Thus by axiom there is a lift on $\mathbf{R}\varprojlim$'s. Again, this is non-unique, but a defining feature of this is that we get a morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^1\varprojlim_i \mathcal{D}(C, K_i[-1]) & \longrightarrow & \mathcal{D}(C, \mathbf{R}\varprojlim_i K_i) & \longrightarrow & \varprojlim_i \mathcal{D}(C, K_i) & \longrightarrow & 0 \\ & & \downarrow & & \vdots & & \downarrow & & \\ 0 & \longrightarrow & R^1\varprojlim_i \mathcal{D}(C, L_i[-1]) & \longrightarrow & \mathcal{D}(C, \mathbf{R}\varprojlim_i L_i) & \longrightarrow & \varprojlim_i \mathcal{D}(C, L_i) & \longrightarrow & 0 \end{array}$$

of exact sequences in **Ab** for all $C \in \mathcal{D}$.

Corollary 2.3.21. *Let (K_n) be an inverse system in a triangulated category whose $\mathbf{R}\varprojlim$ exists.*

(a) For any k, $\mathbf{R}\varprojlim_{i \geq k} K_i$ exists, and any induced natural map $\mathbf{R}\varprojlim_i K_i \to \mathbf{R}\varprojlim_{i \geq k} K_i$ is an isomorphism.

(b) Suppose (K_n) is eventually constant. If k is such that f_n is an isomorphism for $n \geq k$, then the $\mathbf{R}\varprojlim_i K_i \to K_k$ is an isomorphism.

Proof. (a) This follows from the morphism of exact sequences above, the corresponding result for abelian groups, the five lemma, and the Yoneda lemma.

(b) Consider the canonical exact sequence (e.g. the top row of the morphism of exact sequences above). Then the $R^1\varprojlim$ term is zero by the abelian groups case, and so we have isomorphisms

$$\mathcal{D}(C, \mathbf{R}\varprojlim_i K_i) \to \varprojlim_i \mathcal{D}(C, K_i) \to \mathcal{D}(C, K_k)$$

where the composite is the map induced by $\mathbf{R}\varprojlim_i K_i \to K_k$. Thus we are done by the Yoneda lemma again. □

Now let us apply this to the case of derived categories. The first step is to know that the categories we care about actually possess the relevant countable products, and hence derived products. To do this, recall that if $\mathcal{A}$ is an abelian category, then an object $I^\bullet \in \mathrm{K}(\mathcal{A})$ is *K-injective* if for every acyclic complex $A^\bullet$, $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, I^\bullet) = 0$. The following lemma is just summarizing some facts we already know:

Lemma 2.3.22. *Let $\mathcal{A}$ be an abelian category.*

1. *A bounded below complex of injectives is K-injective.*
2. *The following are equivalent for a complex $I^\bullet$:*
   (a) $I^\bullet$ is K-injective.
   (b) The map $\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, I^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, I^\bullet)$ is an isomorphism for all $A^\bullet \in \mathrm{K}(\mathcal{A})$.
   (c) $\underline{\mathrm{Hom}}_{\mathcal{A}}(-, I^\bullet)$ preserves acyclics.
   (d) $\underline{\mathrm{Hom}}_{\mathcal{A}}(-, I^\bullet)$ preserves quasi-isomorphisms.
   In particular, a quasi-isomorphism between K-injective complexes is a homotopy equivalence. More generally, an isomorphism in $D(\mathcal{A})$ between K-injective complexes is represented by a homotopy equivalence.
3. *A (termwise) product of K-injective complexes is K-injective.*
4. *If $I^\bullet$ is K-injective, then $I^\bullet[n]$ is K-injective for $n \in \mathbb{Z}$.*

5. *If $A^\bullet \to B^\bullet \to C^\bullet \to A^\bullet[1]$ is a distinguished triangle in $\mathrm{K}(\mathcal{A})$ and any two of the three objects are K-injective, then the third is as well.*

PROOF. (1) and (2) were proven in Cor. 1.3.5 (for (2), the proof carries over to K-injectives without change). For (3), it is clear that

$$\underline{\mathrm{Hom}}_{\mathcal{A}}(-, \prod_i I_i^\bullet) \cong \prod_i \underline{\mathrm{Hom}}_{\mathcal{A}}(-, I_i^\bullet)$$

because the same is true on individual homs, and products commute with products. The result follows since products are exact in **Ab**.

(4) This follows from the fact that $\underline{\mathrm{Hom}}(-, I^\bullet[n]) = \underline{\mathrm{Hom}}(-, I^\bullet)[n]$ and since $[n]$ preserves acyclicity.

(5) By rotation, we may assume $A^\bullet, C^\bullet$ are K-injective. For any acyclic complex $K^\bullet$, the functor $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(K^\bullet, -)$ is homological, so it follows that $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(K^\bullet, B^\bullet) = 0$ as desired. □

By the way, observe that categorical products in $\mathrm{Ch}(\mathcal{A})$ and $\mathrm{K}(\mathcal{A})$ are indeed termwise. This follows because $\underline{\mathrm{Hom}}_{\mathcal{A}}(-,-)$ preserves termwise products in the second variable, and the fact that products are exact, which means that taking zero cycles $\ker d^0$ and H^0 both commute with products. But these constructions produce $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}$ and $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}$ respectively.

Lemma 2.3.23. *Let $\mathcal{A}$ be an abelian category with countable products.*

(a) Let $(A_i)_{i\geq 1}$ be a family of objects of $D(\mathcal{A})$ such that each A_i is represented by a K-injective complex $I_i^\bullet$. Then $\prod_i A_i$ exists in $D(\mathcal{A})$ and is represented by the K-injective complex $\prod_i I_i^\bullet$.

(b) If $\mathcal{A}$ has enough injectives, then $D(\mathcal{A})$ admits all products of objects in $D^+(\mathcal{A})$. In particular, $D(\mathcal{A})$ admits derived limits of all inverse systems in $D^+(\mathcal{A})$.

PROOF. (a) The product of K-injectives is K-injective by Lemma 2.3.22(3), and since this is also the product in $\mathrm{K}(\mathcal{A})$, we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, \prod_i I_i^\bullet) &= \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X^\bullet, \prod_i I_i^\bullet) \\ &= \prod_i \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X^\bullet, I_i^\bullet) \\ &= \prod_i \mathrm{Hom}_{D(\mathcal{A})}(X^\bullet, I_i^\bullet)\end{aligned}$$

and thus we have the statement. (b) follows directly from (a) and the fact that a bounded above complex of injectives is K-injective. Indeed, once the relevant products exist we can always take cones and rotate. □

Remark 2.3.24. We usually write $\mathrm{K}(\mathcal{I}_{\mathcal{A}}^K)$ for the K-injective complexes, and our results above show that this is a triangulated subcategory. Suppose (K_n) is an inverse system in $D(\mathcal{A})$ for which each K_n is represented by a K-injective complex $I_n^\bullet$. Then each map $f_n \colon K_{n+1} \to K_n$ is represented by a morphism $g_n \colon I_{n+1}^\bullet \to I_n^\bullet$ in $\mathrm{K}(\mathcal{A})$, and the map $1 - s \colon \prod_i I_i^\bullet \to \prod_i I_i^\bullet$ represents the corresponding map

$1 - s\colon \prod_i K_i \to \prod_i K_i$. Therefore we may form its cone C in $\mathrm{K}(\mathcal{A})$, and rotate to obtain the triangle

$$C[-1] \to \prod_i I_i^\bullet \xrightarrow{1-s} \prod_i I_i^\bullet \to C$$

in the homotopy category. It follows that $C[-1]$ is a K-injective complex representing $\mathbf{R}\varprojlim_i K_i$. Observe that if each $I_i^\bullet$ was termwise injective, then the same is true of the product, and hence of the cone C.

Here is the main situation we are going to be interested in. Let $\mathcal{A}$ be an abelian category with countable products and enough injectives. Let $A \in D(\mathcal{A})$ be an object, then we have canonical truncation maps $A \to \tau_{\geq -n}A$ and $\tau_{\geq -n-1}A \to \tau_{\geq -n}A$. Thus there is a natural inverse system

$$\tau_{\geq -1}A \leftarrow \tau_{\geq -2}A \leftarrow \tau_{\geq -3}A \leftarrow \cdots$$

which is called the *Postnikov tower of* A.

Write $A_n \coloneqq \tau_{\geq -n}A$, so $A_n \in D^+(\mathcal{A})$ for each n. Thus by the lemma we may form $\mathbf{R}\varprojlim_n A_n \in D(\mathcal{A})$ equipped with its natural triangle and hence maps $\mathbf{R}\varprojlim_n A_n \to \tau_{\geq -n}A$. Moreover, the canonical morphisms $A \to \tau_{\geq -n}A$ are clearly compatible with the morphisms in the inverse system, thus we obtain some non-unique morphism $A \to \mathbf{R}\varprojlim_n A_n$ making the diagram

$$\begin{array}{ccc} A & \rightsquigarrow & \mathbf{R}\varprojlim_n A_n \\ & \searrow & \downarrow \\ & & \tau_{\geq -n}A \end{array}$$

commute for every n. Note that the following are equivalent:

(a) An induced map $A \rightsquigarrow \mathbf{R}\varprojlim_n A_n$ is an isomorphism.
(b) An induced map $A \rightsquigarrow \mathbf{R}\varprojlim_n A_n$ induces epimorphisms $H^k(A) \to H^k(\mathbf{R}\varprojlim_n A_n)$ for all n.
(c) For all n, $\mathbf{R}\varprojlim_n A_n \to \tau_{\geq -n}A$ induces isomorphisms on H^k for $k \geq n$.
(d) For each k, there exists some $n \geq -k$ such that the maps $\mathbf{R}\varprojlim_n A_n \to \tau_{\geq -n}A$ induce monomorphisms $H^k(\mathbf{R}\varprojlim_n A_n) \to H^k(\tau_{\geq -n}A) = H^k(A)$

Proof. All of these follow from the fact that in the situation of the above diagram, for all $k \geq -n$ we get an induced map

$$H^k(A) \rightsquigarrow H^k(\mathbf{R}\varprojlim_n A_n) \to H^k(\tau_{\geq -n}A)$$

where the composite is an isomorphism (i.e. the identity). The induced map is an isomorphism iff it is one on all cohomology; the sequence shows that the first map is a split monomorphism, and the second is a split epimorphism. So, the first is an epimorphism iff the second is a monomorphism iff they're both isomorphisms. □

In particular, whether $A \rightsquigarrow \mathbf{R}\varprojlim_n A_n$ is an isomorphism is independent of the chosen map, and in fact it's independent of the choice of $\mathbf{R}\varprojlim$, because any two choices admit an isomorphism commuting with the maps to $\tau_{\geq -n}A$. Thus,

we will signify this situation by saying that 'the map $A \rightsquigarrow \mathbf{R}\varprojlim_n \tau_{\geq -n} A$ is an isomorphism', and this statement is independent of all choices.

Example 2.3.25. This is a warmup for the main result. Suppose $\mathcal{A} = \mathrm{Mod}(R)$ for a ring R. Then for any $A \in D(R)$, we claim that $A \rightsquigarrow \mathbf{R}\varprojlim_n \tau_{\geq -n} A$ is an isomorphism. By considering the Milnor sequence for $\mathrm{Hom}_{D(R)}(R[-k], -)$, we obtain a commutative diagram

$$\begin{array}{ccccccc} & & & & H^k(A) & = & H^k(A) \\ & & & & \downarrow & & \downarrow \\ 0 \to R^1 \varprojlim_i H^{k-1}(\tau_{\geq -i} A) & \to & H^k\big(\mathbf{R}\varprojlim_i (\tau_{\geq -i} A)\big) & \to & \varprojlim_i H^k(\tau_{\geq -i} A) & \to & 0 \end{array}$$

with the bottom row exact. But now $H^{k-1}(\tau_{\geq -i} A)$ and $H^k(\tau_{\geq -i} A)$ are eventually constant inverse systems, so the $R^1 \varprojlim$ term is zero, and the right-hand vertical map is an isomorphism. Thus the middle vertical map is an isomorphism as desired.

Finally, we prove the case of interest below, which will require computing the derived limit in $\mathrm{K}(X)$ and then passing to the derived category. To do this, observe the following: if $j \colon U \hookrightarrow X$ is an open inclusion, then we know that $\mathrm{Hom}_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{F}) \cong \mathcal{F}(U)$, thus if $\mathcal{F}^\bullet \in \mathrm{K}(X)$, we have canonically

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(j_! \mathcal{O}_U, \mathcal{F}^\bullet) = \mathcal{F}^\bullet(U) \implies \mathrm{Hom}_{\mathrm{K}(X)}(j_! \mathcal{O}_U[-p], \mathcal{F}^\bullet) = H^p(\mathcal{F}^\bullet(U))$$

by taking H^p of both sides.

Theorem 2.3.26. *Let X be a scheme and $A \in D_{\mathrm{qc}}(X)$. Then the map*

$$A \rightsquigarrow \mathbf{R}\varprojlim_n \tau_{\geq -n} A$$

is an isomorphism.

Proof. Following Remark 2.3.24, represent $A_n \coloneqq \tau_{\geq -n} A \in D_{\mathrm{qc}}(X)$ by a bounded below complex of injectives $\mathcal{I}_n^\bullet$, and write $\mathcal{I}^\bullet$ for the K-injective complex fitting into

$$\mathcal{I}^\bullet \to \prod_n \mathcal{I}_n^\bullet \xrightarrow{1-s} \prod_n \mathcal{I}_n^\bullet \to \mathcal{I}^\bullet[1]$$

in $\mathrm{K}(X)$, so that $\mathcal{I}^\bullet = \mathbf{R}\varprojlim_n A_n$ in $D(X)$. For each open $U \subset X$, the Milnor sequence in either $\mathrm{K}(X)$ or $D(X)$ (since the targets are K-injective), applied to $j_! \mathcal{O}_U[-p]$ gives the exact sequence

$$0 \to R^1 \varprojlim_n H^{p-1}(\Gamma(U, \mathcal{I}_n^\bullet)) \to H^p(\Gamma(U, \mathcal{I}^\bullet)) \to \varprojlim_n H^p(\Gamma(U, \mathcal{I}_n^\bullet)) \to 0$$

by the remark above. In other words:

$$0 \to R^1 \varprojlim_n \mathbb{H}^{p-1}(U, A_n) \to H^p(\Gamma(U, \mathcal{I}^\bullet)) \to \varprojlim_n \mathbb{H}^p(U, A_n) \to 0$$

If U is an affine open, then we know that

$$\mathbb{H}^p(U, \tau_{\geq -n} A) \cong H^0(U, H^p(\tau_{\geq -n} A)),$$

since $A_n \in D_{\mathrm{qc}}(X)$, so this system is essentially constant with value $H^0(U, H^p(A))$, and likewise for $p-1$. Thus the exact sequence shows that the canonical map

$$H^0(U, H^p(A)) \to H^p(\Gamma(U, \mathcal{I}^\bullet))$$

is an isomorphism for all $U \subset X$ affine. We are trying to show that $H^p(A) \to H^p(\mathcal{I}^\bullet) = H^p(\mathbf{R}\varprojlim_n A_n)$ is an isomorphism for all p, and we have a commutative diagram

$$\begin{array}{ccc} H^0(U, H^p(A)) & \xrightarrow{\cong} & H^p(\Gamma(U, \mathcal{I}^\bullet)) \\ & \searrow & \downarrow \\ & & H^0(U, H^p(\mathcal{I}^\bullet)) \end{array}$$

for affine opens U. Taking the filtered colimit over the basis of affine opens of this diagram identifies the two right-hand terms, because $H^p(\mathcal{I}^\bullet)$ is the sheafification of $U \mapsto H^p(U, \mathcal{I}^\bullet)$. Thus $H^p(A)_x \xrightarrow{\sim} H^p(\mathcal{I}^\bullet)_x$ for all x, so $H^p(A) \xrightarrow{\sim} H^p(\mathcal{I}^\bullet)$. □

The operative result here is already contained in the proof of the above theorem. Namely, it says that if X is affine, $A \in D_{\mathrm{qc}}(X)$, and $\mathcal{I}^\bullet$ is the K-injective representative for $\mathbf{R}\varprojlim_n \tau_{\geq -n}A$ induced by injective resolutions for $\tau_{\geq -n}A$, then $\Gamma(X, H^p(A)) \xrightarrow{\sim} H^p(\Gamma(X, \mathcal{I}^\bullet))$.

Now the remaining proofs will be trivial using this. We use two basic general observations. First, let $F\colon \mathcal{D} \to \mathcal{E}$ be an exact functor of triangulated categories and (K_n) be an inverse system in $\mathcal{D}$. Assume that $\mathbf{R}\varprojlim_n K_n$ and $\mathbf{R}\varprojlim_n F(K_n)$ both exist. Then the maps $\mathbf{R}\varprojlim_n K_n \to K_n$ induce a comparison map

$$F(\mathbf{R}\varprojlim_n K_n) \to \mathbf{R}\varprojlim_n F(K_n)$$

making the obvious diagrams commute. The second simple fact that we will use is that the canonical map $A \to \tau_{\geq -n}A$ induces the map

$$\mathrm{Hom}_{D(\mathcal{A})}(\tau_{\geq -n}A, B) \to \mathrm{Hom}_{D(\mathcal{A})}(A, B)$$

which is an isomorphism if $B \in D^{\geq -n}(\mathcal{A})$. We have discussed this before, and it's true because $\mathrm{Hom}_{D(\mathcal{A})}(D^{<-n}, D^{\geq -n}) = 0$ and the canonical truncation triangle

$$\tau_{<-n}A \to A \to \tau_{\geq -n}A \to \tau_{<-n}A[1]$$

for any abelian category $\mathcal{A}$.

Corollary 2.3.27. *Let R be a ring. The functor*

$$(\widetilde{-})\colon D(R) \to D_{\mathrm{qc}}(\mathrm{Spec}\, R)$$

is an equivalence of categories.

Proof. Since $(\widetilde{-})$ is exact, it commutes with the canonical truncation functors and cohomology. Let $A \in D_{\mathrm{qc}}(\mathrm{Spec}\, R)$, then we have the inverse systems $\tau_{\geq -n}A$ and $R\Gamma(\tau_{\geq -n}A)$. First, consider the Milnor sequence

$$\begin{aligned} 0 \to & R^1\varprojlim_n H^{p-1}(R\Gamma(\tau_{\geq -n}A)) \to H^p(\mathbf{R}\varprojlim_n R\Gamma(\tau_{\geq -n}A)) \\ & \to \varprojlim_n H^p(R\Gamma(\tau_{\geq -n}A)) \to 0 \end{aligned}$$

over $D(R)$, but then $\mathbb{H}^p(X, \tau_{\geq -n}A) = \Gamma(X, H^p(A))$ for $n \gg 0$, so the left- and right-hand systems are essentially constant. Thus $R^1 \varprojlim$ vanishes, and we obtain isomorphisms $H^p(\mathbf{R}\varprojlim_n R\Gamma(\tau_{\geq -n}A)) \xrightarrow{\sim} \Gamma(X, H^p(A))$. Applying $(\widetilde{-})$ to both sides and using quasicoherence of $H^p(A)$ shows that

$$\widetilde{H^p(\mathbf{R}\varprojlim_n R\Gamma(\tau_{\geq -n}A))} \xrightarrow{\sim} H^p(A).$$

On the other hand, we may consider the canonical comparison map

$$\mathbf{R}\varprojlim_n \widetilde{R\Gamma(\tau_{\geq -n}A)} \to \mathbf{R}\varprojlim_n R\Gamma\widetilde{(\tau_{\geq -n}A)} = \mathbf{R}\varprojlim_n \tau_{\geq -n}A = A,$$

where the two equalities follow from the bounded below theorem and Thm. 2.3.26. Taking cohomology produces the isomorphism above, and hence this is an isomorphism in $D(\operatorname{Spec} R)$. This shows essential surjectivity.

It remains to show that if $M, N \in D(R)$, then the map

$$\operatorname{Hom}_{D(R)}(M, N) \to \operatorname{Hom}_{D(X)}(\widetilde{M}, \widetilde{N})$$

is an isomorphism. This is true if $M \in D^+(R)$, because in that case the remark above identifies both sides with the Homs from $\tau_{\geq -n}M$ and $\tau_{\geq -n}\widetilde{M}$, and then the result follows from the bounded below case. In the general case, we can write $n = \mathbf{R}\varprojlim_n \tau_{\geq -n}n$ and $\widetilde{N} = \mathbf{R}\varprojlim_n \tau_{\geq -n}\widetilde{N} = \mathbf{R}\varprojlim_n \widetilde{\tau_{\geq -n}N}$, so

$$\begin{array}{ccccc}
R^1\varprojlim_n D(R)(M, \tau_{\geq -n}N[-1]) & \to & D(R)(M, N) & \to & \varprojlim_n D(R)(M, \tau_{\geq -n}N) \\
\downarrow & & \downarrow & & \downarrow \\
R^1\varprojlim_n D(X)(\widetilde{M}, \widetilde{\tau_{\geq -n}N}[-1]) & \to & D(X)(\widetilde{M}, \widetilde{N}) & \to & \varprojlim_n D(X)(\widetilde{M}, \widetilde{\tau_{\geq -n}N})
\end{array}$$

commutes with exact rows (with the four zeros, but they didn't fit into the diagram). By the bounded case, the left- and right-hand maps are isomorphisms, so we are done by the five-lemma. □

This was the main result we cared about, but we can now use the theory above to discuss some aspects of unbounded derived category theory. We will use the notation $\mathrm{K}_{\mathrm{qc}}(X) \subset \mathrm{K}(X)$ for the full subcategory of objects with quasicoherent cohomology.

Corollary 2.3.28. *Let X be a scheme.*

(a) For each $\mathcal{F}^\bullet \in \mathrm{K}_{\mathrm{qc}}(X)$, there is a morphism $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ in $K_{\mathrm{qc}}(\mathcal{O}_X)$ which is a quasi-isomorphism and $\mathcal{I}^\bullet \in \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X^K)$. Moreover, we may choose $\mathcal{I}^\bullet$ to be termwise injective, and $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ to be termwise injective.

(b) The natural functor $\mathrm{K}_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$ is a triangulated localization by the quasi-isomorphisms and the induced map $\mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X^K) \to D_{\mathrm{qc}}(X)$ is a triangulated equivalence.

Proof. (a) Choose an isomorphism $\mathcal{F}^\bullet \to \mathbf{R}\varprojlim_n \tau_{\geq -n}\mathcal{F}^\bullet$ in $D_{\mathrm{qc}}(X)$, then we already explained that we can choose the $\mathbf{R}\varprojlim$ to be represented by a K-injective complex $\mathcal{I}^\bullet$. Thus the corresponding map $\mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is induced from the homotopy category and is a quasi-isomorphism.

(b) The first claim is just because $\mathrm{K}_{\mathrm{qc}}(X)$ is saturated under all quasi-isomorphisms, so the standard localization subcategory criterion applies. For the second statement, the same criterion and (a) imply immediately that $\mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X^K) \to D_{\mathrm{qc}}(X)$ is fully faithful, and then (a) implies again that it is essentially surjective. □

Remark 2.3.29. Since we already proved Postnikov completeness in $D(R)$ for a ring R, the same argument shows that all $M^\bullet \in \mathrm{K}(R)$ admit a quasi-isomorphism to a K-injective complex of R-modules, and $\mathrm{K}(\mathcal{I}_R) \to D(R)$ is a triangulated localization. Note that this is not automatically implied by the affine equivalence because a priori, K-injective representatives could live in $\mathrm{Mod}(\mathcal{O}_{\mathrm{Spec}\, R})$.

Write $Q_X \colon \mathrm{K}_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$ for the localization functor, and suppose $F \colon \mathrm{K}_{\mathrm{qc}}(X) \to \mathcal{E}$ is an exact functor of triangulated categories. A *right derived functor of* F is an exact left Kan extension of F along Q_X. In more detail, it is an exact functor $RF \colon D_{\mathrm{qc}}(X) \to \mathcal{E}$ equipped with a natural transformation

$$\varepsilon \colon F \to RF \circ Q_X$$

satisfying the following universal property: for every triangulated functor $G \colon D_{\mathrm{qc}}(X) \to \mathcal{E}$ and map $\eta \colon F \to G \circ Q_X$, there is a unique map $u \colon RF \to G$ such that $\eta = uQ_X \circ \varepsilon$. Of course, if a right derived functor exists then it is unique up to canonical isomorphism.

Lemma 2.3.30. *Any exact functor $F \colon \mathrm{K}_{\mathrm{qc}}(X) \to \mathcal{E}$ has a right derived functor, which in fact satisfies the left Kan condition against all functors $D_{\mathrm{qc}}(X) \to \mathcal{E}$. The canonical morphism $F \to RF \circ Q_X$ is an isomorphism on the subcategory $\mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X^K)$.*

Proof. Choose an exact inverse $S \colon D_{\mathrm{qc}}(X) \to \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X)$ and an isomorphism u from Id to $D_{\mathrm{qc}} \xrightarrow{S} \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X) \to D_{\mathrm{qc}}(X)$. Define

$$RF \coloneqq D_{\mathrm{qc}}(X) \xrightarrow{S} \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X) \hookrightarrow \mathrm{K}_{\mathrm{qc}}(X) \xrightarrow{F} \mathcal{E},$$

and for each $\mathcal{F}^\bullet \in \mathrm{K}_{\mathrm{qc}}(X)$ we have a natural map $u_{Q_X(\mathcal{F}^\bullet)} \colon \mathcal{F}^\bullet \to S(\mathcal{F}^\bullet)$ in $D_{\mathrm{qc}}(X)$. Since $S(\mathcal{F}^\bullet)$ is a K-injective complex, this comes from a unique map $v_{\mathcal{F}^\bullet} \colon \mathcal{F}^\bullet \to S(\mathcal{F}^\bullet)$ in $\mathrm{K}_{\mathrm{qc}}(X)$. v is a natural transformation because the naturality condition is equality of two maps to a K-injective complex, so it follows from naturality of u. Define $\varepsilon = Fv$, giving a natural transformation

$$\varepsilon \colon F \circ \mathrm{Id} = F \to F \circ \iota \circ S = RF.$$

Note that RF is a composition of exact functors, hence exact. Also, the restriction of $Q_X\varepsilon$ to $\mathrm{K}_{\mathrm{qc}}(X)$ is the isomorphism v in $D_{\mathrm{qc}}(X)$, and since the source and target are K-injective, $\varepsilon|_{\mathrm{K}_{\mathrm{qc}}}$ is also an isomorphism.

To check that (RF, ε) is a left Kan extension, suppose $G \colon D_{\mathrm{qc}}(X) \to \mathcal{E}$ is a functor with a map $\eta \colon F \to G \circ Q_X$. Again letting $\iota \colon \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X) \hookrightarrow \mathrm{K}_{\mathrm{qc}}(X)$ be the inclusion, we obtain a map $\eta\iota \colon F\iota \to G \circ Q_X\iota$, and $\varepsilon\iota \colon F\iota \cong RF \circ Q_X\iota$, and since $Q_X\iota \cong 1$ compatibly, we induce a unique map $RF \to G$ with the desired properties. □

Remark 2.3.31. Note that in the proof, we didn't need to use the fact that $\mathcal{E}$ and F were triangulated except to assert that the resulting derived functor was triangulated. Thus the result more generally allows us to Kan extend functors along the localization map Q_X.

In particular, since the functors $\Gamma\colon \mathrm{K}_{\mathrm{qc}}(X) \to D(\mathcal{O}_X(X))$ and $f_*\colon K_{\mathrm{qc}}(X) \to D(Y)$ for a morphism $f\colon X \to Y$ are triangulated, we obtain unbounded derived functors

$$R\Gamma\colon D_{\mathrm{qc}}(X) \to D(\mathcal{O}_X(X)) \qquad Rf_*\colon D_{\mathrm{qc}}(X) \to D(Y).$$

It is safe to use the same notation as the derived functors we have been working with thus far, because the natural map from the bounded below versions to the restrictions of the unbounded ones are isomorphisms, as is checked on the bounded below complexes of injectives. Of course, the definitions above all apply dually, and to any triangulated subcategory of $\mathrm{K}(\mathcal{A})$ for any abelian category. When there are enough K-injectives, then the above proof applies as well.

Example 2.3.32. Suppose $F\colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ is an adjunction between abelian categories and suppose that F is exact. Then G preserves K-injectives. Indeed, this is because termwise application of F and G descends to an adjunction on the homotopy category, and exact functors preserve acyclic complexes. For example, we can use this to see that the restriction functor $(-)|_U$ preserves K-injectives, and hence unbounded $R\Gamma$ and Rf_* behave well with respect to restriction.

To actually reason about these functors, it will be easiest to reduce them to the bounded below cases using derived limits. Thus we should discuss and prove some compatibilities.

Corollary 2.3.33. *Let X be a scheme. Let (K_n) be an inverse system in $D_{\mathrm{qc}}(X)$.*

(a) All products of objects of $D_{\mathrm{qc}}(X)$ exist in $D(X)$, and $\mathbf{R}\varprojlim_n K_n$ exists in $D(X)$.

(b) ~~The functors $R\Gamma$ and Rf_* commute with derived limits which remain in $D_{\mathrm{qc}}(X)$, meaning that if $\mathbf{R}\varprojlim_n K_n \in D_{\mathrm{qc}}(X)$, then all induced maps~~

$$R\Gamma(X, \mathbf{R}\varprojlim_n K_n) \to \mathbf{R}\varprojlim_n (R\Gamma(X, K_n))$$

~~and~~

$$Rf_*(X, \mathbf{R}\varprojlim_n K_n) \to \mathbf{R}\varprojlim_n Rf_*(K_n)$$

~~are isomorphisms.~~

(c) There is a canonical hypercohomology Milnor sequence

$$0 \to R^1\varprojlim_n \mathbb{H}^{k-1}(U, K_n) \to \mathbb{H}^k(U, \mathbf{R}\varprojlim_n K_n) \to \varprojlim_n \mathbb{H}^k(U, K_n) \to 0$$

for each k and U induced by any presentation of $\mathbf{R}\varprojlim_n K_n \in D_{\mathrm{qc}}(X)$.

Remark 2.3.34. In (c), if $\mathbf{R}\varprojlim_n K_n$ is not necessarily in $D_{\mathrm{qc}}(X)$, then we still have such a sequence except one replaces the middle term with $\mathrm{Hom}_{D(U)}(\mathcal{O}_U, \mathbf{R}\varprojlim_n K_n|_U)$.

PROOF. (a) This follows from Lemma 2.3.23, since every object of $D_{\mathrm{qc}}(X)$ has a K-injective representative.

~~(b) More generally, suppose $F\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathcal{B}$ is a right exact additive functor to an abelian category, then we know that $RF\colon D_{\mathrm{qc}}(X) \to D(B)$ exists. Moreover, if $\mathcal{I}_n^\bullet$ is a K-injective complex representing K_n, then the derived limit is represented by~~

$$\mathbf{R}\varprojlim_n K_n \to \prod_n \mathcal{I}_n^\bullet \to \prod_n \mathcal{I}_n^\bullet \to \mathbf{R}\varprojlim_n K_n[1]$$

~~where the product is termwise, and the middle map is a morphism of complexes. Since the products remain K-injective, any induced map fits into a morphism of triangles in $D(\mathcal{B})$. In particular, if F commutes with countable products then the claim follows, and this applies to f_* and Γ.~~

~~(c) If we apply (b) to the defining triangle to get a derived colimit in $D(\mathcal{O}_X(X))$, and take the cohomological Milnor sequence for a ring, then this is exactly what we get.~~

(c) A presentation is of the form

$$\mathbf{R}\varprojlim_n K_n \to \prod_n K_n \to \prod_n K_n \to \mathbf{R}\varprojlim_n K_n[1],$$

so take the Hom-Milnor sequence for $j_!\mathcal{O}_U[-k]$. It remains to observe that

$$\mathrm{Hom}_{D(X)}(j_!\mathcal{O}_U, A) = \mathbb{H}^0(U, A)$$

for $A \in D_{\mathrm{qc}}(X)$. Indeed, this result is already known to us on $\mathrm{K}(\mathcal{I}_X)$. In the remaining case of the remark, just ignore this observation in the middle part of the sequence. □

Remark 2.3.35. TODO: The reason for the crossed out text is that I naively claimed that $R\Gamma$ and Rf_* commute with products. At this stage, my argument was wrong because there's no guarantee that e.g. $\prod_n f_*(\mathcal{I}_n^\bullet)$ represents the product in $\mathcal{E}$. Instead, we would want to use e.g. that $Lf^* \dashv Rf_*$ which is not available yet. However, I didn't cross out part (b) from the statement because I think we could still prove it using one of the examples below.

Remark 2.3.36. ~~What we showed in (b) should really be broken up into the following two steps: (1) An exact functor $D_{\mathrm{qc}}(X) \to \mathcal{E}$ which commutes with (categorical) products preserves derived limits of inverse systems, and (2) if $F\colon \mathrm{Mod}(\mathcal{O}_X) \to \mathcal{B}$ preserves countable products, then RF commutes with categorical products.~~

Example 2.3.37. If $X = \operatorname{Spec} R$ is an affine scheme, let $A \in D_{\mathrm{qc}}(X)$ and write $A \xrightarrow{\sim} \mathbf{R}\varprojlim_n \tau_{\geq -n}A$. Then for each k,

$$\mathbb{H}^k(X, \tau_{\geq -n}) = H^0(X, H^k(\tau_{\geq -n}A)) = H^0(X, H^k(A))$$

for $n \gg 0$. Thus the Milnor sequence gives $\mathbb{H}^k(X, A) \cong H^0(X, H^k(A))$ for all k, extending the bounded below result. Thus, now let X be arbitrary and consider the map $\mathbb{H}^k(U, A) \to H^0(U, H^k(A))$ for $A \in D_{\mathrm{qc}}(X)$. Since it is an isomorphism

on the basis of affine opens, we deduce that $\mathbf{a}\underline{\mathbb{H}}^k(A) \xrightarrow{\sim} H^k(A)$ is an isomorphism of sheaves.

There is a similar statement for pushforward; note that if $\mathcal{I}^\bullet$ is in $\mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X)$, then we have maps

$$H^p(\Gamma(f^{-1}(U), \mathcal{I}^\bullet)) = H^p(\Gamma(U, f_*(\mathcal{I}^\bullet))) \to \Gamma(U, H^p(f_*\mathcal{I}^\bullet))$$

where the last map is the sheafification unit. Therefore we have a natural map

$$\mathbb{H}^p(f^{-1}(U), A) \to H^p(U, H^p(Rf_*A))$$

for $A \in D_{\mathrm{qc}}(X)$, i.e. a morphism $f_*\underline{\mathbb{H}}^p(A) \to H^p(Rf_*A)$ of presheaves on Y. This morphism is a sheafification, meaning the induced map $\mathbf{a}f_*\underline{\mathbb{H}}^p(A) \to H^p(Rf_*A)$ is an isomorphism, because on K-injectives it is so by definition.

Remark 2.3.38. In the situation of Remark 2.3.36, $RF\colon D_{\mathrm{qc}}(X) \to D(\mathcal{B})$ is called *left t-exact*, which means that it sends $D_{\mathrm{qc}}^{\geq n}(X)$ to $D^{\geq n}(\mathcal{B})$ for all n. This is clear because to compute RF on $D_{\mathrm{qc}}^{\geq n}(X)$ we can just use the classical bounded below RF. We can also say something in the opposite direction, which will be the next focus.

Lemma 2.3.39. *Let X have globally finite* QCoh*-dimension (e.g. X is qcqs). Then for each $A \in D_{\mathrm{qc}}(X)$ and $k \in \mathbb{Z}$, the inverse system $(\tau_{\geq -n}A)_n$ is eventually constant. In particular, the natural map*

$$\mathbb{H}^k(X, A) \xrightarrow{\sim} \varprojlim_n \mathbb{H}^k(X, \tau_{\geq -n}A)$$

is an isomorphism for all $k \in \mathbb{Z}$.

Proof. The first statement (applied to $k-1$) implies the second by the hypercohomology Milnor sequence. For the first, consider the truncation triangle

$$H^{-n}(A)[n] \to \tau_{\geq -n}A \to \tau_{\geq -n+1}A \to H^{-n}(A)[n+1],$$

which induces an exact sequence

$$H^{k+n}(X, H^{-n}(A)) \to \mathbb{H}^k(\tau_{\geq -n}A) \to \mathbb{H}^k(\tau_{\geq -n+1}A) \to H^{k+n+1}(X, H^{-n}(A))$$

for every k (where the X's are suppressed in the middle). For fixed k, the middle map is an isomorphism for $n \gg 0$, because $H^{k+n}(X, H^{-n}(A))$ and $H^{k+n+1}(X, H^{-n}(A))$ are zero once $n+k$ exceeds the global QCoh dimension bound. □

Lemma 2.3.40. *Let $f\colon X \to Y$ be a qcqs morphism of schemes. Then Rf_* maps*

$$Rf_*\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y).$$

Furthermore, if $d \geq 0$ is such that $R^p f_*\mathcal{F} = 0$ for all $p > d$ and $\mathcal{F} \in \mathrm{QCoh}(X)$, then Rf_* maps $D_{\mathrm{qc}}^{\leq b}(X)$ to $D_{\mathrm{qc}}^{\leq b+d}(Y)$. Such a d exists for instance if Y is quasicompact, or X is Noetherian.

Proof. We are going to first prove the boundedness result, and then use it to establish quasicoherence of unbounded pushforward. We have seen that such a d exists if X is Noetherian or if Y is quasicompact. We have also seen previously that

this condition on d implies the result in the bounded below case by considering the spectral sequence on the pushforwards of the cohomology.

Now suppose $A \in D^{\leq b}(X)$. Then $Rf_*(\tau_{\geq -n}A) \in D^{\leq b+d}(Y)$ for all n. For every affine open $U \subset Y$, we know from Prop. 2.3.9 that

$$\mathbb{H}^p(f^{-1}(U), \tau_{\geq -n}A) = H^0(U, H^p(Rf_*\tau_{\geq -n}A)) = 0$$

for $p > b + d$. Therefore, by Lemma 2.3.39,

$$\mathbb{H}^p(f^{-1}(U), A) = \varprojlim_n \mathbb{H}^p(f^{-1}(U), \tau_{\geq -n}A) = 0$$

for $p > b + d$ and U affine, since $f^{-1}(U)$ is qcqs and hence has globally finite QCoh-dimension. Since $H^p(Rf_*A)$ is the sheafification of the presheaf $U \mapsto \mathbb{H}^p(f^{-1}(U), A)$ on the basis of affine opens of Y, we conclude $H^p(Rf_*A) = 0$ for $p > b + d$.

It remains to prove quasicoherence of cohomology of Rf_*A for arbitrary $A \in D_{\mathrm{qc}}(X)$. This is local, so we may assume Y is quasicompact and thus some d as in the first part exists. By the triangle

$$\tau_{<n}A \to A \to \tau_{\geq n}A \to \tau_{<n}A[1],$$

exactness of Rf_* and the result we just proved shows that

$$H^p(Rf_*A) \xrightarrow{\sim} H^p(Rf_*\tau_{\geq n}A) \qquad p \geq n + d.$$

Note that $H^p(Rf_*\tau_{\geq n}A)$ is quasicoherent because of the bounded above case, and since n is arbitrary we can make p arbitrarily small by reducing n. □

Remark 2.3.41. Let $f\colon X \to Y$ be a qcqs morphism of schemes and $A \in D_{\mathrm{qc}}(X)$. Then we have the canonical map

$$H^p(Rf_*A) \to \varprojlim_n H^p(Rf_*\tau_{\geq -n}A)$$

which we claim is an isomorphism. Indeed, restriction commutes with limits (since e.g. it is a right adjoint), so the statement is local on Y. Thus we may assume Y is quasicompact and then this is eventually constant as explained at the end of the proof. This is a simple result which allows us to reduce to the bounded below case when we're working with natural maps. For example, the following corollary.

Example 2.3.42. Suppose $f\colon X \to Y$ is a morphism of schemes. The argument (verbatim) of Cor. 2.3.3 using K-injectives instead of bounded above complexes of injectives shows that there are natural maps $H^p \circ Rf_* \to f_* \circ H^p$ for $p \in \mathbb{Z}$. If f is affine, then these are isomorphisms. Indeed, f is qcqs, and by the remark above and naturality we reduce to testing on $D^+_{\mathrm{qc}}(X)$, which was Cor. 2.3.3(b).

Corollary 2.3.43. *Let $f\colon X \to Y$ be a qcqs morphism of schemes.*

(a) Let $g\colon Y \to Z$ be qcqs as well. Then the natural map

$$R(f \circ g)_* \to Rf_* \circ Rg_*$$

is an isomorphism of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Z)$.

(b) Assume Y (and hence X) is qcqs. Then the natural map

$$R\Gamma(X, A) \to R\Gamma(Y, Rf_*A)$$

is an isomorphism in $D(\mathcal{O}_Y(Y))$.

(c) For each affine open $U \subset Y$, the natural map

$$\mathbb{H}^p(f^{-1}(U), A) \to H^p(Rf_*A)$$

is an isomorphism. In fact, the natural map

$$R\Gamma(\widetilde{f^{-1}(U)}, A) \to Rf_*(A|_{f^{-1}(U)})$$

is an isomorphism in $D_{\mathrm{qc}}(U)$.

Proof. Throughout this proof, we are suppressing some implicit change of scalars morphisms, which is harmless because these functors are exact and hence behave well on the unbounded derived category, and when applying them after unbounded derived functors.

(a) The natural map is induced by the universal property. We have a commutative diagram

$$\begin{array}{ccc} H^p(R(g\circ f)_*A) & \longrightarrow & H^p(Rg_*(Rf_*A)) \\ \downarrow\cong & & \downarrow \\ \varprojlim_n H^p(R(g\circ f)_*(\tau_{\geq -n}A)) & \xrightarrow{\cong} & \varprojlim_n H^p(Rg_*(Rf_*(\tau_{\geq -n}A))) \end{array}$$

by naturality. The bottom map is an isomorphism by the bounded below case, and the left-hand map is the above remark. To see that the right-hand map is an isomorphism, note that it suffices to show

$$H^p(Rg_*Rf_*(\tau_{<-n}A)) = 0 \qquad n \gg 0$$

for all p, locally on Z. We can thus restrict to a qcqs open of Z, and thus assume Z and Y are qcqs since g is qcqs. Thus X is qcqs as well since f is qcqs. Then both Rg_* and Rf_* increase cohomological amplitude by finite numbers, and the claim follows. This implies that the top map of the diagram is an isomorphism for all p.

(b) Same argument as (a), but for the left side use Lemma 2.3.39. Then for the right side, use the fact that Rf_* and $R\Gamma(Y, -)$ (since Y is qcqs) both increase cohomological amplitude on the right by finite numbers.

(c) For the first part, we can argue directly from the bounded below case by applying both the remark above and Lemma 2.3.39. For the second part, by replacing Y with U we have to show that the sheafification of $R\Gamma(X, A)$ in $D(\mathcal{O}_Y(Y))$ is Rf_*A. But by the equivalence $D(\mathcal{O}_Y(Y)) = D_{\mathrm{qc}}(Y)$ for affine schemes, it suffices to show that

$$R\Gamma(X, A) \cong R\Gamma(Y, Rf_*A),$$

which is part (b). □

Remark 2.3.44. Note that this is the first stage where we proved some results strictly less general than what we could state in our framework. Indeed, part (a) is true without g being qcqs, and part (b) is true without Y being qcqs. But these are drawbacks associated with avoiding general existence results for K-injective complexes. In any case, this won't really matter because real results usually rely on schemes being qcqs.

Example 2.3.45. Of course, if X is affine and $\mathcal{F}^\bullet$ is a complex of quasicoherent sheaves on X, then $\Gamma(X, \mathcal{F}^\bullet) \to R\Gamma(X, \mathcal{F}^\bullet)$ is an isomorphism, because $\mathcal{F}^\bullet = \widetilde{M}^\bullet$ and then both sides are $M^\bullet$ by the affine equivalence $D(R) = D(\operatorname{Spec} R)$. It follows that if $f\colon X \to Y$ is an affine morphism and $\mathcal{F}^\bullet$ is a complex of quasicoherent sheaves on X, then $f_*\mathcal{F}^\bullet \to Rf_*\mathcal{F}^\bullet$ is an isomorphism. Indeed, the claim is local on Y so assume X and Y are affine, then (c) of the above corollary reduces us to the initial observation.

Example 2.3.46. If $f\colon X \to Y$ is affine, then $Rf_*\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ is conservative. Indeed, by choosing triangles it suffices to show that $\ker(Rf_*) = 0$. If $Rf_*A = 0$, then for each affine open $U \subset Y$,

$$0 = Rf_*A|_U = Rf_{f^{-1}(U),*}(A|_{f^{-1}(U)})$$

so we reduce to the case where X and Y are affine. But in this case Rf_* computes $R\Gamma(X, -)$ and then restricts scalars to $\mathcal{O}_Y(Y)$, and both of these functors reflect zero as X is affine.

Example 2.3.47. Here is a more specific variant of the previous result; suppose again that $f\colon X \to Y$ is affine. Then in fact

$$Rf_*(D_{\mathrm{qc}}^{[a,b]}(X)) \subset D_{\mathrm{qc}}^{[a,b]}(Y) \qquad Rf_*^{-1}(D_{\mathrm{qc}}^{[a,b]}(Y)) \subset D_{\mathrm{qc}}^{[a,b]}(X)$$

where a and b can possibly be infinite. Indeed, the left-hand statement follows immediately from Example 2.3.42, and the right-hand one also follows from Example 2.3.42 and the fact that $f_*\colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ reflects 0, clear by checking on an affine cover of Y.

Finally, let us briefly discuss unbounded $\mathbf{R}\mathcal{H}om$. This will be especially difficult to work with since we need strong hypotheses for it to be quasicoherent. We have already discussed one case when this works, which is when the first argument is pseudocoherent, but in that case the second argument must be bounded below anyway. The new case we will get is when the first argument is perfect. Then the new formalism will allow us to remove some boundedness hypotheses on our earlier work with perfect objects.

As usual, we start by considering $\underline{\mathcal{H}om}_{\mathcal{O}_X}$ as a functor on the unbounded homotopy categories, and then restrict to a functor

$$\mathrm{K}(\mathcal{O}_X) \times \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X) \to \mathrm{K}(\mathcal{O}_X).$$

When the second variable is fixed, it sends acyclics to acyclics by the criterion for K-injectivity, and therefore (after choosing a quasi-inverse) we obtain an unbounded functor

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\colon D(X) \times D_{\mathrm{qc}}(X) \to D(X)$$

In parallel, we do the same thing for the functor $\mathrm{RHom}_{\mathcal{O}_X}$, and we obtain a functor

$$\mathrm{RHom}_{\mathcal{O}_X} : D(X) \times D_{\mathrm{qc}}(X) \to D(\mathcal{O}_X(X)).$$

We know from before that these are exact in each variable. Now fix $\mathcal{F}^\bullet \in \mathrm{K}(X)$; the functors

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, -) \qquad \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{F}^\bullet, -)$$

are exact, and we have constructed them exactly as in the construction of a derived functor from K-injectives. Therefore these coincide with the right derived functors of

$$\underline{\mathcal{H}om}(\mathcal{F}^\bullet, -) \colon \mathrm{K}_{\mathrm{qc}}(X) \to D(X) \qquad \underline{\mathrm{Hom}}(\mathcal{F}^\bullet, -) \colon \mathrm{K}_{\mathrm{qc}}(X) \to D(X),$$

and we obtain canonical maps (between functors out of $\mathrm{K}_{\mathrm{qc}}(X)$)

$$\underline{\mathcal{H}om}(\mathcal{F}^\bullet, -) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, -) \qquad \underline{\mathrm{Hom}}_{\mathcal{O}_X}(F^\bullet, -) \to \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{F}^\bullet, -)$$

which are isomorphisms on the K-injectives. By the universal property of right derived functors, a map $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ induces a unique natural transformation between these functors. By uniqueness this gives natural maps

$$\underline{\mathcal{H}om}_{\mathcal{O}_X} \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X} \qquad \underline{\mathrm{Hom}}_{\mathcal{O}_X} \to \mathrm{RHom}_{\mathcal{O}_X}$$

of bifunctors $\mathrm{K}(X) \times \mathrm{K}_{\mathrm{qc}}(X) \to D(X)$, which is an isomorphism when the second variable is K-injective. Importantly, and simply by construction, these agree with the derived functors we defined for ringed spaces earlier.

We will wait for the next section to prove any serious results about these functors in the unbounded case. However, since the restriction of K-injective complexes with quasicoherent cohomology retains the same property, we can at least assert that $\mathbf{R}\mathcal{H}om$ behaves as expected upon restriction. What we will show is the following important case where the unbounded $\mathbf{R}\mathcal{H}om$ works well.

Lemma 2.3.48. *Let $E \in D(X)$ be perfect. Then $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, B) \in D_{\mathrm{qc}}(X)$ for each $B \in D_{\mathrm{qc}}(X)$.*

Proof. The claim is local on X, so we may assume E is strictly perfect, thus in particular a bounded complex of locally frees. Write $B = \mathcal{I}^\bullet$ as a K-injective complex with quasicoherent cohomology. By boundedness in the first variable, the Hom-totalization spectral sequence only involves direct sums, so it applies and converges to the cohomology of $\mathbf{R}\mathcal{H}om$. We claim that the E_2-page consists only of quasicoherent sheaves. Indeed, by further restricting, each object in E_2 is the cohomology of a sequence of the form

$$\mathrm{Hom}(\mathcal{O}_X^k, \mathcal{I}^{n-1}) \to \mathrm{Hom}(\mathcal{O}_X^k, \mathcal{I}^n) \to \mathrm{Hom}(\mathcal{O}_X^k, \mathcal{I}^{n+1}),$$

and thus it is a finite power of a quasicoherent sheaf. □

Now suppose X is qcqs and $E \in \mathrm{Perf}(X)$. There is a natural map

$$\mathrm{RHom}_{\mathcal{O}_X}(E, B) \to R\Gamma(X, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, B))$$

which we know is an isomorphism in the usual case, and we are going to use techniques introduced in this section to prove it in this case as well. First, since X is qcqs, there is a global cohomological QCoh-bound d. Since E is perfect and

X is qc, it has amplitude in $[a, b]$. Note that from the spectral sequence in the proof above, it is clear that $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, C)$ is in $D^{\leq r-a}$ whenever $C \in D^{\leq a}$.

The observations above show that $R\Gamma$ and $\mathbf{R}\mathcal{H}om(E, -)$ only increase bounded above amplitude by a finite number, so by the truncation triangle

$$\tau_{<-n}B \to B \to \tau_{\geq -n}B \to \tau_{<-n}B[1]$$

we deduce that for fixed p,

$$\mathbb{H}^p\big(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, B)\big) \to \mathbb{H}^p\big(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, \tau_{\geq -n}B)\big)$$

is an isomorphism for $n \gg 0$.

Now let's consider $\mathrm{RHom}_{\mathcal{O}_X}(E, B)$. We claim that for each p, the inverse system $H^p(\mathrm{RHom}_{\mathcal{O}_X}(E, \tau_{\geq -n}B))$ is eventually constant. Indeed, considering the standard sequential truncation triangle shows that the obstruction to the maps of the system being isomorphisms lives in specific cohomology modules which have to be zero once $n \gg 0$ because X has bounded QCoh-dimensions.

Since this applies to $p - 1$, the corresponding Milnor sequence has vanishing $R^1 \varprojlim$, and thus gives an isomorphism

$$H^p \mathrm{RHom}(E, B) \xrightarrow{\sim} \varprojlim_n H^p(\mathrm{RHom}(E, \tau_{\geq -n}B)),$$

and the right-hand system is eventually constant. Therefore combining the two results gives us the desired property. We've shown:

Lemma 2.3.49. *Let X be qcqs and $E \in \mathrm{Perf}(X)$. Then the natural map*

$$\mathrm{RHom}_{\mathcal{O}_X}(E, B) \to R\Gamma(X, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, B))$$

is an isomorphism in $D_{\mathrm{qc}}(X)$ for all $B \in D_{\mathrm{qc}}(X)$.

2.4. Unbounded derived categories for schemes

2.4.1. Unbounded left derived functors on $D(\mathcal{O}_X)$. Now we discuss the unbounded versions of Lf^* and $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)$. This will be very different from the prior discussion in two respects: First, even though we don't have K-projectives, it will be much easier and much more general (in particular we will be able to construct this on the whole category $D(\mathcal{O}_X)$ of a ringed space). Moreover, we will usually just be able to give direct arguments for compatibilities among the left derived functors, and can avoid derived colimits of truncation towers.

The downside is that the left derived functors are in some sense more classical and less interesting. They behave very well with stalks and thus one can usually reduce them to the case of modules. On the other hand, the way we constructed the unbounded right derived functors was heavily reliant on affine cohomology vanishing, which will not matter in the left case. The main motivation for constructing left derived functors is that they will be adjoint to our right derived functors, and thus we can use them to argue about the right functors. In this sense, we will give new proofs of various facts from the previous section.

We will repurpose our definition from the previous section of a derived functor of an exact functor $\mathrm{K}(\mathcal{O}_X) \to \mathcal{E}$ from $\mathrm{K}(\mathcal{O}_X)$ (or a triangulated subcategory), to a triangulated category $\mathcal{E}$. But we consider the dual case, so now we consider

right Kan extensions along the localization map $Q\colon \mathrm{K}(\mathcal{O}_X) \to D(\mathcal{O}_X)$. We note that for the first part of this section, we have in mind an arbitrary ringed space $(X, \mathcal{O}_X)$, not necessarily a scheme.

Definition 2.4.1. Let X be a ringed space. An object $\mathcal{F}^\bullet \in \mathrm{K}(\mathcal{O}_X)$ is *K-flat* if the complex $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is acyclic for every acyclic object $\mathcal{G}^\bullet \in \mathrm{K}(\mathcal{O}_X)$.

Lemma 2.4.2. *Let X be a ringed space.*

(a) A bounded above complex of flat $\mathcal{O}_X$-modules is K-flat.
(b) If $\mathcal{F}^\bullet, \mathcal{G}^\bullet$ are K-flat, then $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is K-flat.
(c) If $f\colon X \to Y$ is a morphism and $\mathcal{F}^\bullet$ is K-flat over $\mathcal{O}_Y$, then $f^*\mathcal{F}^\bullet$ is K-flat over $\mathcal{O}_X$.
(d) If $\mathcal{F}^\bullet$ is K-flat, then $\mathcal{F}^\bullet[n]$ is K-flat for every n.
(e) If $\mathcal{F}_1^\bullet \to \mathcal{F}_2^\bullet \to \mathcal{F}_3^\bullet \to \mathcal{F}_1^\bullet[1]$ is a distinguished triangle in $\mathrm{K}(\mathcal{O}_X)$ and any two of the three are K-flat, then the third is K-flat.
(f) $\mathcal{F}^\bullet$ is K-flat iff $\mathcal{F}_x^\bullet$ is K-flat over $\mathcal{O}_{X,x}$ for every $x \in X$.
(g) An arbitrary termwise direct sum of K-flat complexes is K-flat.
(h) A termwise filtered colimit of K-flat complexes is K-flat.

Proof. (a) We have shown this already using a spectral sequence. (b) This follows from associativity of tensor product of complexes.

(f) Since stalks commute with sums, they commute with tensor product of complexes. Since stalks preserve and jointly reflect exactness, it is clear that $\mathcal{F}_x^\bullet$ being K-flat for all x implies that $\mathcal{F}^\bullet$ is K-flat. Conversely, if $\mathcal{F}^\bullet$ is K-flat and $M^\bullet \in \mathrm{K}(\mathcal{O}_{X,x})$ is acyclic, then $\mathrm{Sk}_x(M^\bullet)$ is also acyclic, thus $\mathcal{F}^\bullet \otimes \mathrm{Sk}_x(M^\bullet)$ is acyclic and we conclude by taking its stalk at x.

(c) By (f), we reduce to showing that if $A \to B$ is a ring map and $M^\bullet$ is K-flat over A, then $M^\bullet \otimes_A B$ is K-flat over B. Indeed, for an acyclic complex $N^\bullet$,

$$(M^\bullet \otimes_A B) \otimes_B N^\bullet \cong M^\bullet \otimes_A N^\bullet$$

holds as underlying A-modules, so the LHS is in particular acyclic.

(d) This is clear because $\mathcal{F}^\bullet[n] \otimes \mathcal{G}^\bullet = (\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)[n]$.

(e) We have shown previously that $(-) \otimes \mathcal{G}^\bullet$ commutes with cones, and hence is triangulated. Thus the result follows from the 2/3 property for acyclicity.

(g) Since coproducts commute with each other, it follows that $(-)^\bullet \otimes \mathcal{G}^\bullet$ commutes with termwise sums. Then the result follows because direct sums are exact in $\mathrm{Mod}(\mathcal{O}_X)$. □

Remark 2.4.3. Let $\mathcal{A}$ be an abelian category. Then if $K_n^\bullet \in \mathrm{Ch}(\mathcal{A})$, it is clear that

$$\underline{\mathrm{Hom}}^\bullet(\bigoplus_n K_n^\bullet, L^\bullet) = \prod_n \underline{\mathrm{Hom}}^\bullet(K_n^\bullet, L^\bullet)$$

since products commute with one another, where $\bigoplus_n K_n$ denotes the termwise sum. Since products of abelian groups are exact, taking zero cycles and zeroth homology shows that $\bigoplus_n K_n^\bullet$ is also the coproduct in $\mathrm{Ch}(\mathcal{A})$ and in $\mathrm{K}(\mathcal{A})$. Similarly, if $\{K_i^\bullet : i \in I\}$ is a diagram of complexes, denote by $\varinjlim_i K_i$ the termwise colimit.

Then

$$\underline{\mathrm{Hom}}(\varinjlim_{i\in I} K_i^\bullet, L^\bullet) = \varprojlim_{i\in I} \underline{\mathrm{Hom}}(K_i^\bullet, L^\bullet)$$

since limits commute with products, and since limits commute with kernels we can take zero cycles to show that $\varinjlim_i K_i^\bullet$ is also the colimit in the category $\mathrm{Ch}(\mathcal{A})$. However, this does not pass to the homotopy category.

As an application of this, we see that a termwise filtered colimit of K-injective complexes is K-injective. Indeed, a termwise filtered colimit of acyclic complexes is acyclic because filtered colimits are exact. Then, by the tensor-hom adjunction in $\mathrm{Ch}(\mathcal{O}_X)$, $(-)\otimes\mathcal{G}^\bullet$ commutes with colimits, so the result follows.

Lemma 2.4.4. *Let X be a ringed space.*

(a) The map $\mathrm{K}(\mathcal{O}_X)\to D(\mathcal{O}_X)$ preserves arbitrary coproducts.

(b) For a direct system $(\mathcal{F}_n^\bullet)$ in $\mathrm{Ch}(\mathcal{O}_X)$, the canonical sequence of complexes

$$0\to\bigoplus_n \mathcal{F}_n^\bullet \xrightarrow{1-s} \bigoplus_n \mathcal{F}_n^\bullet \to \varinjlim_n \mathcal{F}_n^\bullet \to 0$$

is exact, where $\varinjlim_n \mathcal{F}_n^\bullet$ is the termwise filtered colimit.

Proof. (a) Let $\{\mathcal{F}_i^\bullet : i\in I\}$ be a collection of complexes, and consider maps $f_i\colon \mathcal{F}_i^\bullet\to\mathcal{G}^\bullet$ in $D(\mathcal{O}_X)$; represent this by maps

$$\mathcal{F}_i^\bullet \xleftarrow{q_i} \widetilde{\mathcal{F}}_i^\bullet \xrightarrow{\widetilde{f}_i} \mathcal{G}^\bullet,$$

in $\mathrm{K}(\mathcal{O}_X)$ where the q_i's are quasi-isomorphisms, and then since direct sums are exact, this induces a roof $\bigoplus_i \mathcal{F}_i^\bullet \xleftarrow{q} \bigoplus_i \widetilde{\mathcal{F}}_i^\bullet \xrightarrow{\widetilde{f}} \mathcal{G}^\bullet$ with q a quasi-isomorphism, and it's clear that the corresponding map in $D(\mathcal{O}_X)$ shows the existence part of the universal property.

For uniqueness, suppose we have a map $\bigoplus_i \mathcal{F}_i^\bullet\to\mathcal{G}^\bullet$ in $D(\mathcal{O}_X)$ such that the composites

$$\mathcal{F}_i^\bullet \to \bigoplus_i \mathcal{F}_i^\bullet \xrightarrow{f} \mathcal{G}^\bullet$$

are all zero. Represent f by a roof $\bigoplus_i \mathcal{F}_i^\bullet \xrightarrow{\widetilde{f}} \widetilde{\mathcal{G}}^\bullet \xleftarrow{q} \mathcal{G}^\bullet$ with q a quasi-isomorphism, then the composites

$$\mathcal{F}_i^\bullet \to \bigoplus_i \mathcal{F}_i^\bullet \xrightarrow{\widetilde{f}} \widetilde{\mathcal{G}}^\bullet$$

exist in $\mathrm{K}(\mathcal{O}_X)$ and become 0 in $D(\mathcal{O}_X)$. This means that there are quasi-isomorphisms $\widetilde{\mathcal{F}}_i^\bullet \xrightarrow{s_i} \mathcal{F}_i^\bullet$ whose composites with the above pair are zero in $\mathrm{K}(\mathcal{O}_X)$. Again by exactness of coproducts, this means that the induced diagram

$$\bigoplus_i \widetilde{F}_i^\bullet \xrightarrow{s} \bigoplus_i \mathcal{F}_i^\bullet \xrightarrow{\widetilde{f}} \widetilde{G}^\bullet \xleftarrow{q} \mathcal{G}^\bullet$$

has the properties that s and q are quasi-isomorphisms and the composite of the first two maps is 0 in $\mathrm{K}(\mathcal{O}_X)$. If we invert the last map in $D(\mathcal{O}_X)$ this shows that $f\circ s = 0$ in $D(\mathcal{O}_X)$, hence $f=0$.

(b) (This is unrelated, so I'm not sure why it's in this lemma.) This sequence is exact on the right, because we can check this after taking Hom's, which recovers the universal property that to map into $\varinjlim_n \mathcal{F}_n^\bullet$ we give a sequence of maps into the $\mathcal{F}_n^\bullet$ compatible with the maps $\mathcal{F}_n^\bullet \to \mathcal{F}_{n+1}^\bullet$. For injectivity, since everything is termwise we can assume these are single sheaves. The tuple $(x_n)_{n\geq 1}$ maps to

$$(x_1, x_2 - f_1(x_1), x_3 - f_2(x_2), \dots),$$

so if this is zero, then $x_1 = 0$, so $x_2 - 0 = 0$, and so on. □

Remark 2.4.5. The argument above shows that if $\mathcal{A}$ is an abelian category which admits *exact* coproducts indexed by a set I, then $D(\mathcal{A})$ admits coproducts indexed by I, and they are computed on any representative.

In particular, $D(\mathcal{O}_X)$ has arbitrary direct sums and they are computed termwise on any representative. Also, for any collection $\{A_\alpha\}$ in $D(\mathcal{O}_X)$, the natural map

$$\bigoplus_\alpha H^p(A_\alpha) \to H^p(\bigoplus_\alpha A_\alpha)$$

is an isomorphism. Indeed, this follows by choosing representatives for the A_α and then noting that cohomology of complexes commutes with termwise direct sums because they are exact. It follows e.g. that $D_{\mathrm{qc}}(X)$ for X a scheme also has arbitrary direct sums.

If $\mathcal{E}$ is a triangulated category and (K_n, f_n) is a direct system in $\mathcal{E}$, then we define the *homotopy colimit* of the system to be its derived limit in the opposite category. In other words, it is an object K equipped with a distinguished triangle:

$$\bigoplus_n K_n \xrightarrow{1-s} \bigoplus_n K_n \to K \to \bigoplus_n K_n[1]$$

In this case, we write $K = \operatorname{hocolim}_n K_n$. The dual of the theory we constructed in the previous section applies. For instance, every $L \in \mathcal{E}$ provides a Milnor sequence

$$0 \to R^1 \varprojlim_n \operatorname{Hom}_{\mathcal{E}}(K_n, L[-1]) \to \operatorname{Hom}_{\mathcal{E}}(K, L) \to \varprojlim_n \operatorname{Hom}_{\mathcal{E}}(K_n, L) \to 0$$

of abelian groups. Again, all of the same non-uniqueness issues apply from before. We note immediately that in $D(\mathcal{O}_X)$ (or $D_{\mathrm{qc}}(X)$ for X a scheme), homotopy colimits for all systems exist and are computed by taking the appropriate direct sums and then the cone.

Here is another approach to computing a homotopy colimit. If $(\mathcal{F}_n^\bullet)_n$ is a system of complexes, equipped with actual chain maps $f_n \colon \mathcal{F}_n^\bullet \to \mathcal{F}_{n+1}^\bullet$, then the exact sequence of the previous lemma gives a distinguished triangle

$$\bigoplus_n \mathcal{F}_n^\bullet \xrightarrow{1-s} \bigoplus_n \mathcal{F}_n^\bullet \to \varinjlim_n \mathcal{F}_n^\bullet \to \bigoplus \mathcal{F}_n^\bullet[1]$$

in $D(\mathcal{O}_X)$ where $\varinjlim_n \mathcal{F}_n^\bullet$ is the termwise direct colimit. Therefore, we have

$$\operatorname{hocolim}_n \mathcal{F}_n^\bullet \xrightarrow{\sim} \varinjlim_n \mathcal{F}_n^\bullet$$

as objects in the derived category. Note that since filtered colimits are exact, this proves that the canonical map

$$\varinjlim_n H^p(A_n) \to H^p(\operatorname{hocolim}_n A_n)$$

is an isomorphism for all $p \in \mathbb{Z}$ and systems $(A_n)_n$ in $D(\mathcal{O}_X)$. Dual to the case of derived limits, if $A \in D(\mathcal{A})$ is an object, then the maps

$$\tau_{\leq 1}A \to \tau_{\leq 2}A \to \tau_{\leq 3}A \to \cdots$$

form a canonical directed system, so we have induced maps $\operatorname{hocolim}_n \tau_{\leq n}A \to A$ and we can ask whether these are isomorphisms. For the same reason as the limit case, even though these maps aren't unique, the condition that they be isomorphisms is independent of the choice of presentation of homotopy colimit and choice of map.

Lemma 2.4.6. *Let X be a ringed space.*

(a) For $\mathcal{F}^\bullet \in \operatorname{Ch}(\mathcal{O}_X)$, the canonical map from the (termwise) colimit

$$\varinjlim_n \tau_{\leq n}\mathcal{F}^\bullet \to \mathcal{F}^\bullet$$

is an isomorphism in $\operatorname{Ch}(\mathcal{O}_X)$.

(b) In particular, if $A \in D(\mathcal{O}_X)$, then the canonical map

$$\operatorname{hocolim}_n \tau_{\leq n}A \rightsquigarrow A$$

is an isomorphism in $D(\mathcal{O}_X)$.

Proof. (a) Since it is termwise, we just have to show $\varinjlim_n \tau_{\leq n}\mathcal{F}^k \xrightarrow{\sim} \mathcal{F}^k$ for all k. This is true because this system is eventually constant (i.e. after $n > k$), and that implies the stated result as can be seen e.g. by applying Hom's and using the corresponding statement for inverse limits.

(b) By choosing a representative $A = \mathcal{F}^\bullet$ and the remark above, (a) finishes the proof since the system lifts to $\operatorname{Ch}(\mathcal{O}_X)$. □

Lemma 2.4.7. *Let X be a ringed space and $\mathcal{G}^\bullet \in \operatorname{Ch}(\mathcal{O}_X)$. For each $n \geq 1$, there is a bounded above complex $\mathcal{F}_n^\bullet$ of flat $\mathcal{O}_X$-modules and a commutative diagram*

$$\begin{array}{ccccccc}
\mathcal{F}_1^\bullet & \longrightarrow & \mathcal{F}_2^\bullet & \longrightarrow & \mathcal{F}_3^\bullet & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
\tau_{\leq 1}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 2}\mathcal{G}^\bullet & \longrightarrow & \tau_{\leq 3}\mathcal{G}^\bullet & \longrightarrow & \cdots
\end{array}$$

such that the vertical maps are surjective quasi-isomorphisms.

Proof. We showed that $\mathcal{O}_X$-modules admit surjections from flat $\mathcal{O}_X$-modules functorially, meaning that there's a functor $F\colon \operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(\mathcal{O}_X)$ and a natural transformation $F \to \mathrm{Id}$ giving the surjection. Therefore, since our explicit construction of bounded above resolutions for adapted classes is also functorial, there is a functor $F\colon \operatorname{Ch}(\mathcal{O}_X) \to \operatorname{Ch}(\mathcal{O}_X)$ with a natural transformation $F \to \mathrm{Id}$ that is componentwise a surjective quasi-isomorphism. Applying this to the system $(\tau_{\leq n}\mathcal{G}^\bullet)$ gives the result. □

The previous lemma is the key observation allowing us to construct useful unbounded left derived functors.

Corollary 2.4.8. *Let X be a ringed space.*

(a) For each $\mathcal{G}^\bullet \in \mathrm{K}(\mathcal{O}_X)$ there is a surjective quasi-isomorphism $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ in $\mathrm{K}(\mathcal{O}_X)$ such that $\mathcal{F}^\bullet$ is K-flat.

(b) Denote by $\mathrm{K}(\mathcal{F}_{\mathcal{O}_X}) \subset \mathrm{K}(\mathcal{O}_X)$ the full (triangulated) subcategory consisting of K-flat complexes. Then the natural map

$$\mathrm{K}(\mathcal{F}_{\mathcal{O}_X}) \to D(\mathcal{O}_X)$$

is a triangulated localization by the quasi-isomorphisms.

Proof. Observe that by Lemma 2.4.2, $\mathrm{K}(\mathcal{F}_{\mathcal{O}_X})$ is indeed a full triangulated (replete) subcategory, and we have seen many times why the result of (a) implies the localization statement of (b), from the localization of subcategories criterion.

For (a), we apply the above lemma and set $\mathcal{F}^\bullet \coloneqq \varinjlim_n \mathcal{F}^\bullet_n$ to be the termwise filtered colimit. Then we get an induced map $\mathcal{F} \to \varinjlim_n \tau_{\leq n}\mathcal{G}^\bullet \cong \mathcal{G}^\bullet$, which is first of all surjective because filtered colimits are exact, and is also a quasi-isomorphism, e.g. because $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ represent the hocolim of the two isomorphic systems in $D(\mathcal{O}_X)$. □

We are now going to quickly generalize our proof of the universal property for K-injectives as discussed in the previous section, to use a framework which applies to K-flats.

Remark 2.4.9. This is a brief recap of how we have been using triangulated localizations. Let $\mathcal{A}$ be an abelian category, and suppose we have full triangulated subcategories

$$\mathcal{E} \subset \mathcal{D} \subset \mathrm{K}(\mathcal{A}).$$

(1) Denote by S the localizing class of quasi-isomorphisms in $\mathrm{K}(\mathcal{A})$. Since $\mathcal{E}$ and $\mathcal{D}$ are closed under cones, it follows that $S_\mathcal{E} = S \cap \mathcal{E}$ and $S_\mathcal{D} = S \cap \mathcal{D}$ are both localizing classes in the respective subcategories. The restricted classes are compatible with the triangulation, because the subcategories are closed under shifts, and the other property is clear because they are full subcategories. These are saturated localizing classes because the same was true before restricting.

(2) Then we have the two criteria for localizing subcategories, each dual to the other (in the saturated case): For any quasi-isomorphism $A^\bullet \to D_1^\bullet$ with $D_1^\bullet \in \mathcal{D}$, there exists a quasi-isomorphism $D_2^\bullet \to A^\bullet$ with $D_2^\bullet \in \mathcal{D}$. Then the natural map between localizations is fully faithful. Alternatively, we can reverse the arrows with the same conclusion.

(3) Here are the simple cases where (2) is satisfied for $\mathcal{D}$ that we actually use. First is when $\mathcal{D}$ is saturated under quasi-isomorphisms, meaning that anything isomorphic in the derived category to an object of $\mathcal{D}$ is already in $\mathcal{D}$ (e.g. $\mathrm{K}_{\mathrm{qc}}(X)$ for X a scheme). Another case is (e.g.) $\mathcal{D} = \mathrm{K}^+(\mathcal{A})$, for if $A^\bullet$ has bounded below cohomology, then the map $A^\bullet \to \tau_{\geq k}A^\bullet$ is a quasi-isomorphism for $k \ll 0$.

(4) Now assume that $D_\mathcal{D}(\mathcal{A}) \coloneqq \mathcal{D}[S_\mathcal{D}^{-1}] \to D(\mathcal{A})$ is fully faithful. We describe this situation by saying that $\mathcal{D} \subset \mathrm{K}(\mathcal{A})$ is a *localizing* full triangulated subcategory.

We usually apply the criterion again to the subcategory $\mathcal{E}$ to see when $\mathcal{E}[S_{\mathcal{E}}^{-1}] \to D_{\mathcal{D}}(\mathcal{A})$ is an equivalence. We almost always apply this by saying that for each object $D^\bullet \in \mathcal{D}$, there exists a quasi-isomorphism $E^\bullet \to D^\bullet$ with $E^\bullet \in \mathcal{E}$. Or the other way around, as long as the direction is consistent across different objects. Then the criterion shows that $\mathcal{E} \to D_{\mathcal{D}}(\mathcal{A})$ is a triangulated localization by the quasi-isomorphisms.

(5) Assume we're in the situation of (4) and $F\colon \mathcal{E} \to \mathcal{T}$ is a triangulated functor. Then F sends quasi-isomorphisms to isomorphisms iff it sends acyclic complexes to 0; thus in this case, this induces a triangulated functor on the localization $\widetilde{F}\colon D_{\mathcal{D}}(\mathcal{A}) \to \mathcal{T}$; any map $F \to F'$ of two such functors induces a unique natural transformation of $\widetilde{F} \to \widetilde{F}'$.

We will, going forward, use this remark without further mention. Our next goal is to prove a more general version of our classical adapted classes result which also covers the K-injective result from the last section. Since the application in mind is to K-flats, we will state it in its 'left' version.

Definition 2.4.10. Let $\mathcal{A}$ be an abelian category and $\mathcal{D} \subset \mathrm{K}(\mathcal{A})$ a localizing full triangulated subcategory.

(a) A full triangulated subcategory $\mathcal{E} \subset \mathcal{D}$ is *left preadapted* if for each object $A^\bullet \in \mathcal{D}$ there is a quasi-isomorphism $E^\bullet \to A^\bullet$ with $\mathcal{E}^\bullet \in \mathcal{E}$.

(b) Suppose $F\colon \mathcal{D} \to \mathcal{T}$ is an exact functor of triangulated categories. A left preadapted subcategory $\mathcal{E} \subset \mathcal{D}$ is *left adapted to* F if for every acyclic complex $E^\bullet \in \mathcal{E}$, $F(E^\bullet) = 0$.

We make the dual 'right' definitions as well.

It is immediate from Remark 2.4.9 that a left preadapted subcategory $\mathcal{E}$ satisfies the property that $\mathcal{E} \to D_{\mathcal{D}}(\mathcal{A})$ is a triangulated localization.

Theorem 2.4.11. *Let $\mathcal{A}$ be an abelian category and $\mathcal{D} \subset \mathrm{K}(\mathcal{A})$ a localizing full triangulated subcategory. Let $F\colon \mathcal{D} \to \mathcal{T}$ be an exact functor of triangulated categories and suppose $\mathcal{E} \subset \mathcal{D}$ is left adapted to F.*

(a) F has a left derived functor $(LF\colon D_{\mathcal{D}}(\mathcal{A}) \to \mathcal{T}, \varepsilon)$, which in fact satisfies the right Kan condition against all functors.

(b) If (LF, ε) is a left derived functor of F, then $\varepsilon_{E^\bullet}\colon LF(E^\bullet) \to F(E^\bullet)$ is an isomorphism for all $E^\bullet \in \mathcal{E}$.

The key idea of the proof is the following observation: Suppose there is a morphism $E^\bullet \to D^\bullet$ in $D_{\mathcal{D}}(\mathcal{A})$ represented by a roof $E^\bullet \leftarrow E_1^\bullet \to D^\bullet$, where the assumption means that we can always take the middle object to be in $\mathcal{E}$. Since F preserves quasi-isomorphisms in $\mathcal{E}$, we can apply F to this and obtain

$$F(E^\bullet) \xleftarrow{\cong} F(E_1^\bullet) \to F(D^\bullet) \qquad \text{in } \mathcal{T},$$

which allows us to define a map $F(E^\bullet) \to F(D^\bullet)$. Moreover, this doesn't depend on the choice of roof: To show this, it suffices to show that the map is stable under quasi-isomorphic refinements of $E_1^\bullet$, and we can always choose this to be some

$E_2^\bullet \in \mathcal{E}$. Thus there is a diagram:

$$\begin{array}{ccccc} & & E_2^\bullet & & \\ & \swarrow^{\sim} & \downarrow{\scriptstyle \sim} & \searrow & \\ E^\bullet & \xleftarrow{\sim} & E_1^\bullet & \longrightarrow & D^\bullet \end{array}$$

Upon applying F to this, the three maps labeled with $\sim$ become isomorphisms, and the claim is clear. Therefore we get a well-defined map

$$\mathrm{Hom}_{D(\mathcal{A})}(E^\bullet, D^\bullet) \to \mathrm{Hom}_{\mathcal{T}}(F(E^\bullet), F(D^\bullet))$$

of sets, independent of any choices. Since F sends quasi-isomorphisms in $\mathcal{E}$ to isomorphisms, we can view both sides as functors $\mathcal{E}[S_{\mathcal{E}}^{-1}]^{\mathrm{op}} \times \mathcal{D} \to \mathbf{Set}$, and we further claim that the map above is natural in each variable.

Fix a map $E^\bullet \to D^\bullet$ represented by $E^\bullet \xleftarrow{s} E_1^\bullet \xrightarrow{a} D^\bullet$. If we have $D^\bullet \xrightarrow{b} D_1^\bullet$ in $\mathcal{D}$ then the composite map is represented by

$$E^\bullet \xleftarrow{s} E_1^\bullet \xrightarrow{a} D^\bullet \xrightarrow{b} D_1^\bullet,$$

and its image becomes $F(ba)F(s)^{-1} = F(b)F(a)F(s)^{-1}$, which shows naturality in $\mathcal{D}$. For the other variable, consider a map $E_2^\bullet \to E^\bullet$ represented by $E_2^\bullet \xrightarrow{q} E_3^\bullet \xleftarrow{b} E^\bullet$, then in one direction the resulting map in $\mathcal{T}$ is

$$\big(F(a)F(s)^{-1}\big)\big(F(b)F(q)^{-1}\big).$$

If we compose roofs first, then by definition we form a commutative diagram

$$\begin{array}{ccccccc} & & & & E_4^\bullet & & \\ & & & \overset{\bar{s}}{\swarrow} & & \overset{\bar{b}}{\searrow} & \\ & & E_3^\bullet & & & & E_1^\bullet \\ & \overset{q}{\swarrow} & & \overset{b}{\searrow} & & \overset{s}{\swarrow} & & \overset{a}{\searrow} \\ E_2^\bullet & & & & E^\bullet & & & & D^\bullet \end{array}$$

and take the outer roof, which maps to

$$F(a\bar{b})F(q\bar{s})^{-1} = F(a)\big(F(\bar{b})F(\bar{s})^{-1}\big)F(q)^{-1},$$

and by the commutative square this agrees with the previous direction.

Proof (of Thm. 2.4.11). Let $Q\colon \mathcal{D} \to D_{\mathcal{D}}(\mathcal{A})$ and $Q_{\mathcal{E}}\colon \mathcal{E} \to D_{\mathcal{D}}(\mathcal{A})$ be localizations. Denote by R a quasi-inverse to $\mathcal{E}[S_{\mathcal{E}}^{-1}] \xrightarrow{\sim} D_{\mathcal{D}}(\mathcal{A})$, so that we have a natural isomorphism $u\colon \iota R \xrightarrow{\sim} \mathrm{Id}$.

Since the kernel of $F|_{\mathcal{E}}$ contains the acyclic complexes, this restriction extends to an exact functor which we call

$$LF\colon D_{\mathcal{D}}(\mathcal{A}) \to \mathcal{T}$$

such that $LF \circ Q_{\mathcal{E}} = F|_{\mathcal{E}}$. For each $D^\bullet \in \mathcal{D}$, we have the natural map $u\colon R(D^\bullet) \to D^\bullet$ which is in $D_{\mathcal{D}}(\mathcal{A})$, but by the construction described before the proof and the fact that $R(D^\bullet) \in \mathcal{E}$, this defines a map

$$\varepsilon\colon F(R(D^\bullet)) = LF(D^\bullet) \to F(D^\bullet) \qquad \text{in } \mathcal{T}.$$

The fact that the morphism of Homs was natural in both variables implies that $\mathcal{E}$ is natural in $\mathcal{D}$. Finally, if $E^\bullet \in \mathcal{E}$, then $u\colon R(E^\bullet) \to E^\bullet$ is an isomorphism in $D_{\mathcal{D}}(\mathcal{A})$ by definition of R and u. Thus both arrows in $R(E^\bullet) \leftarrow E_1^\bullet \to E^\bullet$ are quasi-isomorphisms between objects of $\mathcal{E}$, so applying F shows that $\varepsilon\colon LF(E^\bullet) \to F(E^\bullet)$ is an isomorphism.

It remains to prove the universal property: let $G\colon D_{\mathcal{D}}(\mathcal{A}) \to \mathcal{T}$ be a functor equipped with a natural map $\eta\colon G \circ Q \to F$ of functors $\mathcal{D} \to \mathcal{T}$. If we restrict this to $\mathcal{E}$, then we can view this as a map

$$G \circ Q_{\mathcal{E}} \xrightarrow{\eta} F|_{\mathcal{E}} \xleftarrow{\varepsilon} LF \circ Q_{\mathcal{E}}$$

where the second map is an isomorphism. By the universal property of localization, this ascends to a unique map $\widetilde{\eta}\colon G \to LF$ such that

$$\begin{array}{ccc} G \circ Q_{\mathcal{E}} & \xrightarrow{\widetilde{\eta}} & LF \circ Q_{\mathcal{E}} \\ & \searrow^{\eta} & \downarrow{\scriptstyle \varepsilon} \\ & & F|_{\mathcal{E}} \end{array}$$

commutes. We just have to extend this commutativity to $\mathcal{D}$, and uniqueness will follow from uniqueness on $\mathcal{E}$. Let $D^\bullet \in \mathcal{D}$ be arbitrary and $E^\bullet \to D^\bullet$ be a quasi-isomorphism with $E^\bullet \in \mathcal{E}$, then we have a commutative diagram:

$$\begin{array}{ccccc} & & G(D^\bullet) & & \\ & \overset{\cong}{\nearrow} & & \overset{\widetilde{\eta}}{\searrow} & \\ G(E^\bullet) & \xrightarrow{\widetilde{\eta}} & LF(E^\bullet) & \xrightarrow[\cong]{} & LF(D^\bullet) \\ & \searrow^{\eta} & \cong\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ & & F(E^\bullet) & \longrightarrow & F(D^\bullet) \end{array}$$

Because of the displayed isomorphisms, and naturality of η, the composite $\varepsilon \circ \widetilde{\eta}$ is forced to be η as desired. □

Let X be a ringed space. Then we showed that the full subcategory $\mathrm{K}(\mathcal{F}_{\mathcal{O}_X})$ of K-flat complexes is replete, triangulated, and forms a left preadapted class. We will use this to define bounded derived pullbacks and tensor products.

Lemma 2.4.12. *Let Y be a ringed space and $\mathcal{F}^\bullet$ an acyclic K-flat complex.*

(a) For any complex $\mathcal{G}^\bullet \in \mathrm{K}(\mathcal{O}_Y)$, $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is acyclic.

(b) For any morphism $f\colon X \to Y$, $f^*(\mathcal{F}^\bullet)$ is acyclic.

Proof. (a) Let $\mathcal{F}_1^\bullet \to \mathcal{G}^\bullet$ be a K-flat resolution of $\mathcal{G}^\bullet$. Then

$$\mathcal{F}^\bullet \otimes \mathcal{F}_1^\bullet \to \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$$

is a quasi-isomorphism since $\mathcal{F}^\bullet$ is K-flat, but the LHS is acyclic since $\mathcal{F}^\bullet$ is acyclic and $\mathcal{F}_1^\bullet$ is K-flat.

(b) By taking stalks, it suffices to show that if $M^\bullet$ is a K-flat acyclic complex over A and $A \to B$ is a ring map, then $B \otimes_A M^\bullet$ is an acyclic complex of B-modules.

But this is the same as $B \otimes_A M^\bullet$ being acyclic over A, and then the result follows from (a). □

It follows that if $f\colon X \to Y$ is a morphism, then $\mathrm{K}(\mathcal{F}_{\mathcal{O}_Y})$ is left adapted to f^*, so we get a left derived functor

$$Lf^*\colon D(\mathcal{O}_Y) \to D(\mathcal{O}_X).$$

The left derived functors *will* have all of the nice properties, and it will be easy to establish them. For instance, given maps $f\colon X \to Y$ and $g\colon Y \to Z$, $Lf^* \circ Lg^*$ is an exact functor equipped with a map from $(g \circ f)^*$, so there is a natural map $L(g \circ f)^* \to Lf^* \circ Lg^*$. It is clear that this is an isomorphism on the K-flat complexes because pullback preserves K-flatness, so we conclude that $L(g \circ f)^* \cong Lf^* \circ Lg^*$. Similarly, it is easy to prove that it commutes with stalks in the appropriate sense since K-flats are closed under stalks and pullbacks. Finally, since a bounded above complex of flats is K-flat, this construction does extend what we said before.

Now we discuss tensor products. If we restrict the tensor product to

$$\mathrm{K}(\mathcal{F}_{\mathcal{O}_X}) \times \mathrm{K}(\mathcal{O}_X) \to \mathrm{K}(\mathcal{O}_X),$$

then fixing the first variable, we note that the second preserves quasi-isomorphisms by definition of K-flat, and fixing the second shows that the first preserves quasi-isomorphisms by (a) of the lemma above. Thus by localization we obtain a functor

$$(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)\colon D(\mathcal{O}_X) \times D(\mathcal{O}_X) \to D(\mathcal{O}_X).$$

By construction, for each $\mathcal{G}^\bullet$, the functor $(-) \otimes^{\mathbf{L}} \mathcal{G}^\bullet$ is the left derived functor of $(-) \otimes \mathcal{G}^\bullet$, so we obtain canonical natural maps $(-) \otimes^{\mathbf{L}} \mathcal{G}^\bullet \to (-) \otimes \mathcal{G}^\bullet$ between functors $\mathrm{K}(\mathcal{O}_X) \to D(\mathcal{O}_X)$. But, if $\mathcal{G}_1^\bullet \to \mathcal{G}_2^\bullet$ is a map in $\mathrm{K}(\mathcal{O}_X)$, then by the universal property we obtain a natural morphism

$$(-) \otimes^{\mathbf{L}} \mathcal{G}_1^\bullet \to (-) \otimes^{\mathbf{L}} \mathcal{G}_2^\bullet$$

making the obvious diagram commute. In particular, the derived tensor product is equipped with a map

$$\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet \to \mathcal{F}^\bullet \otimes_{\mathcal{O}_X} \mathcal{G}^\bullet$$

natural in both variables, i.e. as functors $\mathrm{K}(\mathcal{O}_X) \times \mathrm{K}(\mathcal{O}_X) \to D(\mathcal{O}_X)$. Our construction says that this map is an isomorphism when $\mathcal{F}^\bullet$ is K-flat, but if $\mathcal{G}^\bullet$ is K-flat, then commutativity of

$$\begin{array}{ccc} \mathcal{F}_1^\bullet \otimes^{\mathbf{L}} \mathcal{G}^\bullet & \xrightarrow[\cong]{} & \mathcal{F}_1^\bullet \otimes \mathcal{G}^\bullet \\ \downarrow\cong & & \cong\downarrow \\ \mathcal{F}^\bullet \otimes^{\mathbf{L}} \mathcal{G}^\bullet & \longrightarrow & \mathcal{F}^\bullet \otimes \mathcal{G}^\bullet \end{array}$$

shows that it is an isomorphism even when $\mathcal{G}^\bullet$ is K-flat (here $\mathcal{F}_1^\bullet \to \mathcal{F}^\bullet$ is a K-flat resolution). Using this map, we can prove all of the standard properties essentially in the same way as before. We will repeat the proof of a few but in general these are easy.

For example, suppose we want a natural isomorphism

$$A \otimes^{\mathbf{L}} (B \otimes^{\mathbf{L}} C) \xrightarrow{\sim} (A \otimes^{\mathbf{L}} B) \otimes^{\mathbf{L}} C$$

as functors $D(\mathcal{O}_X) \times D(\mathcal{O}_X) \times D(\mathcal{O}_X) \to D(\mathcal{O}_X)$. Indeed, if we represent A and B by K-flat complexes then this is clear, also using that the tensor product of K-flats is K-flat. For another example, we show there is a canonical isomorphism

$$Lf^*(A \otimes^{\mathbf{L}} B) \to Lf^*(A) \otimes^{\mathbf{L}} Lf^*(B).$$

To construct it, we have natural maps

$$Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \to Lf^*(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \to f^*(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) \cong f^*(\mathcal{F}^\bullet) \otimes f^*(\mathcal{G}^\bullet)$$

and

$$Lf^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}} Lf^*(\mathcal{G}^\bullet) \to f^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}} f^*(\mathcal{G}^\bullet) \to f^*(\mathcal{F}^\bullet) \otimes f^*(\mathcal{G}^\bullet).$$

If $\mathcal{F}^\bullet, \mathcal{G}^\bullet \in \mathrm{K}(\mathcal{F}_{\mathcal{O}_Y})$, then each of the maps in the bottom row would be isomorphisms because tensor products and pullbacks preserve K-flats. Thus by inverting we get a morphism $Lf^*(\mathcal{F}^\bullet \otimes^{\mathbf{L}} \mathcal{G}^\bullet) \to Lf^*(\mathcal{F}^\bullet) \otimes^{\mathbf{L}} Lf^*(\mathcal{G}^\bullet)$ which is natural as functors $K(\mathcal{O}_Y) \times \mathrm{K}(\mathcal{O}_Y) \to D(\mathcal{O}_X)$. By the universal property of localization this gives a unique map with the desired source and target on the derived category. But also when $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are K-flat, the first row is an isomorphism, so this unique map is an isomorphism.

We mention that since K-flatness can be checked stalkwise, it is in particular a local notion stable under restriction. Therefore we have the standard restriction-to-an-open properties of Lf^* and $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)$ as in the bounded above case.

One of the major reasons to do unbounded derived category theory is that we can use adjunctions in a literal way. In fact, a powerful strategy to prove something about unbounded right derived functors, which is sometimes annoying as we saw, is to prove a different statement about left derived functors and argue by adjointness. Generally, however, we need K-injectives for this, so we will usually do this over schemes. Thus let us first discuss how the unbounded left derived functors behave over schemes, and then we revisit adjunctions.

Let $X = \operatorname{Spec} R$ be an affine scheme. We claim that a complex $M^\bullet \in \mathrm{K}(R)$ is K-flat iff $\widetilde{M}^\bullet$ is K-flat. The latter condition holds iff $M^\bullet_{\mathfrak{p}}$ is K-flat over $R_{\mathfrak{p}}$ for all $\mathfrak{p}$. It is clear that if this condition holds then $M^\bullet$ is K-flat since $(M^\bullet \otimes N^\bullet)_{\mathfrak{p}} = M^\bullet_{\mathfrak{p}} \otimes N^\bullet_{\mathfrak{p}}$ and localization is exact. Conversely, if $M^\bullet$ is K-flat, then as a special case of f^* preserving K-flatness we deduce $M^\bullet_{\mathfrak{p}}$ is K-flat for all $\mathfrak{p}$.

In particular, by the equivalence $D(R) \cong D_{\mathrm{qc}}(\operatorname{Spec} R)$, a quasicoherent sheaf on an affine scheme has a K-flat resolution by quasicoherent sheaves. We use this to give trivial proofs of results like the following one:

Lemma 2.4.13. *Let X be a scheme.*

(a) If $f\colon X \to Y$ is a morphism, then Lf^* maps $D_{\mathrm{qc}}(Y)$ to $D_{\mathrm{qc}}(X)$.
(b) If $A, B \in D_{\mathrm{qc}}(X)$, then $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D_{\mathrm{qc}}(X)$.

Proof. (a) This is local on Y, so assume Y is affine. Then $A \in D_{\mathrm{qc}}(Y)$ is isomorphic to $\mathcal{F}^\bullet$ where $\mathcal{F}^\bullet$ is a K-flat complex of quasicoherent sheaves, and so

$Lf^*(A) \cong f^*\mathcal{F}^\bullet$ is a complex of quasicoherent sheaves and hence has quasicoherent cohomology.

(b) Similarly, the question is local so we may assume X is affine, in which case we can take quasicoherent K-flat resolutions of both terms, and then the tensor product is quasicoherent (since the direct sum of quasicoherent sheaves is quasicoherent). □

Remark 2.4.14. Here is an observation which we essentially already know. Let $\mathcal{E} \subset \mathrm{K}(\mathcal{A})$ be a left preadapted class with the property that $\mathcal{E}_{\mathrm{ac}} \perp B^\bullet$ for some $B^\bullet \in \mathrm{K}(\mathcal{A})$. Here, $\mathcal{E}_{\mathrm{ac}}$ denotes the full subcategory of $\mathcal{E}$ of acyclic complexes, and the notation means that for every $A^\bullet \in \mathcal{E}_{\mathrm{ac}}$ we have $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) = 0$.

If we are in this situation, then in fact the natural map

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) \to \mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$$

is an isomorphism for all $A^\bullet \in \mathcal{E}$. Note that this is easy to see in the presence of K-injectives, but the general proof is easy too. Given a roof $A^\bullet \leftarrow E^\bullet \to B^\bullet$ representing a map in the target, take the cone sequence

$$E^\bullet \to A^\bullet \to \mathrm{Cone} \to E^\bullet[1]$$

and apply $\mathrm{Hom}_{K(\mathcal{A})}(-, B^\bullet)$; by our assumption, the natural induced map $\mathrm{Hom}(A^\bullet, B^\bullet) \to \mathrm{Hom}(E^\bullet, B^\bullet)$ is an isomorphism, and thus there is a lift to a map $A^\bullet \to B^\bullet$ already in $\mathrm{K}(\mathcal{A})$. Conversely, if $A^\bullet \to B^\bullet$ in $\mathrm{K}(\mathcal{A})$ becomes 0 in $D(\mathcal{A})$, there is a quasi-isomorphism $E^\bullet \to A^\bullet$ such that the composite is zero already in $\mathrm{K}(\mathcal{A})$, and we may assume $E^\bullet \in \mathcal{E}$. Then the argument above gives

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(E^\bullet, B^\bullet),$$

so the map $A^\bullet \to B^\bullet$ was already zero.

We note that this is of course an iff statement, since any acyclic $A^\bullet$ is 0 in $D(\mathcal{A})$. This also shows that if $\mathcal{E}_{\mathrm{ac}} \perp B^\bullet$, then $\mathcal{E}$ is left adapted for $\underline{\mathrm{Hom}}_{\mathcal{A}}(-, B^\bullet)$, and thus that the corresponding derived functor computes $\mathrm{RHom}_{\mathcal{A}}(-, B^\bullet)$ as expected.

Given this, let us do the main example of an adjunction. We could formulate this generally, but it will be a bit annoying because our K-injectives only exist on an appropriate subcategory, and we also won't need to do this many times.

Let $f\colon X \to Y$ be a qcqs morphism of schemes. We saw in Lemma 2.3.11 and the lemma prior that we get functors

$$Lf^*\colon D_{\mathrm{qc}}(Y) \rightleftarrows D_{\mathrm{qc}}(X) : Rf_*$$

on the D_{qc}'s. We claim this forms an adjunction. We have already shown that adjunctions of functors descend to the homotopy category, so for each $\mathcal{F}^\bullet$ we have a counit map $f^*f_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet$, so we can obtain a zigzag

$$Lf^*Rf_*\mathcal{F}^\bullet \leftarrow Lf^*f_*\mathcal{F}^\bullet \to f^*f_*\mathcal{F}^\bullet \to \mathcal{F}^\bullet.$$

If $\mathcal{F}^\bullet$ is K-injective, then the left-most map is an isomorphism, and therefore this extends to a natural morphism $Lf^*Rf_* \to \mathrm{Id}_{D_{\mathrm{qc}}(X)}$ whose restriction to the K-injectives agrees with the formula above. Thus we obtain a natural map

$$\mathrm{Hom}_{D_{\mathrm{qc}}(Y)}(A, Rf_*B) \longrightarrow \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(Lf^*A, B)$$

factoring through $\mathrm{Hom}_{D_{\mathrm{qc}}(X)}(Lf^*A, Lf^*Rf_*B)$. To show that this is an isomorphism, take $B = \mathcal{I}^\bullet$ to be K-injective, and $A = \mathcal{F}^\bullet$ to be K-flat. The key observation is that on the homotopy category

$$0 = \mathrm{Hom}_{\mathrm{K}(X)}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet) \cong \mathrm{Hom}_{\mathrm{K}(Y)}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet)$$

if $\mathcal{F}^\bullet$ is acyclic (in $\mathrm{K}(\mathcal{F}_{\mathfrak{X}})$), since then $f^*\mathcal{F}^\bullet$ is acyclic. Therefore, $\mathrm{K}_{\mathrm{ac}}(\mathcal{F}_X) \perp f_*\mathcal{I}^\bullet$, so the remark above applies, and thus

$$\begin{aligned}\mathrm{Hom}_{D_{\mathrm{qc}}(Y)}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet) &= \mathrm{Hom}_{K(Y)}(\mathcal{F}^\bullet, f_*\mathcal{I}^\bullet)\\ &\cong \mathrm{Hom}_{\mathrm{K}(X)}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet)\\ &= \mathrm{Hom}_{D(X)}(f^*\mathcal{F}^\bullet, \mathcal{I}^\bullet),\end{aligned}$$

which is the desired result.

Example 2.4.15. Recall the adjunction $j_! \dashv j^*$ where $j\colon U \hookrightarrow X$ is an open inclusion. By exact adjunctions (which don't even require K-injectives), this descends to an adjunction

$$j_!\colon D(\mathcal{O}_U) \rightleftarrows D(\mathcal{O}_X) : j^*.$$

In particular, as long as $j\colon U \to X$ is qcqs and $A \in D_{\mathrm{qc}}(U)$ we have an isomorphism $A \cong j^*Rj_*A$, and hence an adjoint map $j_!A \to Rj_*A$ in $D(X)$. Suppose that $\operatorname{Supp} A \subset Z$ where Z is closed in X. Then this map is an isomorphism. Indeed, this can be checked on the level of sheaves of abelian groups, where we have closed immersions $i\colon Z \to U$ and $i_1\colon Z \to X$; again the exact adjunction $i^{-1} \dashv i_*$ descends to the derived categories, giving a map $A \to i_*i^{-1}A$. This map is an isomorphism on all cohomology, as that can be checked on stalks. Therefore

$$Rj_*A \cong Rj_*i_*i^{-1}A = i_{1,*}i^{-1}A,$$

and that $j_*A \cong i_{1,*}i^{-1}A$ can be checked after taking cohomology and applying stalks.

We can do the same thing for RHom. Indeed, we have a natural map

$$\underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \underline{\mathcal{H}om}(Lf^*\mathcal{F}^\bullet, Lf^*\mathcal{G}^\bullet) \to \mathrm{RHom}(Lf^*\mathcal{F}^\bullet, Lf^*\mathcal{G}^\bullet),$$

so we obtain a natural map

$$\mathrm{RHom}_{\mathcal{O}_Y}(A, B) \to \mathrm{RHom}_{\mathcal{O}_X}(Lf^*A, Lf^*B)$$

for $B \in D_{\mathrm{qc}}(Y)$. If we replace B with Rf_*B for $B \in D_{\mathrm{qc}}(X)$, we can use this observation and the adjunction counit to obtain

$$\mathrm{RHom}_{\mathcal{O}_Y}(A, Rf_*B) \to \mathrm{RHom}_{\mathcal{O}_X}(Lf^*A, B),$$

which is an isomorphism since it induces an isomorphism on all cohomology by the case above.

These questions become more delicate if we want to involve $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}$, because as we saw it often doesn't land in $D_{\mathrm{qc}}(X)$. We mentioned that if the first argument is pseudocoherent and the second is in $D^+_{\mathrm{qc}}(X)$, then the result is quasicoherent, but this case was already known to us. The other interesting case is when the

first argument is perfect, but most of this case is easy to handle directly from the dualizability properties we discussed earlier.

For instance, this puts a big limitation on proving the unbounded tensor Hom adjunction, because we cannot make sense of a term like

$$\mathbf{R}\mathcal{H}om_{D(X)}(A, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C))$$

unless $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C) \in D_{\mathrm{qc}}(X)$. However, we can use the above strategy to prove the standalone adjunction

$$\mathrm{Hom}_{D(X)}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C) \cong \mathrm{Hom}_{D(X)}(A, \mathbf{R}\mathcal{H}om(B, C))$$

only under the assumption that $C \in D_{\mathrm{qc}}(X)$. This is clear after one notices that if $\mathcal{F}^\bullet$ is K-flat and $\mathcal{I}^\bullet$ is K-injective, then $\underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ is again K-injective. Indeed, if $\mathcal{G}^\bullet$ is acyclic then

$$\mathrm{Hom}_{\mathrm{K}(X)}(\mathcal{G}^\bullet, \underline{\mathcal{H}om}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)) \cong \mathrm{Hom}_{\mathrm{K}(X)}(\mathcal{G}^\bullet \otimes \mathcal{F}^\bullet, \mathcal{I}^\bullet) = 0$$

by the assumptions and the underived adjunction. Therefore, taking $C = \mathcal{I}^\bullet \in D_{\mathrm{qc}}(X)$ to be K-injective, $B = \mathcal{G}^\bullet$ to be K-flat, and $A = \mathcal{F}^\bullet$, we obtain

$$\begin{aligned}
\mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet, \mathcal{I}^\bullet) &= \mathrm{Hom}_{\mathrm{K}(X)}(\mathcal{F}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G}^\bullet, \mathcal{I}^\bullet) \\
&\cong \mathrm{Hom}_{\mathrm{K}(X)}(\mathcal{F}^\bullet, \underline{\mathcal{H}om}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)) \\
&= \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \underline{\mathcal{H}om}(\mathcal{G}^\bullet, \mathcal{I}^\bullet)) \\
&= \mathrm{Hom}_{D(X)}(\mathcal{F}^\bullet, \mathbf{R}\mathcal{H}om(\mathcal{G}^\bullet, \mathcal{I}^\bullet))
\end{aligned}$$

as desired. Note that in this argument, if we want $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ to represent arbitrary objects of $D(X)$, then it's important that we use the fact that $\mathcal{I}^\bullet$ is truly K-injective as an object of $D(X)$ and not just of $D_{\mathrm{qc}}(X)$. But, as explained, we don't have the tools to show (or state) this on the level of RHom or $\mathbf{R}\mathcal{H}om$ unless B is perfect.

Remark 2.4.16. When dealing with perfect complexes over schemes, we usually like to replace the notion of strictly perfect which we discussed in the previous section with 'bounded complex of locally free sheaves'. We can do this for two reasons. First, as we showed, a strictly perfect complex on a locally ringed space is a bounded complex of finite locally free sheaves. Second, because there are only finitely many terms, a bounded complex of finite locally frees is locally a complex of finite free sheaves, hence strictly perfect. Thus, we can also equivalently just work with bounded complexes of finite frees.

The most important property is that bounded complexes of finite locally free sheaves are internally projective in the unbounded sense. In other words, if $\mathcal{F}^\bullet$ is a bounded complex of vector bundles, then the natural map

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

is an isomorphism even for unbounded $\mathcal{G}^\bullet \in \mathrm{K}_{\mathrm{qc}}(X)$. This follows because the corresponding internal Hom spectral sequence works as the first argument is bounded, so $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, -)$ preserves acyclics and hence quasi-isomorphisms.

Let us briefly revisit tor dimension and possibly unbounded perfect objects and show that they have the right properties which extend the cases we studied earlier. Recall our working convention, that whenever we use terms like perfect, tor dimension, etc. for objects of a derived category of a ring R, we mean to consider it as a special case of the earlier sections for the one point ringed space with structure sheaf R.

Lemma 2.4.17. *Let X be a scheme.*

(a) If $X = \operatorname{Spec} R$ is affine, the natural map

$$\widetilde{A} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \widetilde{B} \to \widetilde{A \otimes^{\mathbf{L}}_R B}$$

is an isomorphism for all $A, B \in D(R)$.

(b) Let $A \in D_{\mathrm{qc}}(X)$. Then A has tor-amplitude in $[a, b]$ iff $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{F} \in D^{[a,b]}_{\mathrm{qc}}(X)$ for all $\mathcal{F} \in \mathrm{QCoh}(X)$.

Proof. (a) Writing $A = F^\bullet$ and $B = M^\bullet$ with $F^\bullet$ K-flat, we know that $\widetilde{F}^\bullet$ is also K-flat so the result follows from the underived case. (b) ($\Leftarrow$) Since $\mathcal{O}_X$ is quasicoherent A is bounded. We showed that tor dimension is stalk local, and since stalks commute with cohomology and (a), we reduce to showing that if $A \in D(R)$ has tor amplitude in $[a, b]$, then $A_{\mathfrak{p}}$ has tor amplitude in $[a, b]$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Indeed this is just the fact that tor amplitude respects base change under $A \to A_{\mathfrak{p}}$. □

Lemma 2.4.18. *Let $X = \operatorname{Spec} R$ be an affine scheme and $A \in D(R)$.*

(a) A has tor amplitude in $[a, b]$ iff $\widetilde{A}$ has tor amplitude in $[a, b]$.

(b) $M^\bullet \in \mathrm{K}(R)$ is strictly perfect iff $\widetilde{M}^\bullet$ is strictly perfect.

(c) A is m-pseudocoherent iff $\widetilde{A}$ is m-pseudocoherent.

(d) A is perfect iff $\widetilde{A}$ is perfect.

Proof. (a) This is an immediate consequence of the previous lemma and the fact that $\widetilde{(-)}$ preserves cohomological amplitude. (b) It suffices to show that a module M is strictly perfect iff $\widetilde{M}$ is strictly perfect. One direction is clear because $\widetilde{(-)}$ commutes with sums and preserves frees, and if $\widetilde{M} \oplus \mathcal{F} = \mathcal{O}_X^n$, then by 2/3 $\mathcal{F}$ is quasicoherent and so $\mathcal{F} = \widetilde{N}$, and thus $M \oplus N = R^n$ by full faithfulness.

(c) Since open sets are trivial, we have a triangle $E^\bullet \to A \to C \to E^\bullet[1]$ with $E^\bullet$ strictly perfect and $C \in D^{<m}(R)$. Since $\widetilde{(-)}$ is exact and preserves amplitude, the result follows from (b). Conversely, if $\widetilde{A}$ is m-pseudocoherent, then by the definition and the properties mentioned above and (b), we see that A_{f_i} is m-pseudocoherent over R_{f_i} for $\{f_1, \ldots, f_r\}$ a Zariski cover.

A_{f_i} is m-pseudocoherent over a point hence bounded above, so by finiteness A is bounded above, i.e. $A \in D^{\leq a}(R)$. If $a < m$ then we are done so assume $a \geq m$, then we proceed by upwards induction on a. Indeed, note $A_{f_i} \in D^{\leq a}(R_{f_i})$, so by Example 2.2.44 we have

$$H^a(A_{f_i}) = H^a(A)_{f_i}$$

is finitely generated for $i = 1, \dots, r$, hence $H^a(A)$ is finitely generated. Take F a finite free module with a surjection $F \to H^a(A) \to 0$, and choosing a representative $A = M^\bullet$ with $M^\bullet \in \mathrm{K}^{\leq a}(R)$, we have a surjection $M^a \to H^a(M)$ and hence we have a lift to $F \to M^a$. Again since $M^\bullet \in \mathrm{K}^{\leq a}$, this extends to a map $F[-a] \to M^a[-a] \to M^\bullet = A$ inducing the aforementioned surjection on ath cohomology, and we can take the cone sequence

$$F[-a] \to A \to C \to F[-a+1].$$

The construction implies that $C \in D^{\leq a-1}$, however localizing at all f_i and using that $F_{f_i}[-a]$ remains finite free and hence pseudocoherent, shows that C_{f_i} is m-pseudocoherent for all i. Hence by induction C is m-pseudocoherent. Thus A is m-pseudocoherent.

(d) ($\Rightarrow$) is clear e.g. because we can apply (b) locally. ($\Leftarrow$) If $\widetilde{A}$ is perfect, then since $\operatorname{Spec} R$ is quasicompact, $\widetilde{A}$ is pseudocoherent and has finite tor dimension, thus by (a) and (c) the same hold for A, hence A is perfect. □

Example 2.4.19. Let f be qcqs. Since we have a map

$$Lf^*(Rf_*(A) \otimes^{\mathbf{L}} B) = Lf^*(Rf_*A) \otimes^{\mathbf{L}} Lf^*(B) \to A \otimes^{\mathbf{L}} Lf^*B,$$

there is an induced projection map

$$Rf_*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B \to A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*(B)$$

only by assuming that $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(Y)$. We will show shortly that this is an isomorphism.

Example 2.4.20. If f is qcqs, then Rf_* preserves limits that exist in $D_{\mathrm{qc}}(X)$. We can use this to show that if (K_n) is an inverse system in $D_{\mathrm{qc}}(X)$ for which $\mathbf{R}\varprojlim_n K_n \in D_{\mathrm{qc}}(X)$, then Rf_* commutes with the derived limit. The naive proof which goes through the fact that Rf_* preserves products doesn't work unless $\prod_n K_n$ remains in $D_{\mathrm{qc}}(X)$. If we make the additional assumption that $\mathbf{R}\varprojlim_n Rf_*(K_n)$ is in $D_{\mathrm{qc}}(Y)$, then this has an easy proof, where we just use that Hom is homological and apply the five-lemma and adjunction.

Example 2.4.21. If f, g are qcqs and composable, then we get a trivial proof that $Rg_* \circ Rf_* = R(g \circ f)_*$, simply because this follows for the adjoints. Note that we proved this with some difficulty in the last section by approximation from the bounded below case.

Example 2.4.22. Let X be an R-scheme, and consider the map $p\colon X \to \operatorname{Spec} R$. Here we identify R-modules with their associated quasicoherent sheaves. We can construct a natural map

$$(*) \qquad Lp^* \operatorname{RHom}_{\mathcal{O}_X}(A, B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B)$$

as follows: If $\mathcal{F}^\bullet$ is a complex of $\mathcal{O}_X$-modules and $\mathcal{I}^\bullet$ is K-injective, then we have natural maps

$$Lp^* \operatorname{\underline{Hom}}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \to p^* \operatorname{\underline{Hom}}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \to \underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet),$$

where the second map is adjoint to the isomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet) \xrightarrow{\sim} p_*\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$. Thus we obtain $(*)$. If X (and thus p) is qcqs, and $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \in D_{\mathrm{qc}}(X)$, then this corresponds to a map

$$\mathrm{RHom}_{\mathcal{O}_X}(A, B) \to R\Gamma(X, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B)).$$

We've seen in Lemma 2.3.49 that if A is perfect, then this is an isomorphism. Of course, we also know that we always have this isomorphism under the appropriate boundedness assumptions, for all $\mathcal{O}_X$-modules. We will extend this (in the unbounded case) later.

Example 2.4.23. Here is a simple example of a nontrivial homotopy colimit which can be useful. First let R be a ring and $f \in R$. For each $n \geq 0$, the map $R \to R_f$ sending $1 \mapsto f^{-n}$ induces a morphism

$$\varinjlim (R \xrightarrow{f} R \xrightarrow{f} R \xrightarrow{f} \cdots) \to R_f$$

from the sequential colimit. This map is an isomorphism: it is clear from the definition that a/f^n is in the image for all n and $a \in R$, and on the other hand if $a \mapsto a/f^n$ equals 0 in R_f, then $a/1 = 0$ so $f^k a = 0$ for $k \gg 0$. In other words a becomes zero at a later stage in the directed system.

Let us make this scheme-theoretic and derived. Let $\mathcal{L}$ be a line bundle on a scheme X, $s \in \mathcal{L}(X)$ a global section, and recall that the inclusion $j\colon X_s \hookrightarrow X$ is affine. Indeed, on an affine open $U \subset X$ where $\mathcal{L}$ is trivialized and $s \sim f \in \mathcal{O}_X(U)$, we know that $U \cap X_s = D_U(f)$. In particular we have the derived adjunction $j^* \dashv Rj_*$. We will construct a map

$$\mathrm{hocolim}\,(A \xrightarrow{s} A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L} \xrightarrow{s} A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes 2} \to \cdots) \to Rj_*j^*A$$

for any $A \in D_{\mathrm{qc}}(X)$ and show that it is an isomorphism. Indeed, since s is nowhere vanishing on X_s the map $s\colon \mathcal{O}_{X_s} \to \mathcal{L}|_{X_s}$ is an isomorphism, so we have isomorphisms $\mathcal{L}^{\otimes n}|_{X_s} \to \mathcal{O}_{X_s}$ for each n sending $s^n \mapsto 1$. Tensoring this with A we obtain

$$(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n})|_{X_s} = A|_{X_s} \otimes^{\mathbf{L}}_{\mathcal{O}_{X_s}} \mathcal{L}^{\otimes n}|_{X_s} \to A|_{X_s}$$

so by adjunction this gives maps $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n} \to Rj_*j^*A$. These commute with multiplication-by-s maps since this held before adjunction and the same fact for the maps $\mathcal{L}^{\otimes n}|_{X_s} \to \mathcal{O}_{X_s}$. Thus we obtain the desired map out of the hocolim.

To see that this is an isomorphism, restrict to an affine open $U \subset X$ on which $\mathcal{L}$ is trivialized, so s corresponds to $f \in \mathcal{O}_X(U)$. Then $A|_U$ is represented by a complex of quasicoherent sheaves corresponding to $M^\bullet$, and $j^*\widetilde{M}^\bullet = M^\bullet_f$, so everything reduces to showing

$$\mathrm{hocolim}\,(M^\bullet \xrightarrow{f} M^\bullet \xrightarrow{f} M^\bullet \xrightarrow{f} \cdots) \to M^\bullet_f$$

is an isomorphism inside $D(\mathcal{O}_X(U))$. But homotopy colimits of systems where the maps are chain maps are computed as termwise filtered colimits, and termwise this is an isomorphism by the initial computation. Thus this is an isomorphism. We call this the homotopy colimit localization formula.

In this setup, we can also relatively easily extend the Mayer-Vietoris triangle and the 2-open cover descent result to the unbounded case.

Remark 2.4.24. Suppose we have an exact triangle

$$A \xrightarrow{(b',c')} B \oplus C \xrightarrow{(b,c)} D \to A[1]$$

in a triangulated category, and e.g. c is an isomorphism, then it follows that the map $A \to B$ is an isomorphism. Indeed, consider the diagram of triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B \oplus C & \longrightarrow & D & \longrightarrow & A[1] \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle c^{-1}} & & \downarrow \\ B & \longrightarrow & B \oplus C & \longrightarrow & C & \longrightarrow & B[1] \end{array}$$

where

$$f = \begin{pmatrix} 1 & 0 \\ c^{-1}b & 1 \end{pmatrix} : B \oplus C \to B \oplus C.$$

Now f is an isomorphism because $f \cdot \mathrm{diag}(1,-1)$ squares to the identity matrix. Thus this extends to an isomorphism of triangles for some $r\colon A \to B$. Therefore we deduce

$$(r,0)^T = f(b',c')^T = (b', c^{-1}bb' + c')^T,$$

and in particular $r = b'$ is an isomorphism.

Proposition 2.4.25. *Let X be a scheme and $U, V \subset X$ be quasicompact opens. Suppose $A \in D_{\mathrm{qc}}(U)$, $B \in D_{\mathrm{qc}}(V)$, and we are given an isomorphism $\alpha\colon A|_{U\cap V} \xrightarrow{\sim} B|_{U\cap V}$ in $D_{\mathrm{qc}}(U \cap V)$. Then there exists $C \in D_{\mathrm{qc}}(U \cup V)$ and isomorphisms*

$$\beta_U\colon C|_U \xrightarrow{\sim} A \qquad \beta_V\colon C|_V \xrightarrow{\sim} B$$

making the diagram

$$\begin{array}{ccc} & C|_{U\cap V} & \\ {\scriptstyle \beta_U|_{U\cap V}}\swarrow & & \searrow{\scriptstyle \beta_V|_{U\cap V}} \\ A|_{U\cap V} & \xrightarrow{\alpha} & B|_{U\cap V} \end{array}$$

commute. Moreover, such a (C, β_U, β_V) is unique up to an isomorphism whose restrictions commute with β_U, β_V.

Proof. Let j_U, j_V, j be the (qcqs) inclusions of $U, V, U \cap V$ into the union $U \cup V$. Consider the maps

$$Rj_{U,*}A \xrightarrow{a} Rj_*A|_{U\cap V} \xrightarrow{Rj_*\alpha} Rj_*B|_{U\cap V}$$

and

$$Rj_{V,*}B \xrightarrow{b} Rj_*B|_{U\cap V},$$

and extend the middle map below to an exact triangle:

$$C \xrightarrow{(\beta_U,\beta_V)} Rj_{U,*}A \oplus Rj_{V,*}B \xrightarrow{(Rj_*\alpha\circ a, -b)} Rj_*B|_{U\cap V} \to C[1].$$

Upon restriction to U, the map $-b$ becomes $-\mathrm{Id}$, hence by the observation above the proof it follows that $\beta_U|_U$ is an isomorphism $C|_U \xrightarrow{\sim} A|_U$. Likewise upon

restriction to V, $Rj_*\alpha \circ a$ becomes $Rj_*\alpha|_V$ which is an isomorphism, so $\beta_V|V$ is an isomorphism $C|_V \xrightarrow{\sim} B|_U$. Finally since this is an exact triangle,

$$Rj_*\alpha \circ a \circ \beta_U - b \circ \beta_V = 0,$$

so restricting to $U \cap V$ shows $\alpha \circ \beta_U|_{U\cap V} = \beta_V|_{U\cap V}$.

For uniqueness, now suppose we are given (C', β'_U, β'_V) satisfying the same properties as C. Adjunction will provide a map $C' \to Rj_{U,*}A \oplus Rj_{V,*}B$ whose composite to $Rj_*B|_{U\cap V}$ equals 0, and therefore there is a map $C' \to C$ such that the diagram

$$\begin{array}{ccc} C' & & \\ \vdots & \searrow^{(\beta'_U,\beta'_V)} & \\ C & \xrightarrow{(\beta_U,\beta_V)} & Rj_{U,*}A \oplus Rj_{V,*}B \end{array}$$

commutes. In particular we can split off the two summands, and then restricting to U and V individually shows that $C'|_U \xrightarrow{\sim} C|_U$ and the same for V. □

Remark 2.4.26. We can easily record the following consequence of this proof, which is the Mayer-Vietoris triangle in the unbounded setting. Namely, if we start with $A \in D_{\mathrm{qc}}(U \cup V)$, then the map $A \to Rj_{U,*}A|_U \oplus Rj_{V,*}A|_V$ satisfies the conditions of C as above, so the uniqueness gives a triangle

$$A \to Rj_{U,*}A|_U \oplus Rj_{V,*}A|_V \xrightarrow{(1,-1)} Rj_{U\cap V,*}A|_{U\cap V} \to A[1]$$

mirroring the Mayer-Vietoris triangle from earlier.

Let us briefly check that perfectness still behaves as expected even in the unbounded setting. Most importantly, we want to ensure that duals of perfect complexes still behave well. Now that we have an unbounded tensor hom adjunction, the fact that $\mathcal{O}_X$ is quasicoherent means that we have a natural evaluation map

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} A \to \mathcal{O}_X$$

for any $A \in D(X)$. Therefore, again by adjunction and quasicoherence of $\mathcal{O}_X$, we get a map

$$A \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}\left(\mathbf{R}\mathcal{H}om(A, \mathcal{O}_X), \mathcal{O}_X\right).$$

If $E \in \mathrm{Perf}(X)$ is a possibly unbounded perfect object, then this double dual map is an isomorphism, because the claim is local and hence we may reduce to the bounded case. Next, tensor the evaluation map with $B \in D_{\mathrm{qc}}(X)$. Adjunction gives

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B)$$

for all $A \in D(X)$. If $A = E \in \mathrm{Perf}(X)$, then this is an isomorphism: indeed we can check locally, where E is strictly perfect, and then so is $\underline{\mathcal{H}om}(E|_U, \mathcal{O}_U)$, so both sides reduce to the underived case. Any other properties we care about can be derived from these two.

Now suppose $\mathcal{E}^\bullet$ is a *perfect complex*, i.e. represents a perfect object of $D_{\mathrm{qc}}(X)$. Then if $\mathcal{I}^\bullet \in \mathrm{K}_{\mathrm{qc}}(\mathcal{I}_X)$, we know that

$$\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{I}^\bullet) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{I}^\bullet)$$

is an isomorphism (in $D_{\mathrm{qc}}(X)$). If $f\colon X \to Y$ is a qcqs morphism, we have maps

$$\begin{array}{ccc} f_*\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{I}^\bullet) & \longrightarrow & Rf_*\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{I}^\bullet) \\ & \searrow & \downarrow \cong \\ & & Rf_*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{I}^\bullet) \end{array}$$

and we claim the diagonal is an isomorphism. Indeed, the top map is an isomorphism because we may check this locally, where $\mathcal{E}^\bullet$ admits a quasi-isomorphism from a strictly perfect complex; by naturality of the top map, this is an isomorphism because strictly perfect complexes are K-flat, and hence the corresponding $\underline{\mathcal{H}om}$ is K-injective with quasicoherent cohomology.

To apply this, consider the zigzag

$$\begin{aligned} Rf_*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*\mathcal{E}^\bullet, \mathcal{I}^\bullet) &\leftarrow f_*\underline{\mathcal{H}om}_{\mathcal{O}_X}(f^*\mathcal{E}^\bullet, \mathcal{I}^\bullet) \\ &\cong \underline{\mathcal{H}om}_{\mathcal{O}_Y}(\mathcal{E}^\bullet, f_*\mathcal{I}^\bullet) \\ &\to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}^\bullet, Rf_*\mathcal{I}^\bullet). \end{aligned}$$

The first map is an isomorphism because the same argument as above applies to Lf^*, since $\mathcal{E}^\bullet$ locally admits a quasi-isomorphism from a strictly perfect complex. Therefore there is a natural map

$$Rf_*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*E, B) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E, Rf_*B)$$

for $E \in \mathrm{Perf}(X)$ and $B \in D_{\mathrm{qc}}(X)$. To see that this is an isomorphism, we can apply $R\Gamma(U, -)$ over every affine open $U \subset Y$, and by our earlier result in the qcqs case this reduces to the adjunction on the level of $\mathrm{RHom}_{\mathcal{O}_X}$, which relies on the fact that Lf^*E is perfect as well.

We'll now use this to extend some of our prior isomorphisms. First, if E is perfect, we can use the natural map $\mathcal{O}_Y \to Rf_*\mathcal{O}_X$,

$$E^\vee \to \mathbf{R}\mathcal{H}om(E, Rf_*\mathcal{O}_X) \cong Rf_*\,\mathbf{R}\mathcal{H}om(Lf^*E, \mathcal{O}_Y),$$

so by adjunction we get a map $Lf^*(E^\vee) \to (Lf^*E)^\vee$. This is an isomorphism because locally we can reduce to the bounded case. Therefore, by replacing E with $E^\vee$ in the sheafified adjunction formula above, we obtain

$$Rf_*(B \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*E) \xrightarrow{\sim} Rf_*(B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} E,$$

i.e. the projection formula map is an isomorphism for perfect E and unbounded $B \in D_{\mathrm{qc}}(X)$. Since Lf^* commutes with tensor product, we easily obtain

$$Lf^*\,\mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(E, B) \xrightarrow{\sim} \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^*E, Lf^*B)$$

when $E \in \mathrm{Perf}(Y)$ and $B \in D_{\mathrm{qc}}(X)$.

2.4.2. Generators and Brown representability. In this section, we are going to show that when f is a morphism between qcqs schemes, then there is an adjunction

$$Rf_*\colon D_{\mathrm{qc}}(X) \rightleftarrows D_{\mathrm{qc}}(Y) : f^!,$$

but we will construct $f^!$ through abstract means, and in particular it will *not* be a derived functor of a functor which exists on $\mathrm{K}_{\mathrm{qc}}(Y)$. We are making the

unconventional choice to prove this result before actually doing any algebraic geometry, but the point of this is to emphasize that we can use category theory in a non-trivial way with the introduction of derived categories. We will see later that this abstract result is a generalization of Serre duality.

There are two major steps to this result. We need to prove the *Brown representability theorem* for triangulated categories, and then we need to prove that Rf_* satisfies the hypotheses for this theorem. The main one, which is perfect generation, we will discuss in the next section. Let's do the easier one of the hypotheses as a warmup:

Lemma 2.4.27. *Let $f\colon X \to Y$ be a qcqs morphism of schemes. Then the functor $Rf_*\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ preserves arbitrary coproducts.*

Proof. Since there is a natural comparison map, the claim is local on Y, so we may assume Y is quasicompact. Thus Rf_* has finite right QCoh-dimension d, which means that for any $A \in D_{\mathrm{qc}}(X)$,

$$H^p(Rf_*A) \to H^p(Rf_*(\tau_{\geq n}A))$$

is an isomorphism for $p \geq n + d$. It follows from Prop. 2.2.5, and the fact that coproducts in $D_{\mathrm{qc}}(X)$ are computed termwise on any representative, that Rf_* commutes with arbitrary direct sums of collections of objects of $D_{\mathrm{qc}}(X)$ which are globally bounded below. Moreover, since coproducts of $\mathcal{O}_X$-modules are exact, they commute with truncation. Therefore we have a commutative diagram:

$$\begin{array}{ccc} H^p\big(\bigoplus_\alpha Rf_*A_\alpha\big) & \longrightarrow & H^p\big(Rf_*\bigoplus_\alpha A_\alpha\big) \\ \downarrow & & \downarrow \\ H^p\big(\bigoplus_\alpha Rf_*(\tau_{\geq n}A_\alpha)\big) & \longrightarrow & H^p\big(Rf_*\bigoplus_\alpha \tau_{\geq n}A_\alpha\big) \end{array}$$

The right-hand vertical map is an isomorphism when $p \geq n + d$, and the same holds for the left-hand map because cohomology commutes with direct sums. The bottom map is always an isomorphism by the bounded below case. Therefore for $p \geq n + d$ the top map is an isomorphism. Since n was arbitrary we are done. □

Remark 2.4.28. As an immediate consequence, if $(A_n)_{n\geq 0}$ is a sequential system in $D_{\mathrm{qc}}(X)$ and f is qcqs, then the comparison map

$$\operatorname{hocolim}_n Rf_*A_n \to Rf_*\big(\operatorname{hocolim}_n A_n\big)$$

is an isomorphism in $D_{\mathrm{qc}}(Y)$. Indeed, Rf_* preserves the defining hocolim triangles and the direct sums in these triangles.

Example 2.4.29. This is a continuation of Example 2.4.23, where we proved a canonical isomorphism

$$\operatorname{hocolim}_n(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \xrightarrow{\sim} Rj_*j^*A$$

whenever $\mathcal{L}$ is a line bundle on X, $s \in \mathcal{L}(X)$ a section, $j\colon X_s \hookrightarrow X$ the inclusion, and the transition maps of the system are multiplication by s. Now suppose X is a qcqs R-scheme. Then pushforward to $\operatorname{Spec} R$ is $R\Gamma$, so we obtain an isomorphism

$$\operatorname{hocolim}_n R\Gamma(X, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \xrightarrow{\sim} R\Gamma(X_s, A)$$

inside $D(R)$. Since taking cohomology turns hocolims into filtered colimits, this shows that the natural map

$$\varinjlim_{n\geq 0} \mathbb{H}^p(X, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \xrightarrow{\sim} \mathbb{H}^p(X_s, A)$$

is an isomorphism for any $A \in D_{\mathrm{qc}}(X)$, $s \in \mathcal{L}(X)$, and $p \in \mathbb{Z}$. This simple observation will be very useful for instance when dealing with an ample line bundle. We will also use it in computing the cohomology of projective space.

Let $\mathcal{E}$ be a triangulated category. A collection $\{E_i\} \subset \mathcal{E}$ of objects *generates* $\mathcal{E}$ if for each $A \neq 0 \in \mathcal{E}$ there exists some index i and $n \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathcal{E}}(E_i, A[n]) \neq 0$. This is a somewhat weird notion, so a few observations:

(a) If $\mathcal{E}$ is a subcategory of a derived category which admits unbounded RHom, then the condition is equivalent to the existence of some i such that $\mathrm{RHom}_{\mathcal{A}}(E_i, A) \neq 0$.

(b) If $\mathcal{E}$ has coproducts and $\{E_i\}$ is a (small) set, then $\{E_i\}$ generates $\mathcal{E}$ if and only if $\bigoplus_i E_i$ is a single generator, since

$$\mathrm{Hom}_{\mathcal{E}}(\bigoplus_i E_i, A[n]) = \prod_i \mathrm{Hom}_{\mathcal{E}}(E_i, A[n])$$

and the RHS is nonzero iff one of the factors is nonzero.

(c) If $A \to B$ is a morphism in $\mathcal{E}$, then it is an isomorphism iff

$$\mathrm{Hom}_{\mathcal{E}}(E_i, A[n]) \to \mathrm{Hom}_{\mathcal{E}}(E_i, B[n])$$

is an isomorphism for all i and n. Indeed, one direction is clear, and on the other hand, consider the triangle $A \to B \to C \to A[1]$. Indeed, the corresponding exact sequence shows that $\mathrm{Hom}_{\mathcal{E}}(E_i, C[n]) = 0$ for all $n \in \mathbb{Z}$ and i, and thus $C = 0$.

Of course, the criterion of (b) is in fact equivalent to the set $\{E_i\}$ generating the category, and this is the most reasonable explanation of the terminology. Our main theorem of the next section will be the following one:

Theorem 2.4.30. *Let X be a qcqs scheme. Then $D_{\mathrm{qc}}(X)$ has a perfect generator.*

We instead discuss consequences now. First, if $\mathcal{E}$ is triangulated and has small coproducts, an object $E \in \mathcal{E}$ is *compact* if $\mathrm{Hom}_{\mathcal{E}}(E, -)$ preserves arbitrary coproducts (which is distinct from our general definition of a compact object in an arbitrary category). We say $\mathcal{E}$ is compactly generated if it is generated by a set of compact objects.

First, for any scheme X, if $E \in \mathrm{Perf}(X)$ then $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, -) \cong E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)$ as a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$. By the tensor hom adjunction, this is a left adjoint and hence commutes with colimits in $D_{\mathrm{qc}}(X)$. If X is qcqs, then X has coproducts. Moreover,

$$R\Gamma \circ \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, -) = \mathrm{RHom}_{\mathcal{O}_X}(E, -).$$

since X is qcqs, and as a special case of Lemma 2.4.27 we see that $R\Gamma$ preserves coproducts. Finally, we showed that taking cohomology preserves coproducts, so

we conclude $\mathrm{Hom}_{D_{\mathrm{qc}}(X)}(E, -)$ preserves coproducts (as a functor from $D_{\mathrm{qc}}(X)$), and thus $E \in D_{\mathrm{qc}}(X)$ is compact. We will in the next section show the converse, that all compact objects are perfect.

We emphasize that Thm. 2.4.30 is a remarkably strong statement; it says, for example, that there is a perfect object E such that to check if $A \to B$ is an isomorphism, it suffices to check that

$$\mathbb{H}^p(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} A) \to \mathbb{H}^p(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} B)$$

is an isomorphism for every $p \in \mathbb{Z}$. Indeed, take $E^\vee$ to be a perfect generator, then we know that it suffices to check isomorphisms after applying $\mathrm{RHom}(E^\vee, -)$, but

$$\begin{aligned}\mathrm{RHom}_{\mathcal{O}_X}(E^\vee, -) &= R\Gamma \circ \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-)) \\ &= R\Gamma(X, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} (-))\end{aligned}$$

which proves the statement. The strength of this statement comes from the fact that we are only looking at global cohomology here. In many geometric examples, especially for projective varieties, we will be able to write down explicit generators and can use the principle above.

Let $\mathcal{T}$ have coproducts and $E \in \mathcal{T}_c$. Most immediately, the defining property of compactness says that any map $E \to \bigoplus_i X_i$ corresponds to a collection of maps $E \to X_i$ with all but finitely many nonzero, so in particular the map factors through some summand with only finitely many X_i involved. Here's an alternative way to state this.

Suppose $X = \mathrm{hocolim}_n K_n$ for some direct system (K_n) in $\mathcal{T}$, and take $\mathrm{Hom}(E, -)$ of the associated triangle to obtain an exact sequence

$$\bigoplus_n \mathrm{Hom}^i(E, K_n) \xrightarrow{1-s} \bigoplus_n \mathrm{Hom}^i(E, K_n) \to \mathrm{Hom}^i(E, X) \to \bigoplus_n \mathrm{Hom}^{i+1},$$

but now e.g. as a (very) special case of Lemma 2.4.4, all the left-hand maps are injective, so applying this to $i = 0, 1$ we see

$$0 \to \bigoplus_n \mathrm{Hom}(E, K_n) \xrightarrow{1-s} \bigoplus_n \mathrm{Hom}(E, K_n) \to \mathrm{Hom}(E, X) \to 0$$

is exact. A second application of Lemma 2.4.4 shows that the natural map

$$\varinjlim_n \mathrm{Hom}(E, K_n) \to \mathrm{Hom}(E, X)$$

induced by the maps $K_n \to X$ is an isomorphism. So, this is the replacement for the statement that Hom's out of compact objects should preserve filtered colimits.

Proposition 2.4.31. *Let $\mathcal{E}$ be a triangulated category (with coproducts) and $\{E_i\}$ a generating set of compact objects.*

- *(a) Suppose $\mathcal{D} \subset \mathcal{E}$ is a full replete triangulated subcategory closed under coproducts. If $E_i \in \mathcal{D}$ for each i, then $\mathcal{D} = \mathcal{E}$.*
- *(b) Let $u\colon F \to G$ be a morphism of exact functors $F, G\colon \mathcal{E} \to \mathcal{D}$ which preserve coproducts. If $u_{E_i}\colon F(E_i) \to G(E_i)$ is an isomorphism for each i, then u is an isomorphism.*

PROOF. We will deduce both of these from the following slightly more specific statement.

Lemma 2.4.32. *Under the same assumptions, let $A \in \mathcal{E}$. Then there is a direct system $(K_n)_{n\geq 0}$ where $K_0 = 0$ with morphisms fitting into exact triangles*

$$L_n \to K_n \to K_{n+1} \to L_n[1]$$

such that L_n is a coproduct of shifts of generators and $A \simeq \operatorname{hocolim}_n K_n$.

Indeed, for (a), $L_n \in \mathcal{D}$, so by induction $K_n \in \mathcal{D}$. Since $\mathcal{D}$ is closed under coproducts, it follows that $\operatorname{hocolim}_n K_n \cong A \in \mathcal{D}$ as well. (b) Consider the class of objects $D \in \mathcal{D}$ for which u_D is an isomorphism. By naturality, this is closed under isomorphism. Since F and G commute with shifts and coproducts, it is clear that $\mathcal{D}$ is closed under shifts and coproducts. Finally, by the triangulated 5-lemma, $\mathcal{D}$ is closed under cones. Thus by (a) we conclude. □

PROOF (OF LEMMA 2.4.32). Let K_1 be the direct sum over $E_i[m]$ for all $i, m \in \mathbb{Z}$ and all maps $E_i[m] \to A$. This comes with a natural map $K_1 \to A$. For aesthetic purposes define $K_0 = 0$ and $L_0 = K_1[-1]$ so that we have the $n = 0$ triangle as above.

Inductively, given $K_n \to A$, we construct L_n as the coproduct over all $E_i[m]$ indexed by all i, $m \in \mathbb{Z}$, and maps $E_i[m] \to K_n$ for which the composite $E_i[m] \to K_n \to A$ is zero. This comes with a canonical map to K_n, so define K_{n+1} by the triangle

$$L_n \to K_n \to K_{n+1} \to L_n[1].$$

By definition, the given map $K_n \to A$ is in the kernel of $\operatorname{Hom}(K_n, A) \to \operatorname{Hom}(L_n, A)$, hence it extends to a map $K_{n+1} \to A$ commuting with $K_n \to K_{n+1}$. Thus we obtain a morphism $\operatorname{hocolim}_n K_n \to A$.

To see that this is an isomorphism, we can check this after applying $\operatorname{Hom}(E_i[m], -)$ for all i and m, which by compactness yields the map

$$\varinjlim_n \operatorname{Hom}(E_i[m], K_n) \to \operatorname{Hom}(E_i[m], A)$$

which sends a representative $[\varphi]$ to $E_i[m] \to K_n \to A$. The map was already made surjective when we defined K_1, so suppose $\varphi\colon E_i[m] \to K_n$ represents a class mapping to 0. Then this is a basis element of L_n, and since the composite $L_n \to K_n \to K_{n+1}$ is zero, we have $[\varphi] = 0$ in the source, showing injectivity. □

Remark 2.4.33. Though we loosely assume that all of our categories are locally small, it is crucial that $\mathcal{E}$ be locally small for the construction in Lemma 2.4.32. In the application we have in mind, namely $D_{\mathrm{qc}}(X)$, this is clear because it is equivalent to a subcategory of $\mathrm{K}(X)$. Likewise, it is crucial for the generators to form a small set and not a proper class.

We say that a triangulated category $\mathcal{T}$ is compactly generated if it is generated by a (small) set of compact objects. Now let's discuss some corollaries of the perfect generation theorem.

Corollary 2.4.34. *Let $f\colon X \to Y$ be a qcqs morphism of schemes. Then the natural projection map*

$$Rf_*(A) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} B \to Rf_*(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*B)$$

is an isomorphism for all $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(Y)$.

Proof. The claim is local on Y, so we may assume Y (and thus X) are qcqs. By fixing A, this is a natural transformation in B of exact, coproduct preserving functors $D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(Y)$. Indeed, $(-) \otimes^{\mathbf{L}} (-)$ preserves colimits in each variable, as does Lf^*, since they are left adjoints, and Rf_* preserves arbitrary coproducts. We have seen that it is an isomorphism if $B \in \mathrm{Perf}(Y)$. Thus by Prop. 2.4.31 applied to $D_{\mathrm{qc}}(Y)$ via perfect generation (Thm. 2.4.30), it is an isomorphism. □

Example 2.4.35. We wish to stress that Cor. 2.4.34 is a truly interesting result because it is one of our first examples of statements which are true *only* in the derived category. Namely, we did not prove this by reducing to the underived case, but instead by extending such a case via some category theory. To illustrate this, we'll give an example of a projective morphism $f\colon X \to Y$ of Noetherian schemes and individual quasicoherent sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y where the underived projection formula doesn't hold. In fact, in our example, $\mathcal{G}$ will even be a perfect sheaf.

Let k be a field, $X = \mathbb{P}^1_{k[t]}$, and consider the structure morphism $p\colon X \to \operatorname{Spec} k[t]$. We know (or will see) that

$$\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(-2)) = H^1(X, \mathcal{O}_X(-2)) \cong k[t],$$

so we can choose a nonzero extension class $[\mathcal{F}] \in t \cdot \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(-2))$ in the submodule generated under multiplication by t. Recall that this is the data of a short exact sequence of sheaves

$$(*) \qquad 0 \to \mathcal{O}_X(-2) \to \mathcal{F} \to \mathcal{O}_X \to 0$$

on X (in particular vector bundles) such that the map $\operatorname{Hom}(\mathcal{O}_X, \mathcal{O}_X) \to \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X(-2))$ sends the identity to $[\mathcal{F}]$. On the other hand, regard k as the $k[t]$-module $k[t]/(t)$, and consider the corresponding quasicoherent sheaf $\mathcal{G} = \widetilde{k}$ on $\operatorname{Spec} k[t]$. We claim that the projection map

$$p_*\mathcal{F} \otimes_R \mathcal{G} \to p_*(\mathcal{F} \otimes_{\mathcal{O}_X} p^*\mathcal{G})$$

is *not* an isomorphism.

Indeed, since $R^i p_*$ computes $H^i(X, -)$, we have an exact sequence

$$0 \to (p_*\mathcal{F})(X) \to k[t] \to \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(-2))$$

by $(*)$, and since the Ext term is a torsion free $k[t]$-module and the map from $k[t]$ is $[\mathcal{F}] \neq 0$, this is injective and thus $p_*\mathcal{F} = 0$ since it is quasicoherent. Thus the

left-hand side of the projection map is zero. Next consider the cartesian square:

$$\begin{array}{ccc} \mathbb{P}^1_k & \longrightarrow & \operatorname{Spec} k \\ \downarrow{\scriptstyle i} & \lrcorner & \downarrow{\scriptstyle i'} \\ X & \xrightarrow{p} & \operatorname{Spec} k[t] \end{array}$$

By base change under a closed immersion, $i_*\mathcal{O}_{\mathbb{P}^1_k} = p^* i'_* \mathcal{O}_{\operatorname{Spec} k} = p^*\mathcal{G}$. By closed projection, tensoring $(*)$ with $p^*\mathcal{G}$ produces the bottom row in the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_X(-2) & \longrightarrow & \mathcal{F} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & i_*\mathcal{O}_{\mathbb{P}^1_k}(-2) & \longrightarrow & i_*\mathcal{F}|_{\mathbb{P}^1_k} & \longrightarrow & i_*\mathcal{O}_{\mathbb{P}^1_k} & \longrightarrow & 0 \end{array}$$

Here the zero on the bottom left is due to $L^{-1}p^*(\mathcal{O}_X) = 0$ as $\mathcal{O}_X$ is flat. This means that $[\mathcal{F}] \mapsto [i_*\mathcal{F}|_{\mathbb{P}^1_k}]$ under

$$\operatorname{Ext}^1(\mathcal{O}_X, \mathcal{O}_X(-2)) \xrightarrow{Li^*} \operatorname{Ext}^1(\mathcal{O}_{\mathbb{P}^1_k}, \mathcal{O}_{\mathbb{P}^1_k}(-2)),$$

which is computed by ordinary pullback i^* since both domain and codomain are flat. But we claim that $[\mathcal{F}]$ is also in the kernel of this map. Indeed, this map is isomorphic to $H^1(X, \mathcal{O}_X(-2)) \to H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2))$, and we have a defining exact sequence $0 \to \mathcal{O}_X \xrightarrow{t} \mathcal{O}_X \to i_*\mathcal{O}_{\mathbb{P}^1_k} \to 0$, which follows from the sequence $0 \to k[t] \xrightarrow{t} k[t] \to k \to 0$ and the fact that p is flat, since it is the base change of $\mathbb{P}^1_k \to \operatorname{Spec} k$. Thus the long exact sequence shows that multiplication by $t \cdot H^1(X, \mathcal{O}_X(-2))$ is in the kernel of the map, but we chose $[\mathcal{F}]$ to be in this submodule.

We have shown that the extension class in the bottom row above is zero, so

$$i_*\mathcal{F}|_{\mathbb{P}^1_k} = \mathcal{F} \otimes p^*\mathcal{G} = i_*\mathcal{O}_{\mathbb{P}^1_k} \oplus i_*\mathcal{O}_{\mathbb{P}^1_k}(-2).$$

Applying p_* takes global sections, and hence gives

$$p_*(\mathcal{F} \otimes p^*\mathcal{G}) = H^0(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \oplus 0 = k,$$

which is not zero. Therefore the underived adjunction formula fails. Let us see precisely how the derived version fixes this. Since both sides are quasicoherent on an affine scheme we will identify $D_{\mathrm{qc}}(\operatorname{Spec} k[t]) = D(k[t])$. As $\mathcal{F}$ is locally free and p is flat, the only modification we have to make to the RHS of the projection formula is Rp_* instead of p_*. This gives

$$\begin{aligned} Rp_*(\mathcal{F} \otimes p^*\mathcal{G}) &= Rp_*(i_*\mathcal{O}_{\mathbb{P}^1_k}) \oplus Rp_*(i_*\mathcal{O}_{\mathbb{P}^1_k}(-2)) \\ &\approx R\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \oplus R\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(-2)) \\ &= k[0] \oplus k[-1]. \end{aligned}$$

On the other hand, $Rp_*\mathcal{F} \approx R\Gamma(X, \mathcal{F})$, we showed already that $R^0p_*\mathcal{F} = 0$, and we have the exact sequence

$$0 \to H^0(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X(-2)) \to H^1(X, \mathcal{F}) \to 0.$$

By construction, the first map sends $1 \mapsto [\mathcal{F}]$, so identifying the first two terms as $k[t] \to k[t]$ and $[\mathcal{F}] \in (t)$, write $[\mathcal{F}] = tf(t)$ with $\deg f = n$. Then $H^1(X, \mathcal{F}) \cong k[t]/(tf)$ as a $k[t]$-module. More importantly, this gives

$$Rp_*\mathcal{F} \simeq [k[t] \xrightarrow{tf} k[t]] \in D(k[t]),$$

in degrees $0, 1$, and then applying $\otimes^{\mathbf{L}}_{k[t]}k$ we obtain $k \xrightarrow{0} k$, i.e. $k[0] \oplus k[-1]$, which is what we got before. Note that regardless of which $f(t)$ we pick, $\mathcal{F}$ is a vector bundle of rank 2 which cannot be a sum of line bundles, because it has non-free cohomology $k[t]$-modules.

Example 2.4.36. Let X and Y be qcqs schemes. Denote by

$$p_X \colon X \times Y \to X \qquad p_Y \colon X \times Y \to Y$$

the projection maps, noting that $X \times Y$ is also qcqs. For any $K \in D_{\mathrm{qc}}(X \times Y)$, we define the functor

$$\Phi_K \colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y) \qquad \Phi_K(A) \coloneqq Rp_{Y,*}(K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} Lp_X^*A).$$

A functor of the form Φ_K is called a *Fourier-Mukai functor*, and the object K is its *kernel*. We will see in our geometric examples that almost any real functor which comes up is Fourier-Mukai. To start to see why, let W be a qcqs scheme and take maps $f \colon W \to X$ and $g \colon W \to Y$, then we claim that

$$A \mapsto Rg_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_W} Lf^*A)$$

is a Fourier-Mukai functor for any $K \in D_{\mathrm{qc}}(W)$. Indeed, let (f, g) denote the induced map $W \to X \times Y$, and consider the kernel $R(f, g)_*K$ in $D_{\mathrm{qc}}(X \times Y)$. By the qcqs projection formula, the associated Fourier-Mukai transform is

$$\begin{aligned}
\Phi_{R(f,g)_*K}(A) &= Rp_{Y,*}(R(f, g)_*K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} Lp_X^*A) \\
&= Rp_{Y,*}R(f, g)_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_W} L(f, g)^*Lp_X^*A) \\
&= Rg_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_W} Lf^*A)
\end{aligned}$$

as desired. Therefore, the following functors are all Fourier-Mukai:

1. The functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$ given by $A \mapsto E \otimes^{\mathbf{L}}_{\mathcal{O}_X} A$ for some E, since we can just take $f = g = \mathrm{Id}$ and $E = K$. This includes the identity functor, all shifts $A \mapsto A[n]$, and $A \mapsto \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, A)$ when E is perfect.
2. $Rf_* \colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ for any $f \colon X \to Y$, since we can take g to be f, $f = \mathrm{Id}$, and $K = \mathcal{O}_W$. This in particular includes $R\Gamma$. Likewise, it is clear how to do Lf^*.
3. We will see later that the functor $f^!$ we are going to construct is also a Fourier-Mukai transform, at least for sufficiently nice f.

We also observe that all Fourier-Mukai functors preserve arbitrary coproducts, and hence the result we prove will show that they all have right adjoints.

Now let's return to the general theory. Let $\mathcal{T}$ be a triangulated category. Recall that a strictly full triangulated subcategory $\mathcal{N} \subset \mathcal{T}$ is *thick* if whenever $X, Y \in \mathcal{T}$ and $X \oplus Y \in \mathcal{N}$, we have $X, Y \in \mathcal{N}$. It is clear from the definition that thick, and (strictly full) triangulated subcategories, are conditions closed under arbitrary intersection and that $\mathcal{T}$ itself satisfies both properties; thus if $S \subset \mathcal{T}$ is a collection of objects, we write

$$\mathrm{tria}(S) \qquad \text{and} \qquad \mathrm{thick}(S)$$

for the smallest strictly full triangulated and thick subcategory respectively containing S. So, $\mathcal{N}$ is triangulated/thick iff $\mathrm{tria}(\mathcal{N}) = \mathcal{N}/\mathrm{thick}(\mathcal{N}) = \mathcal{N}$. We sometimes write $\mathrm{thick}_{\mathcal{T}}(S)$ to denote the ambient category; it is clear that $\mathrm{tria}(S)$ can be computed in any ambient (strictly full) subcategory/supercategory containing S. On the other hand, if $\mathcal{N} \subset \mathcal{T}$ is thick and $S \subset \mathcal{N}$, then $\mathrm{thick}_{\mathcal{N}}(S) = \mathrm{thick}_{\mathcal{T}}(S)$, because $\mathrm{thick}(S) \subset \mathcal{N}$, and any $\mathcal{T}$-summand of an object in S must lie in $\mathcal{N}$.

Example 2.4.37. Almost all triangulated subcategories we care about are thick. For instance, we have seen that if $(X, \mathcal{O}_X)$ is a ringed space then $\mathrm{Perf}(X) \subset D(\mathcal{O}_X)$ and $\mathrm{PsCoh}(X) \subset D(\mathcal{O}_X)$ are thick. $D_{\mathrm{qc}}(X)$ and $D_{\mathrm{coh}}(X)$ are also thick inside $D(X)$, since summands of quasicoherent (resp. coherent) sheaves are quasicoherent (resp. coherent), which follows from the fact that we can realize a summand of $\mathcal{F}$ as the image of an idempotent $\mathcal{F} \to \mathcal{F}$. Other examples include the kernel of an exact functor (as is clear from the definition), and the class of objects for which a natural map of exact functors $F \to G$ is an isomorphism, since if we write the triangles

$$FA \to GA \to C \to FA \qquad FA' \to GA' \to C' \to GA'[1],$$

then $F(A \oplus A) \to G(A \oplus A') \to C \oplus C' \to$ is exact, and $C \oplus C'$ is zero iff C and C' are zero. For example, this shows that $\mathcal{T}_c$, the class of compact objects in a triangulated category $\mathcal{T}$ with coproducts, is thick. Other special cases are the so-called 'right-orthogonal'

$$S^{\perp} = \{X \in \mathcal{T} : \mathrm{Hom}^i(A, X) = 0, A \in S, i \in \mathbb{Z}\}$$

and 'left-orthogonal'

$${}^{\perp}S = \{X \in \mathcal{T} : \mathrm{Hom}^i(X, A) = 0, A \in S, i \in \mathbb{Z}\},$$

which we will discuss more in the next section.

Remark 2.4.38. Both $\mathrm{tria}(S)$ and $\mathrm{thick}(S)$ are finitary operations, in the following sense:

$$\mathrm{tria}(S) = \bigcup_{S' \subset S \text{ finite}} \mathrm{tria}(S') \qquad \mathrm{thick}(S) = \bigcup_{S' \subset S \text{ finite}} \mathrm{thick}(S')$$

Indeed, first note that $\mathrm{tria}(S') \cup \mathrm{tria}(S'') \subset \mathrm{tria}(S' \cup S'')$ from the definition, and likewise for thick; so if $A \to B \to C \to A[1]$ is a triangle and $A \in \mathrm{tria}(S)$ and $B \in \mathrm{tria}(S')$, then $A, B \in \mathrm{tria}(S \cup S')$ and $C \in \mathrm{tria}(S \cup S')$. The same thing works for direct summands and thick.

We also note that there is an inductive construction which one can make as follows: let $\mathrm{tria}_0(S) = S$, and if $\mathrm{tria}_n(S)$ is defined then $\mathrm{tria}_{n+1}(S)$ is defined

as the class of all objects which are isomorphic to shifts of objects in $\mathrm{tria}_n(S)$, or cones of morphisms in $\mathrm{tria}_n(S)$. Then in fact $\mathrm{tria}(S) = \bigcup_n \mathrm{tria}_n(S)$. Indeed clearly $\supset$ holds, and if $A \in \mathrm{tria}_n(S)$ then $A[k] \in \mathrm{tria}_{n+1}(S)$ for all $k \in \mathbb{Z}$, and if $A \to B \to C \to A[1]$ is a triangle, then choose n so that $A, B \in \mathrm{tria}_n(S)$, then $C \in \mathrm{tria}_{n+1}(S)$. The same thing can be done with $\mathrm{thick}(S)$, except $\mathrm{thick}_{n+1}(S)$ should consist of all shifts, cones, and direct summands of objects in $\mathrm{thick}_n(S)$.

In general, whenever $\mathcal{N} \subset \mathcal{T}$ is a thick subcategory and $\mathrm{thick}(S) = \mathcal{N}$, we say that S *split-generates* $\mathcal{N}$. What we have just shown means that any object in $\mathcal{N}$ can be written using iterated shifts, summands, and cones finitely many times starting from finitely many objects of S.

Remark 2.4.39. Recall in a triangulated category, all retracts of an identity morphism $1\colon X \to X$ produce direct sum splittings. In fact, the following are equivalent for a triangle $X \to Y \to Z \to X[1]$:

(a) The map $Z \to X[1]$ is 0.
(b) $X \to Y$ has a retract.
(c) $Y \to Z$ has a section.
(d) There is a splitting $Y \cong X \oplus Z$, so that the canonical maps $X \to Y$ and $Y \to Z$ coincide with the ones in the triangle.

Indeed, assume (a), which implies that for all W the sequence

$$0 \to \mathrm{Hom}(W, X) \to \mathrm{Hom}(W, Y) \to \mathrm{Hom}(W, Z) \to 0$$

is exact. If we take $W = Z$, this implies that 1_Z lifts to a map $Z \to Y$ such that $Z \to Y \to Z$ is the identity, i.e. (c). Likewise if we took $\mathrm{Hom}(-, W)$ instead, we would show (b). Then e.g. (c) shows that the exact sequence

$$0 \to \mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y) \to \mathrm{Hom}(-, Z) \to 0$$

splits in the abelian category $\mathrm{Fun}(\mathcal{T}^{\mathrm{op}}, \mathbf{Ab})$, hence there is a retract which by the Yoneda lemma is induced from $Y \to X$. Moreover, also by Yoneda and additivity of Homs we conclude (d). We can do the same in the other direction had we started with (b).

To prove $(d) \Rightarrow (a)$, we will show the stronger statement that $X \to X \oplus Z \to Z \xrightarrow{0} X[1]$ is exact in any triangulated category. Indeed, extend $Z \xrightarrow{0} X[1]$ to triangle $X \to Y \to Z \xrightarrow{0} X[1]$, then the implication we already have shows that $Y \cong X \oplus Z$ with the prescribed maps, so the triangle is isomorphic to $X \to X \oplus Z \to Z \to X[1]$. We call triangles of this form 'split triangles'. It also tells us that the cone of $X \xrightarrow{0} Y$ is $Y \oplus X[1]$.

If $X \in \mathcal{T}$ we define a *homotopy filtration of X by S* to be a system (K_n) in $\mathcal{T}$ such that $\mathrm{hocolim}_n K_n \simeq X$, and all maps in the system fit into triangles of the form

$$L_n \to K_n \to K_{n+1} \to L_n[1]$$

where $L_n \in S$. For example, Lemma 2.4.32 shows that all objects of a category with compact generators $\{E_i\}$ admit a homotopy filtration by coproducts of shifts of the generators. Here is another important application of this:

Lemma 2.4.40. *Let $\mathcal{T}$ be a triangulated category with coproducts and suppose $X \in \mathcal{T}$ admits a homotopy filtration by coproducts of shifts of $\{E_i\} \subset \mathcal{T}_c$. For any $F \in \mathcal{T}_c$, each map $F \to X$ factors through* $\operatorname{tria}(\{E_i\})$.

Proof. Let $\mathcal{E} \coloneqq \operatorname{tria}(\{E_i\})$, and write $X = \operatorname{hocolim}_n K_n$ for the homotopy filtration with triangles

$$L_n \to K_n \to K_{n+1} \to L_n[1]$$

where $K_n = 0$ and L_n a coproduct of shifts of the $\{E_i\}$. Since F is compact, a map $F \to X$ factors through K_n for some n, so we may prove by induction on n that any map from a compact object to K_n factors through $\mathcal{E}$. This is clear when $n = 0$.

Since L_i is a sum of shifts of the E_i, any map from $F \to L_i$ automatically factors through a finite direct sum of shifts of the $\{E_i\}$, and hence through $\mathcal{E}$. In particular, the composite $F \to K_n \to L_{n-1}[1]$ factors through $M_1[1] \in \mathcal{E}$, so we can complete this data to a morphism of triangles,

$$\begin{array}{ccccccc} M_1 & \longrightarrow & F_2 & \longrightarrow & F & \longrightarrow & M_1[1] \\ \downarrow & & \vdots & & \downarrow & & \downarrow \\ L_{n-1} & \longrightarrow & K_{n-1} & \longrightarrow & K_n & \longrightarrow & L_{n-1}[1] \end{array}$$

where $F_2 \in \mathcal{T}_c$ since $M_1 \in \mathcal{E} \subset \mathcal{T}_c$. By induction, $F_2 \to K_{n-1}$ factors through $M_2 \in \mathcal{E}$, then we can consider the composite map $M_1 \to F_2 \to M_2$ and take its cone $M_3 \in \mathcal{E}$. Thus we can factor the morphism of triangles above to obtain

$$\begin{array}{ccccccc} M_1 & \longrightarrow & F_2 & \longrightarrow & F & \longrightarrow & M_1[1] \\ \| & & \downarrow & & \vdots & & \| \\ M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_1[1] \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ L_{n-1} & \longrightarrow & K_{n-1} & \longrightarrow & K_n & \longrightarrow & L_{n-1}[1] \end{array}$$

along with two TR3-extended maps. Denote the map we wish to factor by $\alpha\colon F \to K_n$, the composite $F \to M_3 \to K_n$ by β, and consider the exact sequence

$$\operatorname{Hom}(F, K_{n-1}) \to \operatorname{Hom}(F, K_n) \to \operatorname{Hom}(F, L_{n-1}[1]).$$

The composites of both α and β with $K_n \to L_{n-1}[1]$ are equal to the composite $F \to M_1[1] \to L_{n-1}[1]$ and hence to each other; so, we may write

$$\alpha - \beta = F \xrightarrow{\gamma} K_{n-1} \to K_n$$

for some map γ. By induction, γ factors through $F \to M_4$ for some $M_4 \in \mathcal{E}$, which means that the induced map

$$F \to M_3 \oplus M_4 \to K_n \oplus K_{n-1} \to K_n$$

is equal to $\beta + (\alpha - \beta) = \alpha$, so we are done via $M_3 \oplus M_4 \in \mathcal{E}$. □

Corollary 2.4.41. *Let $\mathcal{T}$ be a compactly generated triangulated category and $\{E_i : i \in I\}$ a collection of compact objects. Then* $\mathrm{thick}(E_i : i \in I) = \mathcal{T}_c$ *if and only if $\{E_i : i \in I\}$ generates $\mathcal{T}$.*

Proof. Since $\mathcal{T}_c$ is thick, we always have $\mathrm{thick}(\{E_i\}) \subset \mathcal{T}_c$. For the first direction, if $\mathrm{Hom}^*(E_i, X) = 0$ for all $i \in I$, i.e. $E_i \in {}^{\perp}X$, then $\mathcal{T}_c \subset {}^{\perp}X$ since the latter subcategory is thick. Since $\mathcal{T}$ is compactly generated, this implies $X = 0$.

For the other direction, let $F \in \mathcal{T}_c$ and consider $1\colon F \to F$; by Lemma 2.4.32 and Lemma 2.4.40, this factors through $\mathrm{tria}(\{E_i\})$ and hence $\mathrm{thick}(\{E_i\})$, meaning (by Remark 2.4.39) that F is a summand of an object of $\mathrm{thick}(\{E_i\})$, hence $F \in \mathrm{thick}(\{E_i\})$. □

Let's now move on to the main theorem of this section, the Brown representability theorem, which is written below in its standard form:

Theorem 2.4.42. *Let $\mathcal{E}$ be a triangulated category with coproducts which is compactly generated. Let*

$$H\colon \mathcal{E}^{\mathrm{op}} \to \mathbf{Ab}$$

be a cohomological functor which preserves products (i.e. coproducts in $\mathcal{E}$). Then H is representable by an object of $\mathcal{E}$.

We usually don't use it in the form above, but instead to construct adjoints as follows:

Corollary 2.4.43. *Let $F\colon \mathcal{E} \to \mathcal{E}'$ be an exact functor of triangulated categories, and suppose $\mathcal{E}$ is compactly generated and F preserves coproducts. Then F has a right adjoint.*

Proof. Recall that having a right adjoint is a pointwise condition. For each $E' \in \mathcal{E}$, the functor $\mathrm{Hom}_{\mathcal{E}'}(F(-), E')$ satisfies the conditions of Thm. 2.4.42 by assumption. Therefore, there exists some $G_{E'}$ such that

$$\mathrm{Hom}_{\mathcal{E}'}(F(E), E') \cong \mathrm{Hom}_{\mathcal{E}}(E, G_{E'})$$

natural in E. This shows that F has a partial right adjoint at E', and since E' was arbitrary it has a global right adjoint (such that $E' \mapsto G_{E'}$). □

Brown representability is surprisingly easy to prove, and we just have to repeat the argument of Lemma 2.4.32 and insert the Yoneda lemma in some places.

Proof (of Thm. 2.4.42). Let $\{E_i : i \in I\}$ be a compact generating set of $\mathcal{T}$, and let K_1 be the sum over all $E_i[m]$ indexed by all $i \in I$, $m \in \mathbb{Z}$, and all elements $a \in H(E_i[m])$. Then

$$H(K_1) = \prod_{(i,m,a)} H(E_i[m]),$$

so there is a canonical distinguished element $a_1 \coloneqq (a)_{(i,m,a)} \in H(K_1)$, which by Yoneda's lemma corresponds to a map $a_1\colon \mathrm{Hom}(-, K_1) \to H$. Inductively, given $a_n\colon \mathrm{Hom}(-, K_n) \to H$, define L_n to be the sum over all $E_i[m]$ indexed by $i \in I$,

$m \in \mathbb{Z}$, and all $a \in \ker(\mathrm{Hom}(E_i, K_n) \to H(E_i))$. Each such a is a map to K_n, so there is a natural map $L_n \to K_n$ and we take the cone

$$L_n \to K_n \to K_{n+1} \to L_n[1],$$

which induces an exact sequence $H(K_{n+1}) \to H(K_n) \to H(L_n)$. The map a_n corresponds to an element of $H(K_n)$ which by construction maps to 0 in $H(L_n)$, thus we obtain an element $a_{n+1} \in H(K_{n+1})$, i.e. a map $\mathrm{Hom}(-, K_{n+1}) \to H$. That $a_{n+1} \mapsto a_n$ corresponds under the Yoneda bijection to the fact that the composite $\mathrm{Hom}(-, K_n) \to \mathrm{Hom}(-, K_{n+1}) \to H$ equals the given map a_n.

Let $X \coloneqq \mathrm{hocolim}_n K_n$, and apply H to the defining triangle to obtain an exact sequence

$$H(X) \to \prod_n H(K_n) \xrightarrow{1-s} \prod_n H(K_n).$$

The commutativity explained above ($a_{n+1} \mapsto a_n$) means that the family $(a_n)_{n \geq 1}$ in the middle maps to zero on the right, and hence is induced from some element $a \in H(X)$, corresponding to a map $\mathrm{Hom}(-, X) \to H$. We claim this map is an isomorphism. Indeed, since both sides are cohomological and send sums to products, it suffices to check this on the generators $E_i[m]$ by Prop. 2.4.31. Since $E_i[m]$ is compact, we have to show that

$$\varinjlim_n \mathrm{Hom}(E_i[m], K_n) \to H(E_i[m])$$

is an isomorphism for all m and i. Explicitly, this sends a representative $[\varphi]$ to the element corresponding to the composite $\mathrm{Hom}(-, E_i[m]) \to \mathrm{Hom}(-, K_n) \to H$, or in other words $H(\varphi)(a_n)$. By definition of K_1, this is surjective. On the other hand, if a representative $E_i[m] \to K_n$ maps to zero, then this map is in the kernel of $\mathrm{Hom}(E_i[m], K_n) \to H(E_i[m])$, i.e. a basis element of L_n. Hence it maps to zero upon composing with $K_n \to K_{n+1}$, which shows injectivity. This proves the claim. □

Remark 2.4.44. A priori we may ask if such a right adjoint as in Cor. 2.4.43 is exact. But indeed if $F \colon \mathcal{T} \rightleftarrows \mathcal{T}' : G$ is an adjunction between triangulated categories, then F is exact iff G is exact. Assume exactness of F, and first note that

$$\begin{aligned} \mathrm{Hom}(A, G(B[r])) &= \mathrm{Hom}(F(A), B[r]) = \mathrm{Hom}(F(A)[-r], B) \\ &= \mathrm{Hom}(F(A[-r]), B) = \mathrm{Hom}(A[-r], G(B)) \\ &= \mathrm{Hom}(A, G(B)[r]), \end{aligned}$$

for all A, so $G(B[r]) = G(B)[r]$. Next take a triangle $A' \to B' \to C' \to A'[1]$ in $\mathcal{T}'$, and then choose a triangle $G(A') \to G(B') \to C \to G(A')$. The adjunction counit gives a morphism of triangles:

$$\begin{array}{ccccccc} FG(A') & \longrightarrow & FG(B') & \longrightarrow & F(C) & \longrightarrow & FG(A')[1] \\ \downarrow & & \downarrow & & \vdots & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

The new map gives a map $C \to G(C')$ which we claim is an isomorphism; indeed, adjunction and the diagram above show that it commutes with $G(B') \to G(C')$ and $G(C') \to G(A')[1]$, so this follows from the long exact sequence and the five lemma.

Example 2.4.45. For any scheme X, we know the inclusion $\iota_X \colon D_{\mathrm{qc}}(X) \hookrightarrow D(X)$ preserves coproducts. If X is qcqs, then by Brown representability there is an exact right adjoint $Q_X \colon D(X) \to D_{\mathrm{qc}}(X)$ which we will by abuse of language call the quasi-coherator. Since the left adjoint is fully faithful, the unit $\mathrm{Id} \to Q_X \circ \iota_X$ is an isomorphism. Suppose that $B \in D^{\geq 0}(X)$, then if $A \in D_{\mathrm{qc}}^{\leq -1}(X)$ we observe that

$$0 = \mathrm{Hom}_{D(X)}(A, B) \cong \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(A, Q_X(B)),$$

meaning $\tau_{\leq -1} Q_X(B) \to Q_X(B)$ is zero, i.e. $Q_X(B) \in D^{\geq 0}$. Thus Q_X has left-dimension 0. In particular, given an $\mathcal{O}_X$-module $\mathcal{G} \in D^{\leq 0}(X)$, the truncation triangle

$$H^0(Q_X(\mathcal{G}))[0] \to Q_X(\mathcal{G}) \to \tau_{\geq 1} Q_X(\mathcal{G}) \to H^0(Q_X(\mathcal{G}))[1]$$

shows that $\mathrm{Hom}_{D(X)}(A, H^0(Q_X(\mathcal{G}))) \cong \mathrm{Hom}_{D(X)}(A, Q_X(\mathcal{G}))$ whenever $A \in D^{\leq 0}$. In particular, if $A = \mathcal{F} \in D_{\mathrm{qc}}(X)$, then

$$\mathrm{Hom}_{D(X)}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{F}, Q_X(\mathcal{G})) \cong \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{F}, H^0(Q_X(\mathcal{G}))).$$

Since $\mathrm{QCoh}(X)$ embeds in $D_{\mathrm{qc}}(X)$ as the H^0-complexes and $\mathrm{Mod}(\mathcal{O}_X)$ embeds in $D(X)$ as the H^0-complexes, we deduce an adjunction

$$\iota_X \colon \mathrm{QCoh}(X) \rightleftarrows \mathrm{Mod}(\mathcal{O}_X) : H^0 \circ Q_X,$$

where we also call the right adjoint the quasi-coherator and sometimes write it as Q_X. This gives a new proof of the existence of the quasicoherator on a qcqs scheme.

Example 2.4.46. We have seen that for any ringed space $(X, \mathcal{O}_X)$, the unbounded derived category $D(\mathcal{O}_X)$ (as well as $D^-(\mathcal{O}_X)$) is a symmetric monoidal category. We have also seen that if X is a scheme, $C \in D_{\mathrm{qc}}(X)$ and $A, B \in D(\mathcal{O}_X)$, then we have a bijection

$$\mathrm{Hom}_{D(X)}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C) \cong \mathrm{Hom}_{D(X)}(A, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C))$$

natural in A, B, C. We cannot a priori deduce a literal adjunction from this, but we can observe the following: if $B \in D_{\mathrm{qc}}(X)$, then $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$ as a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$ preserves coproducts. Indeed, the usual argument shows

$$\mathrm{Hom}_{D(X)}((\bigoplus_{\alpha} A_\alpha) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C) \cong \mathrm{Hom}_{D(X)}(\bigoplus_{\alpha} (A_\alpha \otimes^{\mathbf{L}}_{\mathcal{O}_X} B), C)$$

for $C \in D_{\mathrm{qc}}(X)$, and thus since both sides are in $D_{\mathrm{qc}}(X)$ this implies the result. Therefore, if X is qcqs then $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$ has a right adjoint, which we will denote

$$\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(B, -) \colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X),$$

which also makes $D_{\mathrm{qc}}(X)$ a *closed* monoidal category. Thus by category theory we obtain some facts formally, such as that this extends canonically to a bifunctor

$\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}\colon D_{\mathrm{qc}}(X)^{\mathrm{op}} \times D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X)$, and we upgrade to an internal isomorphism

$$\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(A \otimes^{\mathbf{L}}_{\mathcal{O}_X} B, C) \cong \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(A, \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(B, C)),$$

among other closed/enriched category theory facts.

Note that if $B = E$ is perfect, then $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, -)$ preserves $D_{\mathrm{qc}}(X)$ so we conclude that $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(B, -) = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, -)$, but in general the functors may be different. However, we can 'compute it' in terms of the functor Q_X of Example 2.4.45; indeed, if $A \in D_{\mathrm{qc}}(X)$ then we conclude

$$\mathrm{Hom}_{D(X)}(A, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C)) = \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(A, Q_X(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(B, C))),$$

so the initial formula shows that $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)} = Q_X \circ \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}$. This shows in particular that if ever $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B)$ lies in $D_{\mathrm{qc}}(X)$, then we can always compute $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(A, B)$ using it regardless of whether the condition is satisfied for *all* $B \in D_{\mathrm{qc}}(X)$.

Remark 2.4.47. A lot of the properties we have shown about $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}$ relating to the other derived functors are actually true for $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}$ for formal reasons. For example, let $f\colon X \to Y$ be a morphism of qcqs schemes. The adjunctions

$$Lf^* \dashv Rf_* \qquad (-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \dashv \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(B, -)$$

imply that $Lf^*(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$ is left adjoint to $Rf_*\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(B, -)$. Taking $B = Lf^*A$ and using that Lf^* preserves the monoidal product, we deduce that

$$Lf^*((-) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} A) \dashv Rf_*\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(Lf^*A, -).$$

On the other hand, we observe that the left-hand side is left adjoint to $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(Y)}(A, Rf_*(-))$, so we deduce a natural isomorphism

$$\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(Y)}(A, Rf_*B) \xrightarrow{\sim} Rf_*\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(Lf^*A, B).$$

In particular if both sides can be computed by $\mathbf{R}\mathcal{H}om$, then we reduce to our known statements.

Example 2.4.48. Let X be a qcqs R-scheme and write $p\colon X \to \operatorname{Spec} R$ for the structure map. In Example 2.4.22 we constructed a natural map

$$Lp^* \mathrm{RHom}_{\mathcal{O}_X}(-, -) \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(-, -).$$

By composition and the $\iota_X \dashv Q_X$ adjunction, we obtain a map

$$Lp^* \mathrm{RHom}_{\mathcal{O}_X}(-, -) \to \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(-, -)$$

in $D_{\mathrm{qc}}(X)$, and hence a comparison map

$$\mathrm{RHom}_{\mathcal{O}_X}(-, -) \to R\Gamma(X, \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(-, -))$$

in $D(R)$. We claim this is an isomorphism of functors; indeed, to show this we may prove that $\mathrm{RHom}_{\mathcal{O}_X}(-, B) \to R\Gamma(X, \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(-, B))$ is an isomorphism of functors $D_{\mathrm{qc}}(X) \to D(R)^{\mathrm{op}}$ for all $B \in D_{\mathrm{qc}}(X)$. We have seen it is an isomorphism on $\mathrm{Perf}(X) \subset D_{\mathrm{qc}}(X)$ using Example 2.4.46 and Example 2.4.22. Moreover this is

a natural map of exact coproduct-preserving functors, therefore by Prop. 2.4.31 we conclude the isomorphism.

As a special case of this, we observe that if X is affine, so that $R\Gamma$ is an equivalence, we see $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}$ is the associated sheaf to $\mathrm{RHom}_{\mathcal{O}_X}$.

Now we come to the main interesting result of this section, which is the right adjoint to derived pushforward. This functor is sometimes called the 'twisted inverse image functor', and we will see in nice enough circumstances that it is literally a 'twist' of the derived pullback.

Corollary 2.4.49. *Let $f\colon X \to Y$ be a morphism of qcqs schemes. Then $Rf_*\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ has a right adjoint functor $f^\times\colon D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X)$. Moreover, there are canonical isomorphisms*

$$Rf_* \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(A, f^\times B) \xrightarrow{\sim} \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(Y)}(Rf_* A, B)$$

and

$$\mathrm{RHom}_{\mathcal{O}_X}(A, f^\times B) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{O}_Y}(Rf_* A, B)$$

natural in $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(Y)$.

Proof. The first statement is Cor. 2.4.43, Thm. 2.4.30, and the fact that perfect objects are compact in $D_{\mathrm{qc}}(X)$. The third statement follows from Example 2.4.48 and applying $R\Gamma(Y, -)$ to the second statement. For the second, note that the LHS is the composite of three right adjoints

$$Rf_* \circ \underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(X)}(A, -) \circ f^\times$$

and hence is right adjoint to $Rf_*(Lf^*(-) \otimes^{\mathbf{L}}_{\mathcal{O}_X} A)$, which is isomorphic to $(-) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_* A$ by the qcqs projection formula. But this functor is left adjoint to $\underline{\mathrm{RHom}}_{D_{\mathrm{qc}}(Y)}(Rf_* A, -)$. All the adjunction formulae here are natural in A as well, so the resulting isomorphism is natural in A. □

Example 2.4.50. Let $\Phi\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ be a Fourier-Mukai functor for X and Y qcqs schemes. Suppose Φ is presented in the form $Rg_*(K \otimes^{\mathbf{L}} Lf^*(-))$ where

$$f\colon W \to X \qquad g\colon W \to Y \qquad K \in D_{\mathrm{qc}}(W)$$

for a qcqs scheme W. It is then clear that Φ preserves coproducts, so it automatically has a right adjoint. But we can explicitly write it as

$$A \mapsto Rf_* \underline{\mathrm{RHom}}_{\mathcal{O}_W}(K, g^\times(A))$$

by breaking up Φ as a composition of functors with right adjoints.

If f is a proper morphism, then we will write this as $f^!$ instead of $f^\times$; the notational discrepancy will be explained in more detail later. The counit mentioned in the proof is often called the trace map $\mathrm{tr}_f\colon Rf_* f^\times \to \mathrm{Id}$, to which we will ascribe a geometric meaning later. Note for instance that due to the adjunction with pushforward, it follows that if $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of qcqs schemes, then $(gf)^\times = f^\times g^\times$. Without saying too much, let us explain why this is already nontrivial.

Example 2.4.51. Let k be a field and $p\colon X \to \operatorname{Spec} k$ a qcqs k-scheme. We identify $D(k)$ with the category of $\mathbb{Z}$-graded k-vector spaces. Let $n > 0$ and write $\omega_X \coloneqq p^\times \mathcal{O}_{\operatorname{Spec} k}[n] \in D_{\mathrm{qc}}(X)$. Then there is an isomorphism

$$\operatorname{RHom}_{\mathcal{O}_X}(A, \omega_X[n]) \xrightarrow{\sim} \operatorname{RHom}_k(R\Gamma(X, A), k)$$

over k, i.e. if we take $i - n$th cohomology we get

$$\operatorname{Hom}^i_{D_{\mathrm{qc}}(X)}(A, \omega_X) \cong \operatorname{Hom}_k(\mathbb{H}^{n-i}(X, A), k).$$

If $A = E \in \operatorname{Perf}(X)$, then this becomes

$$\mathbb{H}^i(X, E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} \omega_X) \xrightarrow{\sim} \mathbb{H}^{n-i}(X, A)^\vee,$$

which is an expression of Serre duality, modulo the fact that we don't yet know how to compute what ω_X is. These isomorphisms are induced by the trace map tr_p, which here corresponds to a map $\mathbb{H}^n(X, \omega_X) \to k$.

Remark 2.4.52. This is mainly a recollection. Recall that given $A \in D_{\mathrm{qc}}(X)$, we have $\operatorname{Hom}(D^{\leq a}, A) = 0$ iff $A \in D^{>a}$. Indeed, we have already seen the orthogonality $\operatorname{Hom}(D^{\leq a}, D^{>a}) = 0$, and on the other hand the assumption implies that $\tau_{\leq a} A \to A$ is zero. Since this is an isomorphism on cohomology $\leq a$, $A \in D^{>a}$.

So, given a morphism $f\colon X \to Y$ of qcqs schemes, suppose it has right-QCoh dimension $\leq d$, meaning that $Rf_*(D^{\leq m}_{\mathrm{qc}}(X)) \subset D^{\leq m+d}_{\mathrm{qc}}(Y)$. Then in fact $f^\times$ has *left QCoh-dimension* $\leq d$. To see this, suppose $B \in D^{>m}_{\mathrm{qc}}$, then for $A \in D^{\leq m-d}$, we have

$$\operatorname{Hom}(A, f^\times B) = \operatorname{Hom}(Rf_* A, B) = 0$$

since $Rf_* A \in D^{\leq m}$, thus $f^\times B \in D^{>m-d}$. In particular, we obtain an exact functor

$$f^\times\colon D^+_{\mathrm{qc}}(Y) \to D^+_{\mathrm{qc}}(X).$$

This is especially nice because it means that the adjunction $Rf_* \dashv f^\times$ holds on the bounded above categories as well, which is in contrast to our normal derived adjunctions.

Definition 2.4.53. Let $f\colon X \to Y$ be a qcqs morphism of schemes. f is *quasi-perfect* if $Rf_* E \in \operatorname{Perf}(Y)$ for each $E \in \operatorname{Perf}(X)$, and f is *quasi-proper* if $Rf_* A$ is pseudocoherent in $D_{\mathrm{qc}}(Y)$ for each pseudocoherent $A \in D_{\mathrm{qc}}(X)$.

The relation between being quasi-perfect and quasi-proper is quite a bit more subtle than that between perfect and pseudocoherent, but we will ultimately be able to prove a similar result to the one relating perfectness to pseudocoherence.

Proposition 2.4.54. *Let $f\colon X \to Y$ be a morphism of qcqs schemes.*

(a) If f is quasi-perfect, $E \in \operatorname{Perf}(X)$ and $B \in D_{\mathrm{qc}}(Y)$, we obtain an isomorphism

$$Rf_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(E, f^\times B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(Rf_* E, B)$$

natural in E and B.

(b) If f is quasi-proper, $A \in D_{\mathrm{qc}}(X)$ is pseudocoherent, and $B \in D^+_{\mathrm{qc}}(Y)$, then we obtain an isomorphism

$$Rf_* \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, f^\times B) \cong \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(Rf_* A, B)$$

natural in A and B.

Proof. The hypotheses describe the two main cases where the $\mathbf{R}\mathcal{H}om$ version of the internal statement in Cor. 2.4.49 lies in D_{qc}. Thus the internal hom coincides with $\mathbf{R}\mathcal{H}om$ in these cases. Note that in (b) we are using that $f^\times(D^+) \subset D^+$. □

Prop. 2.4.54(c) is the classical statement of Grothendieck duality, which took quite a bit of work in the past to prove, but is now comparatively trivial given the perfect generation theorem.

The last elementary thing we want to do is relate $f^\times B$ to $f^\times \mathcal{O}_Y$, which we can reliably do when the morphism is quasiperfect. Indeed, for $B, C \in D_{\mathrm{qc}}(Y)$ we have the natural map

$$Rf_*(Lf^*(B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^\times C) = B \otimes^{\mathbf{L}}_{\mathcal{O}_Y} Rf_* f^\times C \xrightarrow{1 \otimes^{\mathbf{L}} \mathrm{tr}_f} B \otimes^{\mathbf{L}}_{\mathcal{O}_Y} C,$$

by projection, and hence by adjunction a map

$$Lf^*(B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^\times C \to f^\times(B \otimes^{\mathbf{L}}_{\mathcal{O}_Y} C).$$

Since Lf^* preserves tensoring, everything already comes from the special case

$$\chi \colon Lf^* B \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^\times \mathcal{O}_Y \to f^\times B,$$

and we wish to see when this comparison map is an isomorphism. Note that in such cases it will show that $f^\times$ is a Fourier-Mukai functor.

Example 2.4.55. Suppose $B = E \in \mathrm{Perf}(Y)$. Then the map χ above is an isomorphism. Indeed, in that case

$$\begin{aligned}
\mathrm{Hom}(K, Lf^* E \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^\times \mathcal{O}_Y) &= \mathrm{Hom}(K, \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(Lf^* E^\vee, f^\times \mathcal{O}_Y)) \\
&= \mathrm{Hom}(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^* E^\vee, f^\times \mathcal{O}_Y) \\
&= \mathrm{Hom}(Rf_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^* E^\vee), \mathcal{O}_Y) \\
&= \mathrm{Hom}(Rf_* K \otimes^{\mathbf{L}}_{\mathcal{O}_Y} E^\vee, \mathcal{O}_Y) \\
&= \mathrm{Hom}(Rf_* K, E) = \mathrm{Hom}(K, f^\times E)
\end{aligned}$$

for every $K \in D_{\mathrm{qc}}(X)$, so χ is an isomorphism by Yoneda.

Lemma 2.4.56. *Let $f \colon X \to Y$ be a morphism of qcqs schemes. The following are equivalent:*

(a) f is quasi-perfect.
(b) $Rf_* E_i \in \mathrm{Perf}(Y)$ for a perfect generating set $\{E_i\}$ of $D_{\mathrm{qc}}(X)$.
(c) $f^\times$ preserves coproducts.
(d) $\chi \colon Lf^*(-) \otimes^{\mathbf{L}} f^\times \mathcal{O}_Y \to f^\times$ is an isomorphism.

PROOF. This uses the fact that $\mathrm{Perf}(X) = D_{\mathrm{qc}}(X)_c$ which we will prove in the next section. Clearly $(d) \Rightarrow (c)$ since the LHS preserves coproducts, and on the other hand if $f^\times$ preserves coproducts, then this is a morphism of exact coproduct-preserving functors; moreover it is an isomorphism on the compact objects by Example 2.4.55, so Prop. 2.4.31 implies that it is an isomorphism.

We will see in Lemma 2.5.28 that (a) and (c) are equivalent, and of course $(a) \Rightarrow (b)$. To see $(b) \Rightarrow (a)$, we observe that $\{A \in D_{\mathrm{qc}}(X) : Rf_*(A) \in \mathrm{Perf}(Y)\}$ is a thick subcategory of $D_{\mathrm{qc}}(X)$ since $\mathrm{Perf}(Y) \subset D_{\mathrm{qc}}(Y)$ is thick. Thus it contains $\mathrm{thick}(\{E_i\}) = \mathrm{Perf}(X)$. □

In general, if X is a qcqs scheme over a base scheme S, we will write $\mathcal{D}_X \coloneqq p^\times \mathcal{O}_S$ for $p\colon X \to S$ the structure map.

Example 2.4.57. Let $i\colon Z \hookrightarrow X$ be a pseudocoherent closed immersion, which means that i is a quasi-proper morphism. We will easily see later that if X is Noetherian then any closed immersion i satisfies this property. Taking $A = \mathcal{O}_Z$ in Prop. 2.4.54 shows us that

$$i_* i^\times B = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(i_*\mathcal{O}_Z, B)$$

for $B \in D^+$, and in particular $i_* i^\times \mathcal{D}_X = \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{D}_X)$. By the composition rule, if $Z \hookrightarrow X \to S$ are S-schemes, then $i^\times \mathcal{D}_X = \mathcal{D}_Z$, so we can at least compute $i_*\mathcal{D}_Z$ as $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{D}_X)$. This doesn't necessarily determine the object $\mathcal{D}_Z$ in the derived category, but for example we can determine its cohomology, and via tools like the projection formula knowing $i_*\mathcal{D}_Z$ is often enough data.

Example 2.4.58. Suppose $i\colon X \hookrightarrow \mathbb{P}^n$ is a projective scheme over a ring R. Then we already know that $\mathcal{D}_X = i^\times \mathcal{D}_{\mathbb{P}^n}$, and thus Example 2.4.55 shows that $i^\times(\mathcal{D}_{\mathbb{P}^n}(d)) = \mathcal{D}_X(d)$. Suppose that X is a hypersurface $V(f)$ of degree d, for $f \in R[x_0, \ldots, x_n]$ a nonzerodivisor. Then by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0,$$

we use the above example to compute $i_*\mathcal{D}_X$ as

$$\begin{aligned} \mathbf{R}\mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_X, \mathcal{D}_{\mathbb{P}^n}) &= \big[\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d)\big] \otimes^{\mathbf{L}} \mathcal{D}_{\mathbb{P}^n} \\ &= \big[\mathcal{D}_{\mathbb{P}^n} \to \mathcal{D}_{\mathbb{P}^n}(d)\big], \end{aligned}$$

which is in degrees 0 and 1, so we can compute this very explicitly. In fact, this case is especially nice because we will show that $\mathcal{D}_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-n-1)[-n]$, so the expression above is exactly

$$\big[\mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}(-n-1+d)\big][-n] = \mathcal{O}_X(-n-1+d)[-(n-1)].$$

Since this has only one cohomology sheaf, we conclude that $\mathcal{D}_X$ has only one cohomology sheaf with

$$i_* H^{n-1}(\mathcal{D}_X) = H^{n-1}(i_*\mathcal{D}_X) = \mathcal{O}_X(d-n-1),$$

hence $\mathcal{D}_X = \mathcal{O}_X(d-n-1)[-n+1]$.

Example 2.4.59. Again let $X \to \operatorname{Spec} k$ be qcqs over a field with dualizing object $\mathcal{D}_X$. We explained previously that

$$\operatorname{Hom}_{D(X)}(A, \mathcal{D}_X) \xrightarrow{\sim} \mathbb{H}^0(X, A)^\vee = \operatorname{Hom}_{D(X)}(\mathcal{O}_X, A)^\vee.$$

In particular, if $E \in \operatorname{Perf}(X)$, then replacing A with $A \otimes^{\mathbf{L}}_{\mathcal{O}_X} E^\vee$ implies

$$\operatorname{Hom}_{D_{\mathrm{qc}}(X)}(A, E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{D}_X) \xrightarrow{\sim} \operatorname{Hom}_{D_{\mathrm{qc}}(X)}(E, A)^\vee.$$

Assume X is quasi-proper over k so that hom-spaces in $\operatorname{Perf}(X)$ are finite-dimensional k-vector spaces, and let us make the additional assumption that $\mathcal{D}_X \in \operatorname{Perf}(X)$. Then we can define a functor

$$\mathcal{S}_X \colon \operatorname{Perf}(X) \to \operatorname{Perf}(X) \qquad E \mapsto E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{D}_X$$

such that we have natural isomorphisms

$$\operatorname{Hom}_{\operatorname{Perf}(X)}(E, \mathcal{S}_X(F)) \xrightarrow{\sim} \operatorname{Hom}_{\operatorname{Perf}(X)}(F, E)^\vee.$$

Such a functor $\mathcal{S}_X$ is called a *Serre functor*, and it has interesting abstract properties which we will discuss later.

2.5. Perfect generation and more

In this section we will start to prove some deeper results in the derived category theory of qcqs schemes. A few of the big theorems in this section are from a 2007 paper by Lipman-Neeman, while the rest are somewhat classical. Many of the arguments will have a similar flavor: We first study the affine case, and then do induction while using some nontrivial triangulated category theory. Many of the affine cases will be a simple observation about the Koszul complex, so we start here.

2.5.1. The Koszul complex. We review some basic aspects of the Koszul complex and Koszul homology. We'll use some easy properties in the construction of a perfect generator, but we will also use this as a foundation for the appendix about commutative algebra. This material will also be useful when computing sheaf cohomology for some schemes from Čech complexes.

Let R be a ring and M an R-module. Recall (e.g. from the notes on the exterior algebra) that one constructs a graded (skew-)commutative R-algebra $\Lambda(M)$ equipped with the wedge product, whose properties we will use. Associated to a map $\phi\colon M \to R$, we define a chain complex

$$\cdots \to \Lambda^3_R(M) \xrightarrow{d} \Lambda^2_R(M) \xrightarrow{d} \Lambda^1_R(M) \to \Lambda^0(M) \to 0$$

called $\operatorname{Kos}^\bullet_R(\phi)$, with $\operatorname{Kos}^i_R(\phi) = \Lambda^{-i}_R(M)$ and differential:

$$d(x_1 \wedge \cdots \wedge x_k) = \sum_{i=1}^{k} (-1)^{i+1} \phi(x_i) x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge x_k.$$

A few basic checks are in order. First, this map is indeed well-defined: if $x_i = x_j$ for $i < j$, then all terms in the RHS vanish except for the two associated to i and j. If the first term is $(-1)^{i+1}(\cdots \wedge \widehat{x_i} \wedge \cdots)$, we can shift the position of the

$x_i = x_j$ term in the second term so that the factor is also $(\cdots \wedge \widehat{x_i} \wedge \cdots)$, which creates the sign $(-1)^{j+1+(i-j+1)} = (-1)^i$, and thus these two terms cancel.

Next, we in fact have $d^2 = 0$, since

$$\begin{aligned} d(x_1 \wedge \cdots \wedge x_k) = &\sum_{j<i} (-1)^{i+j} \phi(x_i)\phi(x_j) \cdots \wedge \widehat{x_j} \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \\ &+ \sum_{j \geq i} (-1)^{i+j} \phi(x_i)\phi(x_{j-1}) \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x}_{j-1} \wedge \cdots , \end{aligned}$$

so if we replace j in the second sum with $j+1$ we get

$$-\sum_{j>i} (-1)^{i+j} \phi(x_i)\phi(x_j) \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x}_j \wedge \cdots ,$$

and now by interchanging i and j we obtain the opposite of the expression in the first sum. $\mathrm{Kos}_R^\bullet(\phi)$ is called the *Koszul complex associated to* ϕ. Let us prove a few more basic properties of this construction. First, it is actually a *differential-graded algebra*, which means that the underlying algebra structure of $\Lambda(M)$ is compatible with d in the following way: for $\alpha \in \Lambda^k(M)$ and $\beta \in \Lambda^l(M)$, the Leibniz rule

$$d(\alpha\beta) = d(\alpha)\beta + (-1)^k \alpha d(\beta)$$

is satisfied, where we view $\alpha\beta = \alpha \wedge \beta$ as an element of $\Lambda^{k+l}(M)$. Indeed, if we write $\alpha = x_1 \wedge \cdots \wedge x_k$ and $\beta = y_1 \wedge \cdots \wedge y_l$, then split the sum as

$$\begin{aligned} d(\alpha \wedge \beta) = &\sum_{i \leq k} (-1)^{i+1} \phi(x_i)(\cdots \wedge \widehat{x_i} \wedge \cdots) \wedge \beta \\ &+ \sum_{i \geq k+1} (-1)^{i+1} \phi(y_{i-k}) x \wedge (\cdots \wedge \widehat{y}_{i-k} \wedge \cdots) \\ &= d(\alpha) \wedge \beta + \sum_{i=1}^{l} (-1)^{i+k+1} \phi(y_i) x \wedge (\cdots \wedge \widehat{y_i} \wedge \cdots) \\ &= d(\alpha) \wedge \beta + (-1)^k \alpha \wedge d(\beta) \end{aligned}$$

as desired. We recall that since $\Lambda(M)$ is graded-commutative, $\mathrm{Kos}_R^\bullet(\phi)$ is a graded-commutative dga, meaning $\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$.

Next, this behaves well with respect to sums. Namely, we showed in the other document that if N is another module then the map

$$\bigoplus_{p+q=n} \Lambda^p M \otimes_R \Lambda^q N \to \Lambda^n(M \oplus N)$$

given by $\alpha \otimes \beta \mapsto (\alpha, 0) \wedge (0, \beta)$ degree-wise is an isomorphism. If $\phi\colon M \to R$ and $\psi\colon N \to R$ are maps, and $(\phi, \psi)\colon M \oplus N \to R$ is induced, then this gives an isomorphism

$$(\mathrm{Kos}_R^\bullet(\phi) \otimes \mathrm{Kos}_R^\bullet(\psi))^{-n} \xrightarrow{\sim} \mathrm{Kos}_R^\bullet(\phi, \psi)$$

of R-modules. We claim that this is a chain map, and hence the above is an isomorphism of complexes. In fact since this came from an isomorphism of graded

algebras, this will show that the above is an isomorphism of dgas. This follows because the composites

$$\alpha \otimes \beta \mapsto \alpha \wedge \beta \mapsto d(\alpha\beta)$$

and

$$\alpha \otimes \beta \mapsto d(\alpha) \otimes \beta + (-1)^{|\alpha|}\alpha \otimes d(\beta) \mapsto d(\alpha)\beta + (-1)^{|\alpha|}\alpha d(\beta)$$

both agree by the Leibniz rule (here the second line is the differential of the tensor product complex).

In practice, the most interesting case is when $M = R^n$, so the map $R^n \to R$ is given by a sequence of elements $(f_1, \dots, f_n)$ of R. We often write $\underline{f}$ to denote such an (ordered) sequence of elements. In this case, we write

$$\mathrm{Kos}^\bullet_R(\underline{f}) = \mathrm{Kos}^\bullet_R(f_1, \dots, f_n)$$

for the corresponding Koszul complex. Observe that in this case, the Koszul complex consists of finite free modules, is concentrated in $[-n, 0]$, with $-i$th term having rank n choose i.

Next, suppose $\phi\colon M \to R$ and $\psi\colon N \to R$ are maps, and suppose there is a map $f\colon M \to N$ over R, meaning that $\psi \circ f = \phi$. Then there is a natural map

$$f_*\colon \mathrm{Kos}^\bullet_R(\phi) \to \mathrm{Kos}^\bullet_R(\psi)$$

of dgas satisfying $f_*(x_1 \wedge \cdots \wedge x_n) = f(x_1) \wedge \cdots \wedge f(x_n)$ in degree n. Indeed, we know (e.g. from the other document) that this is a well-defined morphism of graded algebras, and that it restricts to the corresponding map in each degree. To see that it is a chain map, observe that

$$\begin{aligned} df_*(x_1 \wedge \cdots \wedge x_k) &= \sum_{i=1}^{k} (-1)^{i+1} \psi(f(x_i)) f(x_1) \wedge \cdots \wedge f(x_k) \\ &= \sum_{i=1}^{k} (-1)^{i+1} \phi(x_i) f_*(x_1 \wedge \cdots \wedge x_k) \\ &= f_* d(x_1 \wedge \cdots \wedge x_k) \end{aligned}$$

with the right omissions in the sums. It is clear that this forms a functor out of the category of maps over R, i.e. that this construction respects composition.

Now consider the case where we have $f_1, \dots, f_n$ and $g_1, \dots, g_n$ as sequences in R, and suppose there is an invertible matrix $A\colon R^n \to R^n$ with $f_j = \sum_i a_{ij} g_i$, i.e. $f = gA$. Thus by functoriality, we obtain an isomorphism of the corresponding Koszul complexes.

Example 2.5.1. Let $\underline{f} = (f_1, \dots, f_n)$ in R. The right-most map in the corresponding Koszul complex is

$$R^n \xrightarrow{(f_1, \dots, f_n)} R \to 0,$$

so we conclude that $H^{-1}(\mathrm{Kos}(\underline{f})) = R/(f_1, \dots, f_n)$. The left-most map looks like

$$0 \to \Lambda^n R^n \to \Lambda^{n-1} R^n$$

where the basis vector $e = e_1 \wedge \cdots \wedge e_n$ maps to

$$\sum_{i=1}^{n} (-1)^{i+1} f_i e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_n,$$

which is written naturally in the canonical basis of $\Lambda^{n-1} R^n$. Thus for $a \in R$, $ae \mapsto 0$ iff $af_i = 0$ for all i. In particular, $H^{-1}(\mathrm{Kos}(\underline{f})) = \mathrm{Ann}_R(f_1, \ldots, f_n)$.

Our observation about tensor products above shows that

$$\mathrm{Kos}_R^\bullet(f_1, \ldots, f_n) \cong \mathrm{Kos}_R^\bullet(f_1) \otimes_R \cdots \otimes_R \mathrm{Kos}_R^\bullet(f_n).$$

Therefore, the isomorphism type doesn't depend on the order of the elements. Moreover, on one element the complex is just $\mathrm{Kos}_R^\bullet(f) = R \xrightarrow{f} R$ in degrees $-1, 0$. We can often use this to reduce properties of the complex to the single case. Note also that since each such complex is bounded with finite free terms, we can replace $\otimes_R$ with $\otimes_R^{\mathbf{L}}$ in $D(R)$.

Example 2.5.2. Suppose $(f_1, \ldots, f_n)$ is a *regular sequence*, which means that the image of f_i is regular (i.e. a non-zero divisor) on $R/(f_1, \ldots, f_{i-1})$. Then

$$\mathrm{Kos}_R(f_1) \simeq R/(f_1)$$

by definition, and

$$R/(f_1) \otimes^{\mathbf{L}} \mathrm{Kos}_R(f_2) = R/(f_1) \xrightarrow{f_2} R/(f_1) \simeq R/(f_1, f_2).$$

Thus, inductively,

$$\mathrm{Kos}_R(f_1, \ldots, f_n) \simeq R/(f_1, \ldots, f_{i-1}) \otimes^{\mathbf{L}} \mathrm{Kos}(f_i) \otimes^{\mathbf{L}} \cdots \simeq R/(f_1, \ldots, f_n).$$

In particular the Koszul complex is exact in this scenario after degree zero, and hence provides a finite free resolution of the quotient $R/(f_1, \ldots, f_n)$. We will revisit this notion shortly.

Let $\underline{f} = (f_1, \ldots, f_n)$, and let $a \in R$. Since $d(a\alpha) = ad(\alpha)$, multiplication by a gives an endomorphism $m_a \colon \mathrm{Kos}(\underline{f}) \to \mathrm{Kos}(\underline{f})$ of dgas. Consider the case $a = f_i$ for some i, and let $e_i \in \mathrm{Kos}^{-1}(\underline{f})$ be the degree 1 basis vector with $d(e_i) = f_i$. By the Leibniz rule,

$$d(e_i x) = f_i x - e_i d(x) \implies m_{f_i}(x) = e_i \wedge d(x) + d(e_i \wedge x).$$

In particular, $e_i \wedge (-)$ provides a nullhomotopy for m_{f_i}. In particular, the ideal $(f_1, \ldots, f_n)$ acts on the cohomology modules $H^p(\mathrm{Kos}_R(\underline{f}))$ by 0, so they are canonically (finite) $R/(\underline{f})$-modules.

Remark 2.5.3. Another way we sometimes phrase this result is the following: we often consider the Koszul complex as a complex of quasicoherent sheaves by $\widetilde{\mathrm{Kos}_R^\bullet(\underline{f})}$ on $\mathrm{Spec}\, R$. Since

$$(\underline{f}) \subset \mathrm{Ann}_R(H^p(\mathrm{Kos}_R(\underline{f}))),$$

taking $V(-)$ of both sides gives $\operatorname{Supp} H^p(\operatorname{Kos}_R(\underline{f})) \subset V(\underline{f})$, so in particular letting $U = D(f_1, \ldots, f_n)$, we see that

$$\widetilde{\operatorname{Kos}}_R(f_1, \ldots, f_n)|_U = 0 \qquad \text{in } D_{\mathrm{qc}}(U)$$

because the restriction of the cohomology sheaves is zero. This is an important observation which we will use later.

Next we discuss self-duality of the Koszul complex. Again suppose that M is a general R-module and $\phi\colon M \to R$ is a map. For each $n \geq 0$ and $k \leq n$, we have the canonical wedge pairing

$$\Lambda^k M \otimes \Lambda^{n-k} M \to \Lambda^n M \qquad \alpha \otimes \beta \mapsto \alpha\beta,$$

which we may view as a map $\Lambda^{n-k} M \to \operatorname{Hom}_R(\Lambda^k M, \Lambda^n M)$. Note that another way to write this is as a map

$$\left(\operatorname{Kos}_R^\bullet(\phi)[-n]\right)^k \to \underline{\operatorname{Hom}}_R^k(\operatorname{Kos}_R^\bullet(\phi), \Lambda^n M),$$

and we claim that as written this is a chain map. This is easy to check from the Leibniz rule and the differential in the hom complex. We've seen that if M is finite free of rank n, then the pairing is perfect, which implies that

$$\operatorname{Kos}_R^\bullet(\phi)[-n] \xrightarrow{\sim} \underline{\operatorname{Hom}}_R(\operatorname{Kos}_R^\bullet(\phi), \Lambda^n M)$$

is an isomorphism. In this case, one writes $\Lambda^n M = \det(M)$, and thus since the Koszul complex on a finite free module is strictly perfect, we obtain

$$\operatorname{Kos}_R^\bullet(\phi) \xrightarrow{\sim} \operatorname{Kos}_R^\bullet(\phi)^\vee \otimes_R \det(M)[n].$$

We note that everything here is basis-independent. Choosing an oriented basis gives an isomorphism $\det(M) \cong R$, and in particular

$$\operatorname{Kos}_R^\bullet(f_1, \ldots, f_n)^\vee \cong \operatorname{Kos}_R^\bullet(f_1, \ldots, f_n)[-n],$$

i.e. a sort of 'self-duality' plus shift.

Remark 2.5.4. This is a vague remark which says that the Koszul duality discussed here should remind us of other things in algebraic geometry. Suppose that X is a smooth k-variety, $R = \mathcal{O}_{X,p}$ is the local ring at a k-point, and $M = (\Omega^1_{X/k})_x$ is the cotangent sheaf at this point, which is free of rank $n = \dim(X)$. Then $\det(M) = \omega_{X,x}$ is the canonical bundle at x, and a map $\phi\colon M \to R$ is a choice of element of $(\Omega^1_{X/k})^\vee$ at the point x, i.e. a tangent vector in $T_x X$. Given such a tangent vector v, Koszul duality says

$$\operatorname{Kos}(v) \cong \operatorname{Kos}(v)^\vee \otimes \omega_X[n]$$

at x. This should remind us of Serre duality. Moreover, $\operatorname{Kos}(v)$ is locally

$$0 \to \omega_X \to \Omega^{n-1}_{X/k} \to \cdots \to \Omega^2_{X/k} \to \Omega^1_{X/k} \to \mathcal{O}_X \to 0,$$

which should remind us of a dualized de Rham complex.

We can easily sheafify the construction of the Koszul complex to work over a ringed space $(X, \mathcal{O}_X)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module equipped with a map $\phi\colon \mathcal{F} \to \mathcal{O}_X$, then by functoriality of the Koszul complex, the assignment

$$U \mapsto \mathrm{Kos}^\bullet_{\mathcal{O}_X(U)}(\phi(U))$$

is a complex of presheaves, and hence by sheafifying we obtain a complex

$$\mathrm{Kos}^\bullet_{\mathcal{O}_X}(\phi) \coloneqq [\cdots \to \Lambda^2_{\mathcal{O}_X}\mathcal{F} \to \Lambda^1_{\mathcal{O}_X}\mathcal{F} \to \mathcal{O}_X]$$

such that $\mathrm{Kos}^\bullet_{\mathcal{O}_X}(\phi)_x = \mathrm{Kos}^\bullet_{\mathcal{O}_{X,x}}(\phi_x)$ for every $x \in X$ (see exterior power notes section 2 for why this is clear). If X was a scheme and $\mathcal{F}$ was quasicoherent, then for $U \subset X$ an affine open we see

$$H^0(U, \mathrm{Kos}^\bullet_{\mathcal{O}_X}(\phi)) = \mathrm{Kos}^\bullet_{\mathcal{O}_X(U)}(\phi(U))$$

as expected (see the same source). We almost always use this when $\mathcal{F} = \mathcal{E}$ is a vector bundle of constant rank n, so then ϕ is identified with a global section of $\mathcal{E}^\vee$, and the complex stops at the nth stage:

$$0 \to \det(\mathcal{E}) \to \Lambda^{n-1}_{\mathcal{O}_X}\mathcal{E} \to \cdots \to \Lambda^2_{\mathcal{O}_X}\mathcal{E} \to \mathcal{E} \to \mathcal{O}_X \to 0$$

Note that, in this case, the Koszul complex is stalkwise isomorphic to a Koszul complex given by a sequence of elements in $\mathcal{O}_{X,x}$. Even more specifically, we usually take $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{L}_1$ to be a direct sum of line bundles, so that ϕ is identified with a sequence of global sections $s_i \in \mathcal{L}_i^\vee$. In this case we write $\mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_1, \ldots, s_n)$ for the associated Koszul complex.

Then in this case, we know that

$$\begin{aligned}\Lambda^k(\mathcal{E}) &= \bigoplus_{I \subset \{1,\ldots,n\}, |I|=k} \bigotimes_{i \in I} \mathcal{L}_i \\ &= \bigoplus_{1 \le i_1 < \cdots < i_k \le n} \mathcal{L}_{i_1} \otimes \cdots \otimes \mathcal{L}_{i_n}\end{aligned}$$

for $k \le n$, which we can use to write out the Koszul complex explicitly. For instance, the $n = 2$ and $n = 3$ cases look like:

$$0 \to \mathcal{L}_1 \otimes \mathcal{L}_2 \xrightarrow{\begin{pmatrix}-s_2 & s_1\end{pmatrix}} \mathcal{L}_1 \oplus \mathcal{L}_2 \xrightarrow{\begin{pmatrix}s_1 & s_2\end{pmatrix}} \mathcal{O}_X \to 0$$

and

$$0 \to \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_3 \to \bigoplus_{i<j} \mathcal{L}_i \otimes \mathcal{L}_j \to \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \to \mathcal{O}_X \to 0,$$

and so on. Just as before (with the same proof) we have

$$\begin{aligned}\mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_1, \ldots, s_n) &= \mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_1) \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} \mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_n) \\ &= \mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_1) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \cdots \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_n).\end{aligned}$$

Recall that for a section $s \in \mathcal{L}^\vee$, the subscheme $V(s)$ cut out by s has ideal sheaf given by the image of the corresponding map $\mathcal{L} \to \mathcal{O}_X$. Likewise for the general case with more sections, so

$$H^0(\mathrm{Kos}^\bullet_{\mathcal{O}_X}(s_1, \ldots, s_n)) = \mathcal{O}_{V(s_1,\ldots,s_n)} = \mathcal{O}_{V(s_1) \cap \cdots \cap V(s_n)}.$$

Therefore, the same proof as earlier shows that if $s_1, \ldots, s_n$ forms a regular sequence in X, meaning that $V(s_1, \ldots, s_i) \subset V(s_1, \ldots, s_{i-1})$ is an effective Cartier divisor for each $i = 1, \ldots, n$ (and the final quotient is nonzero),

$$\mathrm{Kos}^{\bullet}_{\mathcal{O}_X}(s_1, \ldots, s_n) \simeq \mathcal{O}_{V(s_1,\ldots,s_n)}$$

in $D(X)$. We want to think of $\mathrm{Kos}^{\bullet}_{\mathcal{O}_X}(s_1, \ldots, s_n)$ as a derived replacement for the intersection of these subschemes. For instance, if $V(s_1), \ldots, V(s_n)$ are all effective Cartier divisors, then

$$\mathrm{Kos}_{\mathcal{O}_X}(s_1, \ldots, s_n) = \mathcal{O}_{V(s_1)} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \cdots \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{V(s_n)}$$

by the equations above. If these form a complete intersection in X, then this agrees with $V(s_1, \ldots, s_n)$, which is the underived tensor product, but otherwise they may not. In section 3.1, we will discuss more criteria for handling such intersections correctly.

Remark 2.5.5. In general, it is always true that

$$H^0(\mathcal{O}_{Z_1} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \cdots \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{Z_n}) = \mathcal{O}_{Z_1 \cap \cdots Z_n}.$$

Indeed, by induction, we may assume $n = 2$. Then this coincides with

$$\mathcal{T}or_0^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_2} = \mathcal{O}_{Z_1 \cap \cap Z_2}.$$

Of course, all intersections here are scheme-theoretic.

As a special case of this, we can write down the Koszul complex for the regular sequence $x_0, \ldots, x_n \in H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(1))$ of projective space. By the formulas above, this will look like

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}(-n)^{\binom{n+1}{1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\binom{n+1}{1}} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Note that this is quasi-isomorphic to the structure sheaf of $V(x_0, \ldots, x_n) = \varnothing$, meaning it is an exact sequence. We will call this the canonical Koszul sequence for $\mathbb{P}^n$; it will appear again when we compute the cohomology of projective space.

Next we discuss a simple but important result about perfect complexes which will provide the inductive step of some arguments in the rest of this section.

Lemma 2.5.6. *Let R be a ring, $f_1, \ldots, f_r \in R$, and $A \in D^{\leq 0}(R)$ such that $\operatorname{Supp} H^q(A) \subset V(f_1, \ldots, f_r)$ for $q \in [-r+1, 0]$. Then every map $R \to A$ factors through the canonical map*

$$R \to \mathrm{Kos}^{\bullet}_R(f_1^{n_1}, \ldots, f_r^{n_r})$$

in degree 0 for some $n_1, \ldots, n_r \geq 1$.

Proof. We prove this by induction but with a stronger base case: namely, we'll show that for any $E \in \mathrm{Perf}(R)$ with E_f having tor-amplitude in $[a, \infty)$, and $A \in D(R)$ with $A_f \in D^{<a}(R_f)$, any map $E \to A$ factors as

$$E \to \mathrm{Kos}^{\bullet}_R(f^n) \otimes^{\mathbf{L}}_R E \to A$$

for some $n > 0$. Indeed, since E is perfect over an affine scheme, we have

$$\mathrm{RHom}_R(E, A)_f = (E^\vee \otimes^{\mathbf{L}}_R A)_f = E_f^\vee \otimes^{\mathbf{L}}_{R_f} A_f \in D^{-a+a-1} = D^{\leq -1}(R),$$

Therefore taking H^0 shows $\mathrm{Hom}_{D(R)}(E, A)_f = 0$, so for each $\alpha\colon E \to A$ there exists $n > 0$ such that $f^n\alpha = 0$. Thus, tensoring the canonical triangle for $\mathrm{Kos}(f^n)$ with E gives

$$E \xrightarrow{f^n} E \to \mathrm{Kos}_R(f^n) \to E[1],$$

so $E \xrightarrow{f^n} E \xrightarrow{\alpha} A$ being zero implies that the map factors through $\mathrm{Kos}_R(f^n)$.

Now for the general case induct on r, where $r = 1$ is handled by the above. In general, we in particular have $\operatorname{Supp} H^q(A) \subset V(f_1, \dots, f_{r-1})$ for $q \in [-r+2, 0] \subset [-r+1, 0]$, so factor any such map as

$$R \to \mathrm{Kos}_R(f_1^{n_1}, \dots, f_{r-1}^{n_{r-1}}) \to A.$$

The Koszul complex above has perfect amplitude in $[-r+1, 0]$, which remains true of its localization at f_r, and by assumption $A_{f_r} \in D^{<-r+1}$, so the second map factors through

$$\mathrm{Kos}(f_1^{n_1}, \dots, f_{r-1}^{n_{r-1}}) \otimes^{\mathbf{L}}_R \mathrm{Kos}_R(f_r^{n_r}) = \mathrm{Kos}(f_1^{n_1}, \dots, f_r^{n_r})$$

which is what we wanted. □

Let's also relate the powered Koszul complexes of the lemma above to the standard Koszul complex $\mathrm{Kos}(f_1, \dots, f_n)$. In general, if $\mathcal{T}$ is a triangulated category and $S \subset \mathcal{T}$ is a collection of objects, we define the *extension closure* $\mathrm{ExtCl}_{\mathcal{T}}(S)$ to be the smallest strictly full subcategory $\mathcal{C}$ containing 0 and S such that if $A, C \in \mathcal{C}$ and $A \to B \to C \to A[1]$ is exact, then $B \in \mathcal{C}$. This is *not* a triangulated subcategory, as it is not closed under shifts.

Example 2.5.7. Let R be a ring and $f_1, \dots, f_r \in R$. Then

$$\mathrm{Kos}_R(f_1^{n_1}, \dots, f_r^{n_r}) \in \mathrm{ExtCl}\left(\mathrm{Kos}_R(f_1, \dots, f_r)\right).$$

Indeed, since the composite $R \xrightarrow{f_i^{n-1}} R \xrightarrow{f_i} R$ equals $R \xrightarrow{f_i^n} R$, the octahedral axiom gives triangles

$$\mathrm{Kos}_R(f_i^{n-1}) \to \mathrm{Kos}_R(f_i^n) \to \mathrm{Kos}_R(f_i) \to \mathrm{Kos}_R(f^{n-1})[1].$$

Tensoring this with $\mathrm{Kos}_R(\dots, \widehat{f_i}, \dots)$ shows inductively that if

$$\mathrm{Kos}_R(f_1^{n_1}, \dots, f_i, \dots, f_r^{n_r}) \in \mathrm{ExtCl}\left(\mathrm{Kos}_R(f_1, \dots, f_r)\right)$$

then $\mathrm{Kos}_R(f_1^{n_1}, \dots, f_r^{n_r}) \in \mathrm{ExtCl}\left(\mathrm{Kos}_R(f_1, \dots, f_r)\right)$. Repeating this across the various i one at a time proves the claim.

Example 2.5.8. Let $\mathcal{T}$ be a triangulated category, S a collection of objects, and $H\colon \mathcal{T} \to \mathcal{A}$ a cohomological functor. Write $\ker H$ for the strictly full (not necessarily triangulated) subcategory of objects A such that $H(A) = 0$. If $S \subset \ker H$, then $\mathrm{ExtCl}(S) \subset \ker H$. Indeed, $H(0) = 0$, and if $A \to B \to C \to A[1]$ is a triangle and $H(A) = H(C) = 0$, then $H(A) \to H(B) \to H(C)$ is exact so $H(B) = 0$. Of course, the same thing applies for homological functors.

Let's combine the simple observations above to obtain the following result:

Lemma 2.5.9. *Let R be a ring, $\underline{f} \coloneqq f_1, \dots, f_r \in R$, and*

$$A \in D(R) \qquad \operatorname{Supp} A \subset V(\underline{f}).$$

If $H^m(A) \neq 0$, then $\operatorname{Hom}_{D(R)}(\operatorname{Kos}_R^\bullet(\underline{f})[-m], A) \neq 0$.

Proof. By replacing A with $A[m]$, we may assume $m = 0$. Note moreover that $\operatorname{Hom}_{D(R)}(\operatorname{Kos}(\underline{f}), D^{>0}(R)) = 0$, so

$$\operatorname{Hom}_{D(R)}(\operatorname{Kos}_R^\bullet(\underline{f}), A) = \operatorname{Hom}_{D(R)}(\operatorname{Kos}_R^\bullet(\underline{f}), \tau_{\leq 0} A),$$

and thus we may assume $A \in D^{\leq 0}$. Take a nonzero element of $H^0(A) = \operatorname{Hom}_R(R, A)$, which factors as

$$R \to \operatorname{Kos}_R(f_1^{n_1}, \dots, f_r^{n_r}) \to A$$

so that both maps above are nonzero. Thus $\operatorname{Hom}_{D(R)}(\operatorname{Kos}(\underline{f}), A) \neq 0$: indeed, if it were zero, then $\operatorname{Kos}(\underline{f})$ lies in the kernel of the homological functor $\operatorname{Hom}_{D(R)}(-, A)$, so by both of the examples above this would imply

$$\operatorname{Hom}_{D(R)}(\operatorname{Kos}_R(f_1^{n_1}, \dots, f_r^{n_r}), A) = 0$$

a contradiction. □

2.5.2. Verdier quotients and localizing subcategories. Let $\mathcal{T}$ be a triangulated category, and $\mathcal{N} \subset \mathcal{T}$ be a strictly full triangulated subcategory which is not necessarily thick. We say a morphism $s\colon X \to Y$ in $\mathcal{T}$ is an $\mathcal{N}$-*equivalence* if a (or equivalently every) cone of s lies in $\mathcal{N}$. Write $S_{\mathcal{N}}$ for the collection of $\mathcal{N}$-equivalences.

Lemma 2.5.10. *Let $\mathcal{T}$ be a triangulated category with $\mathcal{N} \subset \mathcal{T}$ a strictly full triangulated subcategory. Then $S_{\mathcal{N}}$ is a multiplicative, saturated, localizing class containing all isomorphisms, closed under shifts, and compatible with the triangulation.*

Proof. As $0 \in \mathcal{N}$ and $\mathcal{N}$ is closed under shifts, it is clear that $S_{\mathcal{N}}$ has the isomorphisms and is closed under shifts. If $s\colon X \to Y$ and $t\colon Y \to Z$ are morphisms then the octahedral axiom gives a triangle

$$\mathrm{Cone}(s) \to \mathrm{Cone}(ts) \to \mathrm{Cone}(t) \to \mathrm{Cone}(s)[1].$$

Since membership in $\mathcal{N}$ has 2-out-of-3 for triangles, it follows that of s, t, st any two are in $S_{\mathcal{N}}$ iff the third is. This proves that $S_{\mathcal{N}}$ is a saturated multiplicative class containing isomorphisms and closed under shifts.

To check that S is compatible with the triangulation, it remains only to show that in TR3 if the first two morphisms are in S then we can take the third to be in S. Indeed, given a commutative diagram as in the top left, we can use the 3x3 lemma to extend it to:

$$\begin{array}{ccccc}
X & \longrightarrow & Y & \dashrightarrow & Z \\
\downarrow & & \downarrow & & \vdots \\
X' & \longrightarrow & Y' & \dashrightarrow & Z' \\
\vdots & & \vdots & & \vdots \\
X'' & \longrightarrow & Y'' & \dashrightarrow & Z''
\end{array}$$

Then since $X'', Y'' \in \mathcal{N}$, $Z'' \in \mathcal{N}$ as desired.

It remains to see that $S_{\mathcal{N}}$ is a localizing class. First, start with $A \xrightarrow{f} X \xleftarrow{s} B$ and complete a diagram of the map $A \oplus B \xrightarrow{(f,-s)} C$ to a triangle

$$P \xrightarrow{(a,b)} A \oplus B \xrightarrow{(f,-s)} X \to P[1],$$

thus the successive composite is $fa - sb = 0$, i.e. the diagram

$$\begin{array}{ccc}
P & \xrightarrow{a} & A \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle f} \\
B & \xrightarrow{s} & X
\end{array}$$

commutes. It remains to show $\mathrm{Cone}(a) \in \mathcal{N}$. Indeed, a is the composite $P \to A \oplus B \to A$, and because of the triangle

$$A \oplus B \to A \to B[1] \to A \oplus B[1]$$

the octahedral axiom gives a triangle

$$X \to \mathrm{Cone}(a) \to B[1] \xrightarrow{-s[1]} X[1].$$

$\mathrm{Cone}(a)[1] \in \mathcal{N}$ as $-s[1] \in \mathcal{N}$, thus $\mathrm{Cone}(a) \in \mathcal{N}$. The lifting condition for left roofs follows similarly.

Finally, let $f, g\colon X \to Y$ be parallel maps, and suppose $s \in S_{\mathcal{N}}$ with $sf = sg$. Take an exact triangle

$$Y \xrightarrow{s} Y' \to Y'' \to Y[1],$$

(so that $Y'' \in \mathcal{N}$) and observe that $s(f-g) = 0$, so since $\mathrm{Hom}(X, -)$ is cohomological this means $f - g$ factors as a composite

$$f - g = X \to Y''[-1] \to Y.$$

Thus we may complete this to $X' \to X \to Y''[-1] \to X'[1]$ so that the first map is in $S_{\mathcal{N}}$, and its composition with $f - g$ is zero. The converse follows dually. □

Definition 2.5.11. Let $\mathcal{T}$ be a triangulated category and $\mathcal{N} \subset \mathcal{T}$ a strictly full triangulated subcategory. We define

$$\mathcal{T}/\mathcal{N} \coloneqq \mathcal{T}[S_{\mathcal{N}}^{-1}]$$

to be the triangulated localization of $\mathcal{T}$ with respect to the saturated localizing class $S_{\mathcal{N}}$ compatible with the triangulation. $\mathcal{T}/\mathcal{N}$ is the *Verdier quotient*. We call the localization map $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ the quotient map.

Corollary 2.5.12. *Let $\mathcal{N} \subset \mathcal{T}$ be a strictly full triangulated subcategory of $\mathcal{T}$ and let $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ be the Verdier quotient map.*

(a) Then q is exact and essentially surjective, and if $X \to Y \to Z \to X[1]$ is a sequence of morphisms in $\mathcal{T}$, then

$$q(X) \to q(Y) \to q(Z) \to q(X)[1]$$

is exact in $\mathcal{T}/\mathcal{N}$ iff the original sequence was exact.

(b) If $f\colon X \to Y$ is a morphism in $\mathcal{T}$, the following are equivalent:

i. $q(f) = 0$.
ii. f factors through $\mathcal{N}$.
iii. $fs = 0$ for some $s \in S_{\mathcal{N}}$.
iv. $sf = 0$ for some $s \in S_{\mathcal{N}}$.

(c) If $X \in \mathcal{T}$ the following are equivalent:

i. $q(X) = 0$.
ii. id_X factors through $\mathcal{N}$.
iii. X is a summand of an object of $\mathcal{N}$.
iv. $X \in \mathrm{thick}\,\mathcal{N}$.

In particular, $\ker(q) = \mathrm{thick}(\mathcal{N})$.

(d) If $f\colon X \to Y$ is a morphism in $\mathcal{T}$, the following are equivalent:

(a) $q(f)$ is an isomorphism.
(b) $\mathrm{Cone}(f) \in \mathrm{thick}(\mathcal{N})$.

PROOF. (a) is the basic theorem on triangulated localizations, Prop. 1.2.10.

(b) The equivalence of $(i), (iii), (iv)$ is already known to us from the theory of saturated localizing classes. Next, $q(\mathcal{N}) = 0$ because if $X \in \mathcal{N}$, then $X \to 0 \to X[1] \to$ is exact, showing $X \to 0$ becomes an isomorphism in the quotient. Thus $(ii) \Rightarrow (i)$. Conversely, assume $fs = 0$, and write

$$X \xrightarrow{s} Y \to C \to X[1]$$

with $C \in \mathcal{N}$. If $f\colon Y \to Z$, then since $\mathrm{Hom}(-, Z)$ is cohomological we see that $fs = 0$ implies s factors through C.

(c) $q(X) = 0$ iff $q(\mathrm{id}_X) = 0$. Thus $(i) = (ii)$ by (b). But (ii) implies by Remark 2.4.39 that X is a summand of an object of $\mathcal{N}$, i.e. (iii). But if this holds, then $X \in \mathrm{thick}(\mathcal{N})$. Finally $\ker(q)$ is a thick subcategory, so $\ker(q) \subset \mathrm{thick}(\mathcal{N})$, and hence $q(X) = 0$. The equivalence implies $\ker q = \mathrm{thick}(\mathcal{N})$.

(d) $q(f)$ is an isomorphism iff $\mathrm{Cone}(q(f)) = q(\mathrm{Cone}(f)) = 0$, which happens iff $\mathrm{Cone}(f) \in \mathrm{thick}(\mathcal{N})$ by (c). □

Example 2.5.13. Let $\mathcal{A}$ be an abelian category. Then we denote by $\mathrm{K}_{\mathrm{ac}}(\mathcal{A}) \subset \mathrm{K}(\mathcal{A})$ the strictly full subcategory of acyclic objects. This is clearly triangulated, and it is also clearly thick. Thus we obtain a Verdier quotient $\mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})/\mathrm{K}_{\mathrm{ac}}(\mathcal{A})$. Note that a morphism f is a quasi-isomorphism iff $\mathrm{Cone}(f) \in \mathrm{K}_{\mathrm{ac}}(\mathcal{A})$, so it follows that the Verdier quotient is $D(\mathcal{A})$.

Proposition 2.5.14. *Let $\mathcal{N} \subset \mathcal{T}$ be a strictly full triangulated subcategory. The quotient map $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ has the universal property that composition with q induces a 1-to-1 correspondence*

$$(-) \circ q\colon \{\widetilde{F}\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}'\} \xrightarrow{\sim} \{F\colon \mathcal{T} \to \mathcal{T}' : \mathcal{N} \subset \ker(F)\}$$

where on both sides we consider only exact functors. Moreover, if either side is viewed as the full subcategory of $\mathrm{Fun}(\mathcal{T}/\mathcal{N}, \mathcal{T}')$ *and* $\mathrm{Fun}(\mathcal{T}, \mathcal{T}')$ *spanned by the exact functors, then this is an equivalence of categories.*

PROOF. This is just the universal property of localization. Namely, all functors $\mathcal{T}/\mathcal{N} \to \mathcal{T}'$ correspond to all functors $\mathcal{T} \to \mathcal{T}'$ such that elements of $S_{\mathcal{N}}$ are sent to isomorphisms. Since this is induced by an exact functor, the correspondence preserves exact functors in one direction. On the other hand if $F\colon \mathcal{T} \to \mathcal{T}'$ is exact and $F = \widetilde{F}q$, then by definition of the triangles in $\mathcal{T}/\mathcal{N}$ we see that $\widetilde{F}$ is exact.

Finally, if $F\colon \mathcal{T} \to \mathcal{T}'$ is exact, then F sends $\mathcal{N}$-equivalences to isomorphisms iff $\mathrm{Cone}(F(f)) = F(\mathrm{Cone}(f)) = 0$ whenever $\mathrm{Cone}(f) \in \mathcal{N}$. This condition is implied by $F(\mathcal{N}) = 0$, and conversely every object of $\mathcal{N}$ is a cone. □

Remark 2.5.15. Here is our usual full-subcategory localization result translated into this language. Suppose $\mathcal{T}', \mathcal{N} \subset \mathcal{T}$ are strictly full triangulated subcategories. Then $\mathcal{N} \cap \mathcal{T}' \subset \mathcal{T}'$ is also a strictly full triangulated subcategory, and thus by the universal property we obtain an exact functor

$$\mathcal{T}' \,/\, \mathcal{N} \cap \mathcal{T}' \to \mathcal{T} \,/\, \mathcal{N}$$

between the Verdier quotients. This is fully faithful in either of the following circumstances:

(a) Every map from $\mathcal{T}' \to \mathcal{N}$ factors through $\mathcal{N} \cap \mathcal{T}'$.
(b) Every map from $\mathcal{N} \to \mathcal{T}'$ factors through $\mathcal{N} \cap \mathcal{T}'$.

Indeed, let us see how it follows from the usual criterion. First, from the definition we see that $S_{\mathcal{N}\cap\mathcal{T}'} = S_{\mathcal{N}} \cap \mathcal{T}'$ and both are localizing classes. Now, suppose e.g. we're in the situation of (a) and are given a map $X \to T'$ in $S_{\mathcal{N}}$ with $T' \in \mathcal{T}'$. Write the triangle $X \to T' \to N \to X[1]$ with $N \in \mathcal{N}$, then the second map factors as $T' \to B \to N$ for $B \in \mathcal{N} \cap \mathcal{T}'$. Apply the octahedral axiom to this composition, which gives a triangle

$$\mathrm{Cone}(T' \to B) \to X[1] \to \mathrm{Cone}(B \to N) \to .$$

Thus the first term is in $\mathcal{T}'$ and the third is in $\mathcal{N}$, so shifting down one provides an $\mathcal{N}$-equivalence from an object of $\mathcal{T}'$ to X.

This description makes it very easy to apply the criterion: For instance to show that $\mathrm{K}^+(\mathcal{A})/\mathrm{K}^+_{\mathrm{ac}}(\mathcal{A}) = D^+(\mathcal{A})$ embeds into $D(\mathcal{A})$, we observe that every map $A^\bullet \to X^\bullet$, where $A^\bullet$ is acyclic and $X^\bullet$ is bounded below, factors through the smart truncation $A \to \tau_{\geq n}A \to X^\bullet$ for some n such that $X^i = 0$ for $i < n$.

Here are some more elementary results, which we have essentially seen before in the specific case of the derived category. We note that the whole point of introducing this is that we are going to take quotients of the derived category itself. For instance, recall that if $S \subset \mathcal{T}$ is a collection of objects in a triangulated category, we define

$$S^\perp \coloneqq \{T \in \mathcal{T} : \mathrm{Hom}^*(S, T) = 0\}$$

and

$${}^\perp S \coloneqq \{T \in \mathcal{T} : \mathrm{Hom}^*(T, S) = 0\}$$

as the right- and left- orthogonals of S. Then it is immediate that these are both thick subcategories of $\mathcal{T}$.

Lemma 2.5.16. *Let $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ be a Verdier quotient.*

(a) For $Y \in \mathcal{N}^\perp$, the natural map

$$\mathrm{Hom}_{\mathcal{T}}(-, Y) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(q(-), q(Y))$$

is an isomorphism.

(b) For $X \in {}^\perp\mathcal{N}$, the natural map

$$\mathrm{Hom}_{\mathcal{T}}(X, -) \to \mathrm{Hom}_{\mathcal{T}/\mathcal{N}}(q(X), q(-))$$

is an isomorphism.

Proof. Let's prove (a) as (b) is dual. Represent $q(X) \to q(Y)$ by a roof

$$X \xleftarrow{s} X' \xrightarrow{f} Y$$

with a triangle $X' \xrightarrow{s} X \to N \to X[1]$ so that $N \in \mathcal{N}$. By assumption,

$$\mathrm{Hom}(X, Y) \xrightarrow{\sim} \mathrm{Hom}(X', Y)$$

so there's already $g\colon X \to Y$ in $\mathcal{T}$ commuting with s, which proves surjectivity. On the other hand if $f\colon X \to Y$ and $q(f) = 0$, then f factors through $\mathcal{N}$ by Cor. 2.5.12, and hence it must be zero since a map $\mathcal{N} \to Y$ is zero. □

Here is a basic but important consequence of this simple lemma, which helps characterize what we will call 'admissibility'.

Proposition 2.5.17. *Let $i\colon \mathcal{N} \hookrightarrow \mathcal{T}$ be the inclusion of a strictly full triangulated subcategory, and $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ the corresponding Verdier quotient. Then the following conditions are equivalent:*

(a) The inclusion $\bar{i}\colon \operatorname{thick}(\mathcal{N}) \hookrightarrow \mathcal{T}$ has a right adjoint.
(b) q has a right adjoint.
(c) The composite $\mathcal{N}^{\perp} \to \mathcal{T} \to \mathcal{T}/\mathcal{N}$ is an equivalence.
(d) Each object $X \in \mathcal{T}$ fits into an exact triangle of the form

$$N_X \to X \to L_X \to N_X[1]$$

with $N_X \in \operatorname{thick}(\mathcal{N})$ and $L_X \in \mathcal{N}^{\perp}$.

When these conditions hold, denote by $i^!$ and r any right adjoints as in (a) and (b) respectively. Then $i^!$ and r canonically induce inverses to the functor of (c), and a triangle as in (d) can be written

$$\bar{i}i^!X \to X \to rqX \to \bar{i}i^!X[1].$$

Conversely any such prescription of exact triangles in (d) for all $X \in \mathcal{T}$ defines right adjoints $i^!$ and r as above, and r is always fully faithful.

The dual statements hold where we replace 'right' by 'left', $\mathcal{N}^{\perp}$ by ${}^{\perp}\mathcal{N}$, and the triangles in (d) with $R_X \to X \to N_X \to R_X[1]$ where $N_X \in \operatorname{thick}(\mathcal{N})$ and $R_X \in {}^{\perp}\mathcal{N}$.

Proof. First note that $\mathcal{N} \to \mathcal{T} \to \mathcal{T}/\mathcal{N}$ is always fully faithful by Lemma 2.5.16. If each object has an exact triangle as in (d), then since $\operatorname{thick}(\mathcal{N}) = \ker(q)$ we deduce that $q(X) \xrightarrow{\sim} q(L_X)$, showing essential surjectivity. Thus $(d) \Rightarrow (c)$.

If we assume (c), take an inverse $\mu\colon \mathcal{T}/\mathcal{N} \to \mathcal{N}^{\perp}$, then

$$\operatorname{Hom}_{\mathcal{T}/\mathcal{N}}(q(X), Y) \cong \operatorname{Hom}_{\mathcal{N}^{\perp}}(\mu(q(X)), \mu(Y)) \cong \operatorname{Hom}_{\mathcal{T}}(X, \mu(Y)),$$

so the composite $\mathcal{T}/\mathcal{N} \to \mathcal{N}^{\perp} \to \mathcal{T}$ is a right adjoint.

Assuming (b), let $q \dashv r$, then for $N \in \mathcal{N}$ we have

$$\operatorname{Hom}^*_{\mathcal{T}}(N, r(A)) = \operatorname{Hom}^*_{\mathcal{T}/\mathcal{N}}(q(N), A) = 0$$

which means that r lands in $\mathcal{N}^{\perp}$. Thus r is fully faithful because

$$\operatorname{Hom}_{\mathcal{T}/\mathcal{N}}(qA, B) \cong \operatorname{Hom}_{\mathcal{T}}(A, rB) \cong \operatorname{Hom}_{\mathcal{T}}(qA, qrB),$$

where the second isomorphism is by Lemma 2.5.16, and q is essentially surjective, so the counit $B \to qrB$ is an isomorphism. Complete the unit to a triangle

$$Q_X \to X \to rqX \to Q_X[1],$$

then upon applying q the middle map is an isomorphism, so $q(Q_X) = 0$, i.e. $Q_X \in \operatorname{thick}(\mathcal{N})$. Next $rq(X) \in \mathcal{N}^{\perp}$, so $\mathcal{N} \subset {}^{\perp}(rqX)$ and thus $\operatorname{thick}(\mathcal{N}) \subset {}^{\perp}(rqX)$ since this subcategory is thick. Therefore applying $\operatorname{Hom}_{\mathcal{T}}(\bar{i}(N), -)$ for $N \in \operatorname{thick}(\mathcal{N})$ gives an isomorphism

$$\operatorname{Hom}_{\operatorname{thick}(\mathcal{N})}(N, Q_X) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{T}}(\bar{i}N, X),$$

showing Q_X is a partial right adjoint to $\bar{i}$ at X, and since X was arbitrary $\bar{i}$ has a right adjoint assembled out of such Q_X's.

Note that this along with the exact triangle proves one direction of the claim at the end of the proof. For the converse, note that the isomorphism above only depended on $rqX \in \mathcal{N}^\perp$, so given triangles as in (d) proves that $\bar{i}$ has a right adjoint $i^!$ with $i^!(X) = N_X$ and with counit $N_X \to X$. Thus by uniqueness of cones and the isomorphism above, we deduce $rqX \cong L_X$ and can replace r with a naturally isomorphic r' with counits the maps $X \to L_X$.

Finally, assume (a) and for any $X \in \mathcal{T}$ complete the counit to a triangle

$$\bar{i}i^!X \to X \to C \to \bar{i}i^!X[1].$$

For each $N \in \mathcal{N} \subset \operatorname{thick}(\mathcal{N})$, note that

$$\operatorname{Hom}^*(\bar{i}N, \bar{i}i^!X, iN) \xrightarrow{\sim} \operatorname{Hom}^*(N, i^!X) \xrightarrow{\sim} \operatorname{Hom}^*(\bar{i}N, X)$$

is an isomorphism, so $\operatorname{Hom}^*(N, C) = 0$, hence $C \in \mathcal{N}^\perp$. This proves (d). □

A thick subcategory $\mathcal{N} \subset \mathcal{T}$ is called *right admissible* (resp. *left admissible*) if it satisfies the equivalent conditions of the first (resp. second) orientation of the theorem. It is *admissible* if it is both right and left admissible.

Remark 2.5.18. We note that if $\mathcal{N}$ is any strictly full triangulated subcategory and the inclusion $\mathcal{N} \hookrightarrow \mathcal{T}$ has a right adjoint, then $\mathcal{N}$ is automatically thick. Indeed, using the notation as in Prop. 2.5.17, if $X \to i(N) \to X$ is an idempotent splitting in $\mathcal{T}$ with N in $\mathcal{N}$, then since this is a fully faithful adjoint applying $i^!$ gives $i^!(X) \to N \to i^!(X)$ as a split idempotent in $scrN$. Thus $i^!(X) \cong X$ in $\mathcal{T}$ and we conclude $X \in \mathcal{N}$.

Remark 2.5.19. Suppose that $\mathcal{N} \hookrightarrow \mathcal{T}$ is right admissible and let $\mathcal{M} \coloneqq \mathcal{N}^\perp$, which is another thick subcategory. We know that the right adjoint $r\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}$ to q is identified with the inclusion $\mathcal{M} \hookrightarrow \mathcal{T}$, so this in particular has a left adjoint. Thus $\mathcal{M}$ is left admissible. The kernel of this left adjoint is supposed to be ${}^\perp\mathcal{M}$, but on the other hand we know that $\ker q = \mathcal{N}$. The same statements hold dually, and thus $\mathcal{N}$ is right admissible iff $\mathcal{M} = \mathcal{N}^\perp$ is left admissible, in which case $\mathcal{N} = {}^\perp\mathcal{M}$.

Here's a related statement which gives a simple criterion for computing a Verdier quotient.

Proposition 2.5.20. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be an exact functor of triangulated categories with a fully faithful right adjoint. Then the induced map*

$$\mathcal{T}/\ker(F) \to \mathcal{T}'$$

is a triangulated equivalence.

Proof. Let $G\colon \mathcal{T}' \to \mathcal{T}$ be the right adjoint, then we claim that the composite qG is an inverse to this functor. One direction is clear: Since G is fully faithful, $FG \cong \mathrm{Id}$, so

$$\overline{F} \circ q \circ G \cong F \circ G \cong \mathrm{Id}.$$

On the other hand, let $\eta = \eta_A \colon A \to GFA$ be the unit of the adjunction, then we know that $F\eta_A \colon FA \to FGFA$ is an isomorphism, hence

$$0 = \operatorname{Cone}(F\eta_A) = F(\operatorname{Cone}(\eta_A))$$

which means $\operatorname{Cone}(\eta_A) \in \ker F$ for all A. Therefore $q(\eta_A)$ is an isomorphism, which is a natural map $q(A) \to (qG)F(A)$. Both sides preserve $S_{\ker(F)}$-equivalences, so by the universal property of localization this descends to a natural isomorphism $\mathrm{Id}_{\mathcal{T}/\ker(F)} \xrightarrow{\sim} (qG)\overline{F}$ as desired. □

Example 2.5.21. Let X be a scheme and $j \colon U \hookrightarrow X$ a quasicompact inclusion, so that j is qcqs and $Rj_* \colon D_{\mathrm{qc}}(U) \to D_{\mathrm{qc}}(X)$ is well-defined. Then $j^* \dashv Rj_*$, and the counit $j^*Rj_*A \to A$ is an isomorphism by the behavior of derived pushforward under restriction. Therefore

$$j^* \colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(U)$$

satisfies the hypotheses of the proposition, and shows that $D_{\mathrm{qc}}(X)/\ker(j^*) = D_{\mathrm{qc}}(U)$. We usually write

$$D_Z(X) \coloneqq \ker(j^*) = \{A \in D_{\mathrm{qc}}(X) : A|_U = 0\} = \{A \in D_{\mathrm{qc}}(X) : \operatorname{Supp} A \subset Z\},$$

where $Z = X \setminus U$. Thus $D_{\mathrm{qc}}(X)/D_Z(X) = D_{\mathrm{qc}}(U)$. Since the quotient map $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(U)$ has a right adjoint, we know by Prop. 2.5.17 that $D_Z(X) \subset D_{\mathrm{qc}}(X)$ is right admissible, and has its own right adjoint $i^! \colon D_{\mathrm{qc}}(X) \to D_Z(X)$ which provides functorial triangles

$$i^!A \to A \to Rj_*j^*A \to i^!A[1].$$

These are called *local cohomology triangles*. We will see later that in this special case, $i^!$ is induced by a derived functor of a right exact functor. Notice that moreover the result shows $Rj_* \colon D_{\mathrm{qc}}(U) \hookrightarrow D_{\mathrm{qc}}(X)$ gives an embedding whose essential image is $D_Z(X)^\perp$. We will later write $D_{\mathrm{qc}}(X) = \langle D_Z(X), D_{\mathrm{qc}}(U)\rangle$ and call this a *semiorthogonal decomposition*.

Next let $\mathcal{T}$ be a triangulated category with (arbitrary) coproducts. We wish to study subcategories $\mathcal{A} \subset \mathcal{T}$ which are strictly full, triangulated, and closed under coproducts in $\mathcal{A}$. Such a subcategory is called a *localizing subcategory*, though we will only use this terminology in this section since we have (short-sightedly) used it to mean other things.

It is clear that an arbitrary intersection of such subcategories has the same property, and that $\mathcal{T}$ itself is localizing. Thus, given a collection $S \subset \mathcal{T}$ of objects, we write $\operatorname{Loc}(S)$ for the smallest localizing subcategory containing S.

Lemma 2.5.22. *Let $\mathcal{T}$ be a triangulated category with countable coproducts. Then $\mathcal{T}$ is idempotent-complete.*

Proof. Let $e \colon X \to X$ be an idempotent in $\mathcal{T}$ so that $e^2 = e$. Write

$$Y \coloneqq \operatorname{hocolim}(X \xrightarrow{e} X \xrightarrow{e} \cdots)$$

so that we have a triangle

$$\bigoplus_n X \xrightarrow{1-s} \bigoplus_n X \xrightarrow{q} Y \to \bigoplus_n X[1],$$

where s is given by $s\iota_n = \iota_{n+1}e$. Define

$$\ell\colon \bigoplus_n X \to \bigoplus_n X \qquad \ell\iota_n = \iota_n(1-e) - \sum_{k<n} \iota_k e.$$

Since we have $(1-e)e = 0$, note that for each $n \geq 1$,

$$\begin{aligned}\ell(1-s)\iota_n &= \iota_n(1-e) - \sum_{k<n} \iota_k e - \ell\iota_{n+1}e \\ &= \iota_n(1-e) - \sum_{k<n} \iota_k e - \sum_{k<n+1} \iota_k e \\ &= \iota_n(1-e) + \iota_n e = \iota_n,\end{aligned}$$

meaning $\ell(1-s) = \mathrm{Id}$. Thus $1-s$ is a split monomorphism, and exactness of $\mathrm{Hom}(-, \bigoplus_n X)$ shows in the defining triangle that $Y \to \bigoplus_n X[1]$ is 0. Therefore the triangle splits as a direct sum and q is a split epimorphism.

Let $j_n = q\iota_n\colon X \to Y$. Note that

$$0 = q(1-s)\iota_n = q(\iota_n - \iota_{n+1}e) \implies j_n = j_{n+1}e.$$

Since e is idempotent, all of the j_n's are the same map $j\colon X \to Y$, satisfying $je = j$. Finally, the maps $(e\colon X \to X)_n$ commute with themselves as $e^2 = e$, hence induce $r\colon Y \to X$ such that $rq\iota_n = e$ for all n. Thus $rj = e$, and note that

$$jrq\iota_n = jrj = je = j = q\iota_n$$

for all n, hence $jrq = q$. Since q is split epi, this means $jr = 1$. Thus we have split e. □

Corollary 2.5.23. *Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{A} \subset \mathcal{T}$ a localizing subcategory. Then $\mathcal{A}$ is thick.*

Proof. Indeed, $\mathcal{A}$ has coproducts and hence is idempotent-complete. So if $X \oplus Y \in \mathcal{A}$, the idempotent $X \oplus Y \to X \to X \oplus Y$ splits in $\mathcal{A}$ factoring through some Z. By uniqueness, $Z \cong X$ in $\mathcal{T}$ and hence $X \in \mathcal{A}$ by strict fullness. □

When a triangulated category $\mathcal{T}$ has coproducts indexed by I, then they are necessarily exact, meaning that if $X_i \to Y_i \to Z_i \to X_i[1]$ is a collection of exact triangles, then

$$\bigoplus_i X_i \to \bigoplus_i Y_i \to \bigoplus_i Z_i \to \bigoplus_i X_i[1]$$

is also exact. To see this, let Z be a cone of the direct map between direct sums. For each $i \in I$, the canonical maps from X_i and Y_i to the direct sums induce (by TR3) a family of maps $Z_i \to Z$, and hence a map $\bigoplus_i Z_i \to Z$. To check this is an isomorphism, we can check after applying $\mathrm{Hom}(-, W)$ for every W; then 5-lemma, the fact that $\mathrm{Hom}(-, W)$ turns sums to products, and that products of abelian groups are exact, give the result.

Lemma 2.5.24. *Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{N} \subset \mathcal{T}$ be a localizing subcategory, and consider the Verdier quotient $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$. Then $\mathcal{T}/\mathcal{N}$ has coproducts, and q preserves coproducts.*

Proof. We copy the proof of Lemma 2.4.4. Let $\{X_i : i\}$ be a family of objects in $\mathcal{T}$ and suppose we are given maps $f_i\colon q(X_\alpha) \to q(Y)$ in the quotient. Represent these by roofs and take the direct sum:

$$X_i \xleftarrow{s_i} X_i' \xrightarrow{g_i} Y \qquad \bigoplus_i X_i \xleftarrow{(s_i)} \bigoplus_i X_i' \xrightarrow{(g_i)} Y$$

Since $\mathrm{Cone}(s_i) \in \mathcal{N}$, what we showed above the lemma implies that $\mathrm{Cone}(\bigoplus_i s_i) \in \mathcal{N}$ since $\mathcal{N}$ is localizing. This represents a morphism in the quotient making the relevant diagram commute, and shows the existence part of the universal property.

Next uniqueness: Suppose we have a map $f\colon q(\bigoplus_i X_i) \to q(Y)$ such that the composites $q(X_i) \to q(\bigoplus_i X_i) \to q(Y)$ are all 0. Represent f by a right roof $\bigoplus_i X_i \xrightarrow{g} Y' \xleftarrow{s} Y$, then the hypothesis is that the roof $X_i \xrightarrow{g_i} Y' \xleftarrow{s} Y$ is 0, i.e. that $q(X_i \xrightarrow{g_i} Y') = 0$. Therefore there are $\mathcal{N}$-equivalences $t_i\colon X_i' \to X_i$ such that $X_i' \to X_i \to Y'$ is 0, so again by exactness of coproducts, the induced diagram

$$\bigoplus_i X_i' \xrightarrow{(t_i)} \bigoplus_i X_i \xrightarrow{g} Y' \xleftarrow{s} Y$$

is f followed by an isomorphism and an isomorphism followed by 0 in the quotient. Thus $f = 0$. □

Example 2.5.25. Suppose $\mathcal{T}$ is a *compactly generated* triangulated category and $\mathcal{N} \subset \mathcal{T}$ is a localizing subcategory. Then since $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ preserves coproducts, we deduce by Brown representability that q has a right adjoint $q_*\colon \mathcal{T}/\mathcal{N} \to \mathcal{T}$. By Prop. 2.5.17, q_* is fully faithful and $\mathcal{N} = \mathrm{thick}(\mathcal{N})$. Hence $i\colon \mathcal{N} \hookrightarrow \mathcal{T}$ has a right adjoint $i^!$ and we have functorial triangles

$$ii^!A \to A \to q_*q(A) \to ii^!A[1]$$

breaking up each object into its $\mathcal{N}$ part and its $\mathcal{N}^\perp$ part. Note moreover that if $\{E_i\}$ is a compact generating set for $\mathcal{N}$, then we in fact have

$$\mathrm{Loc}(\{E_i\}) = \mathcal{T}/\mathcal{N}.$$

Indeed, using Lemma 2.4.32, we may write any $X \in \mathcal{T}$ as $\mathrm{hocolim}_n K_n$ where $K_0 = 0$, and the transition maps fit into triangles

$$L_n \to K_n \to K_{n+1} \to L_n[1]$$

such that L_n is a coproduct of shifts of the E_i. Apply q to this; since it preserves coproducts, it preserves each successive triangle and the hocolim, and writes $q(L_n)$ as a sum of shifts of $q(E_i)$. Thus $q(L_i) \in \mathrm{Loc}(\{q(E_i)\})$, and since $K_0 = 0$ we inductively have $K_n \in \mathrm{Loc}(\{q(E_i)\})$. Thus $\mathrm{hocolim}\, q(K_n) = q(\mathrm{hocolim}\, K_n) = q(X) \in \mathrm{Loc}(q(\{E_i\}))$.

Localizing subcategories control (weak) generation in the sense defined earlier. In particular, we have the following result:

Lemma 2.5.26. *Let $\mathcal{T}$ be a triangulated category and S a collection of objects of $\mathcal{T}$. If $\mathrm{Loc}(S) = \mathcal{T}$, then S generates $\mathcal{T}$. Conversely if S is a set of compact objects and generates $\mathcal{T}$, then $\mathrm{Loc}(S) = \mathcal{T}$.*

Proof. Let $A \in \mathcal{T}$ and consider the subcategory $\mathcal{E}$ of E for which $\mathrm{Hom}^*(E, A) = 0$. Then this is triangulated, but moreover

$$\mathrm{Hom}(\bigoplus_i E_i, A) = \prod_i \mathrm{Hom}(E_i, A)$$

is zero iff all terms are zero, so $\mathcal{E}$ is closed under coproducts and hence is a localizing subcategory. So, if $\mathrm{Loc}(S) = \mathcal{T}$, then $\mathrm{Hom}^*(E, A) = 0$ for $E \in S$ would imply $\mathrm{Hom}^*(E, A) = 0$ for all $E \in \mathcal{T}$, i.e. $A = 0$ by the Yoneda lemma. We proved the converse as Prop. 2.4.31(a). □

Example 2.5.27. Let $\mathcal{T}$ be a triangulated category with coproducts and $\mathcal{N} \subset \mathcal{T}$ a localizing subcategory generated by objects in $\mathcal{T}_c$. Denote the generators by $\{E_i\}$, then these are compact in $\mathcal{N}$ as well, so every object of $\mathcal{N}$ admits a homotopy filtration by coproducts of shifts of the $\{E_i\}$, by Lemma 2.4.32. By Lemma 2.4.40, all maps from $\mathcal{T}_c$ to $\mathcal{N}$ factor through $\mathrm{thick}_{\mathcal{T}}(\{E_i\})$, which is in particular contained in $\mathcal{T}_c$ and in $\mathcal{N}$ (since $\mathcal{N}$ is thick). Thus every map from $\mathcal{T}_c$ to $\mathcal{N}$ factors through $\mathcal{T}_c \cap \mathcal{N}$, which implies that the induced map

$$\mathcal{T}_c/\mathcal{N} \cap \mathcal{T}_c \to \mathcal{T}/\mathcal{N}$$

is fully faithful, by Remark 2.5.15.

Lemma 2.5.28. *Let $F\colon \mathcal{T} \rightleftarrows \mathcal{T}' : G$ be an exact adjunction between triangulated categories with coproducts.*

(a) If G preserves coproducts, then $F(\mathcal{T}_c) \subset \mathcal{T}'_c$.

(b) If $\mathcal{T}$ is compactly generated and $F(\mathcal{T}_c) \subset \mathcal{T}'_c$, then G preserves coproducts.

Proof. (a) If $E \in \mathcal{T}_c$, then

$$\mathrm{Hom}_{\mathcal{T}'}(F(E), -) \cong \mathrm{Hom}_{\mathcal{T}}(E, G(-))$$

is a composition of functors which preserve coproducts.

(b) Let $\{B_i\} \in \mathcal{T}'$ and consider the map $\bigoplus_i G(B_i) \to G(\bigoplus_i B_i)$. For each $E \in \mathcal{T}_c$, we have

$$\begin{aligned}\mathrm{Hom}^*_{\mathcal{T}}(E, \bigoplus_i G(B_i)) &= \bigoplus_i \mathrm{Hom}^*_{\mathcal{T}}(E, G(B_i)) = \bigoplus_i \mathrm{Hom}^*_{\mathcal{T}}(F(E), B_i) \\ &= \mathrm{Hom}_{\mathcal{T}'}(F(E), \bigoplus_i B_i) = \mathrm{Hom}_{\mathcal{T}}(E, F(\bigoplus_i B_i))\end{aligned}$$

since $F(E) \in \mathcal{T}'_c$. Since $\mathcal{T}$ is compactly generated, isomorphisms are detected by $\mathcal{T}_c$, so the comparison map is an isomorphism. □

Proposition 2.5.29. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ be an exact, coproduct-preserving functor of compactly generated triangulated categories. If F restricts to an equivalence $\mathcal{T}_c \xrightarrow{\sim} \mathcal{T}'_c$, then F is an equivalence.*

PROOF. By Brown representability, choose a right adjoint $G\colon \mathcal{T}' \to \mathcal{T}$. By the lemma above, G preserves coproducts. For $E_1, E \in \mathcal{T}_c$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}(E_1, E) & \xrightarrow{\cong} & \mathrm{Hom}(F(E_1), F(E)) \\ & \searrow & \downarrow \cong \\ & & \mathrm{Hom}(E_1, GFE) \end{array}$$

commutes by adjunction, and the upper map is an isomorphism since $F|_{\mathcal{T}_c}$ is fully faithful. Therefore $\mathrm{Hom}(E_1, -)$ applied to $E \to GF(E)$ is an isomorphism for all compact E_1, so $E \xrightarrow{\sim} GF(E)$ by compact generation. Since this is a natural map of coproduct-preserving functors, it follows that $\mathrm{Id} \xrightarrow{\sim} GF$, meaning F is fully faithful. This means that the essential image is a triangulated subcategory of $\mathcal{T}'$, and it is localizing since F preserves coproducts. It also contains $\mathcal{T}'$, so we conclude that F is essentially surjective by compact generation of $\mathcal{T}'$. □

Example 2.5.30. This is a continuation of Example 2.5.25; suppose $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ is a quotient of a compactly generated category by a localizing subcategory of the form $\mathcal{N} = \mathrm{Loc}(\{K_i\})$ where $\{K_i\} \subset \mathcal{T}_c$, i.e. so that $\mathcal{N}$ is generated by objects compact in $\mathcal{T}$. Continue writing $i^!$ and q_* for the relevant right adjoints.

1. Then $\mathcal{N}^\perp$ is also localizing. Indeed, to check $A \in \mathcal{N}^\perp$ we just check $\mathcal{N} \subset {}^\perp A$. But ${}^\perp A$ is obviously localizing since Hom's send coproducts in the first variable to products. Thus it is equivalent to check $K_i \in {}^\perp A$ for all i, i.e. $A \in K_i^\perp$ for all i. But now K_i is compact, so $K_i^\perp$ is localizing. Therefore in fact
$$\mathcal{N}^\perp = \bigcap_i K_i^\perp$$
is an intersection of localizing subcategories hence localizing.
2. Then $q(\mathcal{T}_c) \subset (\mathcal{T}/\mathcal{N})_c$. Indeed, to see this, we just need to show that q_* preserves coproducts, and q_* is isomorphic to the inclusion $\mathcal{N}^\perp \hookrightarrow \mathcal{T}$, so it suffices to show that $\mathcal{N}^\perp$ is closed under coproducts which was (1).
3. Thus if $\{E_i\}$ is a compact generating set for $\mathcal{T}$, then $\{q(E_i)\}$ is a compact generating set for $\mathcal{T}/\mathcal{N}$ by Example 2.5.25. In particular the quotient is compactly generated.
4. Moreover, the inclusion $i\colon \mathcal{N} \to \mathcal{T}$ also preserves compact objects, i.e. $i^!$ preserves coproducts. Indeed, we have functorial triangles
$$ii^!A \to A \to q_*qA \to ii^!A[1]$$
where $A \mapsto A$ and $A \mapsto q_*qA$ preserve coproducts, hence $A \mapsto ii^!A$ preserves coproducts. Since i preserves coproducts and is fully faithful, it follows that $i^!$ does as well. Thus $\mathcal{N}_c \subset \mathcal{N} \cap \mathcal{T}_c$, and conversely an object of $\mathcal{N}$ which is compact in $\mathcal{T}$ is a fortiori compact in $\mathcal{N}$, thus $\mathcal{N}_c = \mathcal{N} \cap \mathcal{T}_c$.
5. We know by Remark 2.5.19 that $\mathcal{M} = \mathcal{N}^\perp$ is left admissible and that $i^!$ is isomorphic to the quotient map $\mathcal{T} \to \mathcal{T}/\mathcal{M}$. Since $i^!$ preserves coproducts, by Brown representability it has a right adjoint ρ, which means that $\mathcal{M}$ is also right admissible and $\rho\colon \mathcal{T}/\mathcal{M} \cong \mathcal{N} \to \mathcal{T}$ is another fully faithful embedding. Writing $\mathcal{C}$ for its essential image, we know that $\mathcal{C} = \mathcal{M}^\perp$

and ${}^{\perp}\mathcal{C} = \mathcal{M}$, where $\mathcal{C}$ and $\mathcal{N}$ are equivalent categories. This remarkable phenomenon is called a TTF triple.

In exactly the setting of the example above, there is a more interesting thing we can say, called the Neeman localization lemma.

THEOREM 2.5.31. *Let $\mathcal{T}$ be a compactly generated triangulated category, $\mathcal{N}$ a localizing subcategory generated by compact objects of $\mathcal{T}$, and $q\colon \mathcal{T} \to \mathcal{T}/\mathcal{N}$ the quotient. Then $\mathcal{T}_c \to (\mathcal{T}/\mathcal{N})_c$ induces a fully faithful functor*

$$\mathcal{T}_c/\mathcal{N}_c = \mathcal{T}_c/(\mathcal{N} \cap \mathcal{T}_c) \to (\mathcal{T}/\mathcal{N})_c$$

which is an idempotent completion, and in fact for each $Q \in (\mathcal{T}/\mathcal{N})_c$ there exists $P \in \mathcal{T}_c$ such that $q(P) \cong Q \oplus Q[1]$.

PROOF. The fact that this proposition makes sense is explained in Example 2.5.30. The fact that the induced functor is fully faithful is Example 2.5.27, so denote its essential image by $\mathcal{E} \subset (\mathcal{T}/\mathcal{N})_c$, which is of course also the essential image of $\mathcal{T}_c \to (\mathcal{T}/\mathcal{N})_c$. By Example 2.5.30(3), $\mathcal{E}$ contains a compact generating set for $\mathcal{T}/\mathcal{N}$, so by Cor. 2.4.41 we have $\operatorname{thick}(\mathcal{E}) = (\mathcal{T}/\mathcal{N})_c$. This immediately shows the idempotent-completion statement because $(\mathcal{T}/\mathcal{N})_c$ is a thick subcategory of an idempotent complete category, and since $\mathcal{E}$ is triangulated we know all objects of $\operatorname{thick}(\mathcal{E})$ are summands of objects of $\mathcal{E}$.

Therefore let $Q \in (\mathcal{T}/\mathcal{N})_c$ and choose $F \in \mathcal{T}_c$ such that $q(F) \cong Q \oplus Q'$ for some $Q' \in \mathcal{T}/\mathcal{N}$. Consider the two triangles

$$Q \xrightarrow{0} Q \to Q \oplus Q[1] \to Q[1] \qquad Q' \xrightarrow{1} Q' \to 0 \to Q'[1]$$

and take their direct sum, which is isomorphic to

$$q(F) \to q(F) \to Q \oplus Q[1] \to q(F)[1].$$

Since $\mathcal{E}$ is triangulated, $Q \oplus Q[1] \in \mathcal{E}$. □

2.5.3. Perfect generation and approximation. In this section we will prove that if X is a qcqs scheme, then $D_{\mathrm{qc}}(X)$ has a perfect generator, and thus that $D_{\mathrm{qc}}(X)$ is a compactly generated triangulated category. First, let us give an explicit example of generators in some of the cases which are relevant to us.

Proposition 2.5.32. *Let X be a (quasicompact) scheme with an ample line bundle $\mathcal{L}$. Then*

$$\{\mathcal{L}^{\otimes -d} : d \geq 0\}$$

is a generating set for $D_{\mathrm{qc}}(X)$. In particular, $\operatorname{thick}(\mathcal{L}^{\otimes d} : d \leq 0) = \operatorname{Perf}(X)$.

Proof. Since each $\mathcal{L}^{\otimes -d}$ is perfect, we have

$$\operatorname{Hom}_{D_{\mathrm{qc}}(X)}(\mathcal{L}^{\otimes -d}, A[q]) = \mathbb{H}^q(X, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes d})$$

and we just need to show this is nonzero for some q and $d \geq 0$ if $A \neq 0$. Suppose that this expression is zero for all q and $d \geq 0$. Thus $R\Gamma(X, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes d}) = 0$ for $d \geq 0$. Let $p \geq 0$ and $s \in \mathcal{L}^{\otimes p}(X)$ a global section for which X_s is affine. Then

$$R\Gamma(X_s, A|_{X_s}) = \operatorname{hocolim}_d R\Gamma(X, A \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes pd}) = 0,$$

by Example 2.4.29. Since $R\Gamma$ is an equivalence on an affine scheme, $A|_{X_s} = 0$. Since $\mathcal{L}$ is ample, affine opens of the form X_s cover X, so $A = 0$.

Of course $\operatorname{thick}(\mathcal{L}^{\otimes d} : d \leq 0) \subset \operatorname{Perf}(X)$, however $\operatorname{Perf}(X) \subset D_{\mathrm{qc}}(X)_c$, and the result implies that $\operatorname{thick}(\mathcal{L}^{\otimes d}) = D_{\mathrm{qc}}(X)_c$, so all three things are equal. □

Example 2.5.33. Let R be a ring. Then $D(R)_c = \operatorname{Perf}(R) = \operatorname{thick}(R)$. Indeed, $\mathcal{O}_{\operatorname{Spec} R}$ is an ample line bundle on $\operatorname{Spec} R$.

Example 2.5.34. Let R be a ring and $X = \mathbb{P}^n_R$ with its ample line bundle $\mathcal{O}_X(1)$. Then the result shows that $\operatorname{Perf}(X) = \{\mathcal{O}_X, \mathcal{O}_X(-1), \dots\}$. However, in this case, consider the Koszul complex

$$0 \to \mathcal{O}_X(-n-1) \to \mathcal{O}_X(-n)^{\binom{n+1}{1}} \to \cdots \to \mathcal{O}_X(-1)^{\binom{n+1}{1}} \to \mathcal{O}_X \to 0.$$

discussed in the previous section, and define

$$\mathcal{E} \coloneqq \operatorname{thick}\left(\mathcal{O}_X(-n), \mathcal{O}_X(-n+1), \dots, \mathcal{O}_X(-1), \mathcal{O}_X\right) \subset \operatorname{Perf}(X).$$

Then the exact sequence above shows that $\mathcal{O}_X(-n-1) \in \mathcal{E}$ since $\mathcal{E}$ is triangulated. Now twist the sequence by $\mathcal{O}_X(-1)$, then all terms are in $\mathcal{E}$ except for the left most $\mathcal{O}_X(-n-2)$. Thus inductively $\mathcal{O}(-d) \in \mathcal{E}$ for all $d \geq 0$, and hence $\mathcal{E} = \operatorname{Perf}(X)$. Of course, this also proves that these $n+1$ line bundles generate the derived category $D_{\mathrm{qc}}(X)$. Likewise, we can twist by any $d \in \mathbb{Z}$ to conclude that

$$\operatorname{Perf}(X) = \operatorname{thick}(\mathcal{O}_X(-n+d), \dots, \mathcal{O}_X(d))$$

and that $D_{\mathrm{qc}}(X)$ is generated by $\{\mathcal{O}_X(-n+d), \dots, \mathcal{O}_X(d)\}$. We will see later that this gives

$$\operatorname{Perf}(\mathbb{P}^n_R) = \langle \mathcal{O}_{\mathbb{P}^n_R}(-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n} \rangle$$

as a *semiorthogonal decomposition* of $\operatorname{Perf}(X)$.

Now let's prove the main result in three steps, most of which come from our triangulated work in the previous section.

Lemma 2.5.35. *Let $X = \operatorname{Spec} R$ for R a ring, $f_1, \dots, f_r \in R$, and set*

$$U = D(f_1) \cup \cdots \cup D(f_r) \qquad \mathcal{K}^\bullet := \widetilde{\mathrm{Kos}}^\bullet(f_1, \dots, f_r) \in D_{\mathrm{qc}}(X).$$

Then for every $A \in D_{\mathrm{qc}}(X)$, the unit map $A \to Rj_*A|_U$ is an isomorphism if and only if $\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{K}^\bullet, A) = 0$.

Proof. We use throughout the fact that $\mathcal{K}^\bullet$ is strictly perfect and hence behaves well with respect to Homs. First, if the map is an isomorphism, then

$$\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{K}^\bullet, A) = \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{K}^\bullet|_U, A|_U) = 0$$

because $\mathcal{K}^\bullet|_U = 0$ in $D(\mathcal{O}_U)$ by Remark 2.5.3. Conversely, there are two cases:

Case 1: $A|_U = 0$. We show $A = 0$. In this case, note by Koszul duality that

$$\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{K}^\bullet, A) = (\mathcal{K}^\bullet)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} A = \mathcal{K}^\bullet \otimes^{\mathbf{L}} A[-r],$$

and since $R\Gamma(X, -)$ is an equivalence this implies $\mathcal{K}^\bullet \otimes^{\mathbf{L}}_{\mathcal{O}_X} A[-r] = 0$. Thus

$$0 = (\mathcal{O}_X \xrightarrow{f_1} \mathcal{O}_X) \otimes \cdots \otimes (\mathcal{O}_X \xrightarrow{f_r} \mathcal{O}_X) \otimes^{\mathbf{L}} A.$$

The tensor product of the right-most r terms still restricts to 0 on U, so by induction we may show that if $B|_U = 0$ and $(\mathcal{O}_X \xrightarrow{f_i} \mathcal{O}_X) \otimes^{\mathbf{L}} B = 0$, then $B = 0$. But this just says that $B \xrightarrow{f_i} B = 0$, i.e. f_i is an isomorphism on B, contradicting $B|_{D(f_i)} = 0$ unless $B = 0$.

Case 2: $A \in D_{\mathrm{qc}}(X)$ arbitrary. Extend the unit map to a triangle

$$C \to A \to Rj_*A|_U \to C[1]$$

and apply $\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{K}^\bullet, -)$. By the first direction, this kills the $Rj_*A|_U$ term and hence gives an isomorphism

$$\mathrm{RHom}_{\mathcal{O}_X}(\mathcal{K}^\bullet, C) = \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{K}^\bullet, A) = 0,$$

by hypothesis. Since $A \to Rj_*A|_U$ is an isomorphism on U, $C|_U = 0$. Thus by Case 1 we conclude $C = 0$. Hence $A \xrightarrow{\sim} Rj_*A|_U$. □

Corollary 2.5.36. *Let X be a qcqs scheme of the form $X = U \cup V$ for U quasicompact and V affine. For each $F \in \mathrm{Perf}(U)$ there exists $E \in \mathrm{Perf}(X)$ such that $E|_U \cong F \oplus F[1]$.*

Proof. First assume X is itself affine, and consider the Verdier quotient

$$D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X) \,/\, D_Z(X) = D_{\mathrm{qc}}(U)$$

of Example 2.5.21 where $Z = X \setminus U$. Write $U = D(f_1) \cdots \cup D(f_r)$ for some global sections (f_i). Let $\mathcal{K}^\bullet$ be the corresponding Koszul complex as in Lemma 2.5.35. By definition, $D_Z(X)$ is closed under coproducts, so $\mathrm{Loc}(\mathcal{K}^\bullet) \subset D_Z(X)$; on the other hand, Lemma 2.5.35 shows that $\mathcal{K}^\bullet$ generates $D_Z(X)$, and it is compact in $D_Z(X)$ as it was perfect and thus compact in $D_{\mathrm{qc}}(X)$. Hence by Lemma 2.5.26 we deduce $\mathrm{Loc}(\mathcal{K}^\bullet) = D_Z(X)$. Finally, by Example 2.5.33, $\mathcal{O}_X$ is a compact generator for $D_{\mathrm{qc}}(X)$, and its restriction to $D_{\mathrm{qc}}(U)$ is perfect and hence compact. Thus by Thm. 2.5.31, for every compact (i.e. perfect) object F in $D_{\mathrm{qc}}(U)$, $F \oplus F[1]$ is the

restriction of a compact object of $D_{\mathrm{qc}}(X)$. Example 2.5.33 again shows that such an object is perfect on X.

Now if $X = U \cup V$ with V affine and $F \in \mathrm{Perf}(U)$, then $F|_{U\cap V} \in \mathrm{Perf}(U \cap V)$ and $U \cap V \subset V$ is a qc open, so we may choose $E' \in \mathrm{Perf}(V)$ such that $E|_{U\cap V} \cong F \oplus F[1]$. Therefore $F \oplus F[1] \in D_{\mathrm{qc}}(U)$ and $E' \in D_{\mathrm{qc}}(V)$ agree on $U \cap V$, and hence glue to an object $E \in D_{\mathrm{qc}}(X)$ by Prop. 2.4.25. Then E is perfect since this holds locally, and $E|_U = F \oplus F[1]$. □

We will be able to come back and strengthen this result after proving perfect generation, so that we can in fact take U to be any quasicompact open of X. For the next few results, it will be useful to write $\mathrm{Perf}_Z(X) \coloneqq \mathrm{Perf}(X) \cap D_Z(X)$.

THEOREM 2.5.37. *Let X be a qcqs scheme, and $Z \subset X$ a closed subset with quasicompact complement. There exists $E \in \mathrm{Perf}_Z(X)$ such that $\mathrm{Loc}(E) = D_Z(X)$. In particular, every qcqs scheme is compactly generated by a single perfect object.*

PROOF. We prove the claim by induction on the number of affine opens needed to cover X. If X is affine, we're done by Lemma 2.5.35 (as shown in Cor. 2.5.36). In the general case, we can write by induction $X = U \cup V$ where U satisfies the result and V is affine.

Thus there exists $F \in \mathrm{Perf}_{Z\cap U}(U)$ such that $\mathrm{Loc}(F) = D_{Z\cap U}(U)$, i.e. F generates $D_{Z\cap U}(U)$. Since $F|_{U\setminus Z} = 0$, we can glue F and 0 to a perfect object $\widetilde{F} \in \mathrm{Perf}(U \cup X \setminus Z)$ by Prop. 2.4.25. Writing $X = (U \cup X \setminus Z) \cup V$, Cor. 2.5.36 gives $P \in \mathrm{Perf}(X)$ such that $P|_{U\cup X\setminus Z} \cong \widetilde{F} \oplus \widetilde{F}[1]$, so in particular

$$P|_U \cong F \oplus F[1] \qquad \text{and} \qquad P \in \mathrm{Perf}_Z(X).$$

$X \setminus U$ is a closed subset of X contained in V, and thus $Z \cap (X \setminus U)$ is a closed subset of V with quasicompact complement $D(f_1, \dots, f_r)$ for $f_i \in \mathcal{O}_X(V)$. Let $\mathcal{K}^\bullet \in D_{\mathrm{qc}}(V)$ be the complex associated to the Koszul complex on $f_1, \dots, f_r$. We claim that

$$E \coloneqq P \oplus Rj_{V,*}\mathcal{K}^\bullet$$

satisfies the conclusion.

1. $P \in \mathrm{Perf}_Z(X)$ so we have to check $Rj_{V,*}\mathcal{K}^\bullet \in \mathrm{Perf}_Z(X)$. Indeed it is supported on $Z\cap(X\setminus U)$ and hence Z, and it is perfect since its restriction to V is $\mathcal{K}^\bullet$ and its restriction to U is 0 (in the derived category).
2. Let $A \in D_Z(X)$ and assume $\mathrm{RHom}(E, A) = 0$. Observe that since $\mathcal{K}^\bullet$ is supported on $Z \cap (X \setminus U) \subset V$ which is closed in X, the natural map $j_{V,!}\mathcal{K}^\bullet \to Rj_{V,*}\mathcal{K}^\bullet$ is an isomorphism (Example 2.4.15). Thus
$$0 = \mathrm{RHom}_{\mathcal{O}_X}(j_{V,!}\mathcal{K}^\bullet, A) = \mathrm{RHom}_{\mathcal{O}_V}(\mathcal{K}^\bullet, A|_V),$$
which means that the natural map
$$A|_V \to Rj_{V\cap(U\cup X\setminus Z),*}A|_{V\cap(U\cup X\setminus Z)}$$
is an isomorphism by Lemma 2.5.35. Thus $A \to Rj_{U\cup X\setminus Z,*}A|_{U\cup X\setminus Z}$ is an isomorphism since this holds on the open cover $\{U \cup X \setminus Z, V\}$. Writing $j' \colon U \hookrightarrow U \cup (X \setminus Z)$, note that $A|_{U\cup X\setminus Z} \to Rj'_*A|_U$ is an isomorphism

since both sides are zero upon restriction to $X \setminus Z$ (as $\operatorname{Supp} A \subset Z$). By composition, $A \to Rj_{U,*}A$ is an isomorphism.

3. Therefore $0 = \operatorname{RHom}_{\mathcal{O}_X}(E, Rj_{U,*}A|_U) = \operatorname{RHom}_{\mathcal{O}_U}(E|_U, A|_U)$, and F is a summand of $E|_U$ and $A|_U \in D_{Z\cap U}(U)$. Since F generates $D_{Z\cap U}(U)$, this implies $A|_U = 0$, hence $A \cong Rj_{U,*}A|_U = 0$.

Finally, E is compact in $D_{\mathrm{qc}}(X)$ hence a fortiori in $D_Z(X)$, and $D_Z(X)$ is localizing. Thus to prove $\operatorname{Loc}(E) = D_Z(X)$ it suffices to show E is a generator (Lemma 2.5.26), which we just did. □

Corollary 2.5.38. *Let X be a qcqs scheme and $Z \subset X$ a closed subset with quasicompact complement.*

(a) $D_Z(X)_c = \operatorname{Perf}_Z(X)$, and the functor

$$\operatorname{Perf}(X)/\operatorname{Perf}_Z(X) \to \operatorname{Perf}(X \setminus Z)$$

is an idempotent completion.

(b) Let $U \subset X$ be a quasicompact open. Then for each $F \in \operatorname{Perf}_{Z\cap U}(U)$, there exists $E \in \operatorname{Perf}_Z(X)$ such that $E|_U \cong F \oplus F[1]$.

(c) The functor $i_!\colon D_{\mathrm{qc}}(X) \to D_Z(X)$ has a fully faithful right adjoint $\rho\colon D_Z(X) \to D_{\mathrm{qc}}(X)$, and we have the relations

$$^{\perp}\operatorname{im}(Rj_*) = D_Z(X) \qquad \operatorname{im}(\rho)^{\perp} = \operatorname{im}(Rj_*)$$
$$D_Z(X)^{\perp} = \operatorname{im}(Rj_*) = {}^{\perp}\operatorname{im}(\rho)$$

where j is the inclusion $X \setminus Z \hookrightarrow X$. In particular, $D_{\mathrm{qc}}(X \setminus Z)$ embeds as an admissible subcategory of $D_{\mathrm{qc}}(X)$ via Rj_*.

Proof. (a) If X is qcqs, then $\operatorname{Perf}(X) \subset D_{\mathrm{qc}}(X)_c$, and conversely $D_{\mathrm{qc}}(X)_c = \operatorname{thick}(E)$ for E a perfect generator of $D_{\mathrm{qc}}(X)$, hence $D_{\mathrm{qc}}(X)_c = \operatorname{Perf}(X)$. By Thm. 2.5.37, $\mathcal{N} = D_Z(X) \subset D_{\mathrm{qc}}(X)$ satisfies the hypotheses of Example 2.5.30 and Thm. 2.5.31. Thus $D_Z(X)_c = \operatorname{Perf}_Z(X)$, and since $D_{\mathrm{qc}}(X)/D_Z(X) = D_{\mathrm{qc}}(X \setminus U)$ the second statement follows as well.

(b) First if $Z = X$, then this result is known to us from the same result above (applied to $Z = X \setminus U$). In the general case, take $F \in \operatorname{Perf}_{Z\cap U}(U)$, then since $F|_{U\setminus Z} = 0$, F and 0 glue together to extend to $\widetilde{F} \in \operatorname{Perf}(U \cup X \setminus Z)$. Then $\widetilde{F} \oplus \widetilde{F}[1]$ extends to a perfect object of X, which is necessarily supported on Z, and whose restriction to U is $F \oplus F[1]$.

(c) This remark follows from Example 2.5.30(5). □

There is also a morphism-lifting statement similar to (b) which is easy to deduce from this result, and which will be used shortly.

Corollary 2.5.39. *Let X be a qcqs scheme, $U \subset X$ a quasicompact open, and $Z \subset X$ a closed subset. Given $A \in D_{\mathrm{qc}}(X)$ and $F \in \operatorname{Perf}_Z(X)$, for each map $\alpha\colon F|_U \to A|_U$ in $D_{\mathrm{qc}}(U)$ there exists $E \in \operatorname{Perf}_Z(X)$ and maps $A \leftarrow E \to F$ such*

that the diagram

$$\begin{array}{ccc} E|_U & \xrightarrow{\cong} & F|_U \\ & \searrow & \downarrow{\scriptstyle \alpha} \\ & & A|_U \end{array}$$

commutes and the upper restriction is an isomorphism.

Proof. α corresponds to $\alpha^\sharp\colon F \to Rj_*A|_U$, and consider a triangle $A \to Rj_*A|_U \to C \to A[1]$ and the composite map $F \to C$, so C is supported in $X \setminus U$. We know from Example 2.5.27 and the results above that this composite factors through an object $K \in \mathrm{Perf}_{X\setminus U}(X)$. Writing the first map as $\mathcal{O}_X \otimes^{\mathbf{L}}_{\mathcal{O}_X} F \to K$, just as in any adjunction it factors through the counit

$$F \to \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(F, K) \otimes^{\mathbf{L}}_{\mathcal{O}_X} F \cong F^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} F \to K,$$

so the middle term $L \coloneqq F^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \otimes^{\mathbf{L}}_{\mathcal{O}_X} F$ is perfect, supported in $Z \cap (X \setminus U)$, and $F \to C$ factors through L. Thus we can form a morphism of triangles

$$\begin{array}{ccccccc} E & \longrightarrow & F & \longrightarrow & L & \longrightarrow & E[1] \\ \vdots & & \downarrow & & \downarrow & & \vdots \\ A & \longrightarrow & Rj_*A|_U & \longrightarrow & C & \longrightarrow & A[1] \end{array}$$

and get an extension $E \to A$. Here, $F, L \in \mathrm{Perf}_Z(X)$ so $E \in \mathrm{Perf}_Z(X)$, and since $L|_U = 0$ we have $E|_U \xrightarrow{\sim} F|_U$. Finally since the unit $A \to Rj_*A|_U$ restricts to the identity on U, the desired diagram commutes. □

In the rest of this section we will prove the so-called perfect approximation theorem for qcqs schemes, which is a powerful result allowing us to approximate pseudocoherent objects of $D_{\mathrm{qc}}(X)$ *globally* by a perfect complex above any fixed degree (analogous to the definition). This is newly important given our perfect generation theorem, because we will be able to use triangulated category theory to manipulate perfect objects, and then we may want to discuss consequences for pseudocoherence. Alternatively, we will give another more elementary (and stronger) way of doing this in the next section if e.g. the scheme admits an ample line bundle.

A motivation for doing this is to characterize when a morphism is quasi-perfect and quasi-proper. Here is a simple example of this.

Lemma 2.5.40. *Let $f\colon X \to Y$ be a qcqs morphism of schemes. If f is quasi-proper and has locally finite tor dimension, then f is quasi-perfect.*

Proof. The claim is local on Y so we may assume Y is quasicompact and hence Rf_* is bounded. Let $E \in \mathrm{Perf}(X)$, then we have to show Rf_*E has finite tor dimension, since it is already pseudocoherent. Indeed, for $\mathcal{F} \in \mathrm{QCoh}(Y)$,

$$Rf_*(E) \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F} \cong Rf_*(E \otimes^{\mathbf{L}} Lf^*\mathcal{F})$$

by the projection formula; f having finite tor dimension means that Lf^* is bounded, hence $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*\mathcal{F}$ is bounded, hence after applying Rf_* it remains bounded. □

The converse of this fact is indeed true, and we will be interested in proving the result, which will rely on the perfect approximation theorem. One step will require a generalization of the perfect generation theorem, so we will prove this now.

THEOREM 2.5.41. *Let X be a qcqs scheme and $Z \subset X$ a closed subset with quasicompact complement. There exists $r \geq 0$ and $E \in \mathrm{Perf}_Z(X)$ with the property that*

$$\mathrm{RHom}_{\mathcal{O}_X}(E, A) \in D^{<m} \implies A \in D^{<m+r}$$

for every $A \in D_Z(X)$ and $m \in \mathbb{Z}$.

PROOF. In this proof we will denote the property above by saying that E satisfies the property (Q_r) with respect to $Z \subset X$.

Case 0: If X is affine, and $Z = V(f_1, \ldots, f_r)$, then the Koszul complex

$$K = \widetilde{\mathrm{Kos}}^\bullet(f_1, \ldots, f_r) \in \mathrm{Perf}_Z(X)$$

satisfies (Q_0). Indeed, we showed in Lemma 2.5.9 that if $H^m(A) \neq 0$, then $\mathrm{Hom}_{D_{\mathrm{qc}}(X)}(K[-m], A) \neq 0$, i.e. $\mathrm{RHom}_{\mathcal{O}_X}(K, A)$ has nonzero mth cohomology, which clearly implies (Q_0).

Case 1: Now induct on the number of affine opens required to cover X. We just proved the base case, so we may write $X = U \cup V$ where V is affine, U is quasicompact, and the inclusion $Z \cap U \subset U$ admits a perfect object $F \in \mathrm{Perf}_{Z\cap U}(U)$ satisfying (Q_r). By Cor. 2.5.38(b) we may choose $P \in \mathrm{Perf}_Z(X)$ such that $P|_U \cong F \oplus F[1]$. On the other hand, let K be the Koszul complex of Case 0 for the inclusion $Z \cap (X \setminus U) \subset V$ to the affine scheme V, and set

$$E \coloneqq P \oplus Rj_{V,*}K.$$

As before, $E \in \mathrm{Perf}_Z(X)$ and $Rj_{V,*}K = j_{V,!}K$. Choose e such that $P^\vee$ has tor amplitude in $(-\infty, e]$, and let d be the right QCoh-amplitude of Rf_*. We claim that E satisfies (Q_{r+e+d}) with respect to $Z \subset X$.

Indeed, suppose $\mathrm{RHom}(E, A) \in D^{<m}$. First take a localization triangle $C \to A \to Rj_{U,*}A|_U \to C[1]$. Upon restricting to U the second map is an isomorphism, and upon restricting to $X \setminus Z$ both A and $Rj_{U,*}A|_U$ become 0, which shows that C is supported $Z \cap (X \setminus U)$. Note that

$$\mathrm{RHom}(Rj_{V,*}K, Rj_{U,*}A|_U) = \mathrm{RHom}_{\mathcal{O}_U}(Rj_{U\cap V,*}K|_{U\cap V}, A|_U) = 0$$

since $K|_{U\cap V} = 0$ as it is supported in $X \setminus U$. Thus

$$\mathrm{RHom}_{\mathcal{O}_V}(K, C|_V) = \mathrm{RHom}_{\mathcal{O}_X}(j_{V,!}K, C) \xrightarrow{\sim} \mathrm{RHom}(j_{V,!}K, A) \in D^{<m}$$

so by Case 0 we have $C|_V \in D^{<m}$, and thus $C \in D^{<m}$ as C is already supported on an X-closed set contained in V.

On the other hand, applying $\mathrm{RHom}(P, -)$ to the triangle above gives

$$P^\vee \otimes^{\mathbf{L}} C \to \mathrm{RHom}(P, A) \to \mathrm{RHom}_{\mathcal{O}_U}(P|_U, A|_U) \to,$$

where the assumption on the tor amplitude of $P^\vee$ and $C \in D^{<m}$ implies $P^\vee \otimes^{\mathbf{L}} C \in D^{<m+e}$. Since the middle term is by assumption in $D^{<m} \subset D^{<m+e}$, we conclude the third term is in $D^{<m+e}$. Thus $\mathrm{RHom}_{\mathcal{O}_U}(F, A|_U) \in D^{<m+e}$, hence

$A|_U \in D^{<m+e+r}$. Therefore $Rj_{U,*}A|_U \in D^{<m+e+r+d}$. Since we already have $C \in D^{<m}$, this proves $A \in D^{<m+e+r+d}$ completing the proof. □

Note that this trivially implies perfect generation, which is the case that $\mathrm{RHom}(E, A) \in D^{<m}$ for every m, so $A \in D^{<m+r}$ for every m and thus $A = 0$. Thus we will call this the quantitative perfect generation theorem.

Now we move on to perfect approximation. The elements of the proof will be similar to the last few proofs. The idea is that, given an m-pseudocoherent object A, we can construct an $(m + r)$-isomorphism $E \to A$ (meaning its cone lies in $D^{<m+r}$) with E perfect, where r is some constant only depending on the ambient space. We will again prove a more general result depending on support. We begin with the affine case:

Lemma 2.5.42. *Let R be a ring and $I \subset R$ an ideal generated by r elements. Suppose $A \in D(R)$ is m-pseudocoherent and $\operatorname{Supp} H^q(A) \subset V(I)$ for $q > m$. Then there exists $E \in \mathrm{Perf}_{V(I)}(R)$ and an exact triangle*

$$E \to A \to C \to E[1]$$

with $C \in D^{<m+r}(R)$.

Proof. We proceed by (ascending) induction on the top cohomology degree t of A. If $t < m + r$, then the triangle $0 \to A \to A \to 0$ satisfies the conclusion, so assume $t \geq m+r$. Since A is m-pseudocoherent, this implies by Example 2.2.44 that $H^t(A)$ is finitely generated, say by elements $x_1, \ldots, x_n$. Viewing the generators as maps $R \to A[t]$ and writing $I = (f_1, \ldots, f_r)$, observe that $A[t] \in D^{\leq 0}$ and $H^q(A[t]) = H^{q+t}(A)$ is supported on $V(I)$ for $q > m - t \geq -r$. Therefore by Lemma 2.5.6, each map $R \to A[t]$ corresponding to x_i factors as

$$R \to \mathrm{Kos}^\bullet_R(f_1^{n_1}, \ldots, f_r^{n_r}) \to A[t].$$

The middle term is supported on $V(f_1^{n_1}, \ldots, f_1^{n_r}) = I$ and concentrated in $[-r, 0]$. Thus, we may find complexes of finite free modules $F_i^\bullet$ supported on $V(I)$ and concentrated in $[-r + t, t]$, with maps $F_i^\bullet \to A$ such that x_i is in the image of the induced map on H^t.

Therefore the direct sum $F^\bullet = \bigoplus_i F^\bullet$ satisfies the same properties and induces a map $F^\bullet \to A$ which is surjective on H^t. Taking the cone $F^\bullet \to A \to A_1 \to F^\bullet[1]$, it follows that $A_1 \in D^{<t}$. Since $F^\bullet$ is perfect and A is m-pseudocoherent, A_1 is m-pseudocoherent, and since $F^\bullet$ is supported in $V(I)$ we find that $\widetilde{A}|_{D(I)} \xrightarrow{\sim} \widetilde{A}|_{D(I)}$, so A_1 satisfies the same support condition as A. By induction, choose a triangle $P \to A_1 \to C \to P[1]$ with $P \in \mathrm{Perf}_{V(I)}(R)$ and $C \in D^{<m+r}(R)$. Applying the octahedral axiom to the composite of the two middle maps gives a triangle

$$E \to A \to C \to E[1]$$

where E fits into $F^\bullet \to E \to P \to F^\bullet[1]$. The latter implies that $E \in \mathrm{Perf}_{V(I)}(R)$, so the former completes the proof. □

Theorem 2.5.43. *Let X be a qcqs scheme and $Z \subset X$ a closed set with quasicompact complement. There exists a constant $r \geq 0$ (depending only on X*

and Z) such that whenever $A \in D_{\mathrm{qc}}(X)$ is m-pseudocoherent and $\operatorname{Supp} H^q(A) \subset Z$ *for $q > m$, there exists $E \in \operatorname{Perf}_Z(X)$ and an exact triangle*

$$E \to A \to C \to E[1]$$

where $C \in D_{\mathrm{qc}}^{<m+r}(X)$.

Proof. We induct on the number of affine opens covering X. If X is affine, we are done by Lemma 2.5.42, so in general we may write $X = U \cup V$ where U is a quasicompact open satisfying the conclusion of the theorem for all closed subsets, and V is affine. Let r_U be the corresponding number for the inclusion $Z \cap U \subset U$, and r_V be for $Z \cap (X \setminus U) \subset V$. We claim that $r_X := r_U + r_V$ works for $Z \subset X$.

We prove this by induction on $t = \max\{q : H^q(A)|_U \neq 0\}$. First suppose that $t \leq m + r_U$. This means that for $q > m + r_U$, $H^q(A)$ is supported in $Z \cap (X \setminus U)$, and $A|_V$ is a fortiori $(m + r_U)$-pseudocoherent, so the affine case provides $P \in \operatorname{Perf}_{Z\cap(X\setminus U)}(V)$ and $P \to A|_V \to C' \to P[1]$ with $C' \in D_{\mathrm{qc}}^{<m+r_X}(V)$. Since $Z \cap (X \setminus U)$ is X-closed and contained in V, we know from Example 2.4.15 that $Rj_{V,*}P = j_{V,!}P$ so adjunction provides a map $j_{V,!}P \to A$ which we can extend to a triangle

$$Rj_{V,*}P \to A \to C \to Rj_{V,*}[1].$$

Restricting to V, this becomes the previous triangle, and restricting to U the first term vanishes. Thus $Rj_{V,*}P \in \operatorname{Perf}(X)$, and $(Rj_{V,*}P)|_{X\setminus Z} = 0$ since P is supported on $V \setminus Z$. On the other hand this means $C|_V \in D^{<m+r_X}$ and $C|_U \cong A|_U \in D^{<t} \subset D^{<m+r_X}$ since $t \leq m + r_U$. That finishes this case.

Now assume $t > m + r_U$, which is the interesting case. Then $t - r_U > m$ so $A|_U$ is $(t - r_U)$-pseudocoherent with cohomology sheaves supported in $Z \cap U$ above degree $t - r_U$. Therefore we may choose a triangle

$$F \to A|_U \to C_1 \to F[1]$$

where $F \in \operatorname{Perf}_{Z\cap U}(U)$ and $C_1 \in D_{\mathrm{qc}}^{<t}(U)$. Adding the triangle $F[1] \to 0 \to F[2] \to F[2]$ produces

$$F \oplus F[1] \to A|_U \to C_1 \oplus F[2] \to F[1] \oplus F[2].$$

The triangle $A|_U[1] \to C_1[1] \to F[2] \to A[2]$ and the fact that $A|_U[1], C_1[1] \in D^{<t}$ imply that $F[2] \in D^{<t}$, thus $C_1 \oplus F[2] \in D^{<t}$. By Cor. 2.5.38(b), there exists $P \in \operatorname{Perf}_Z(X)$ such that $P|_U \cong F \oplus F[1]$. By Cor. 2.5.39, we may modify P to a new object in $\operatorname{Perf}_Z(X)$ which still satisfies $P|_U \cong F \oplus F[1]$, but now the first map in the triangle above is the restriction of a map $P \to A$.

Extend this to a triangle

$$P \to A \to C_2 \to P[1],$$

then $C_2|_U \cong C_1 \oplus F[2] \in D^{<t}$, C_2 is still m-pseudocoherent since P is perfect, and $P|_{X\setminus Z} = 0$ implies that C_2 satisfies the same support condition as A. By induction, we may find a triangle

$$Q \to C_2 \to C \to Q[1]$$

where $Q \in \mathrm{Perf}_Z(X)$ and $C \in D^{<m+r_X}$. By the octahedral axiom applied to the composite $A \to C_2 \to C$, there is a triangle

$$E \to A \to C \to E[1]$$

where E fits into a triangle $P \to E \to Q \to P[1]$. The latter shows that $E \in \mathrm{Perf}_Z(X)$, so the former completes the proof. □

Remark 2.5.44. As we have alluded to, one downside of the construction of $f^\times$ is that it is not local on the target. However we can say something: if $f\colon X \to Y$ is a morphism of qcqs schemes, and $U \subset Y$ is a quasicompact open, consider $f_U\colon f^{-1}(U) \to U$. Then we know that

$$j_U^* \circ Rf_* = Rf_{U,*} \circ j_{f^{-1}(U)}^*,$$

so by taking right adjoints we observe $f^\times \circ Rj_{U,*} = Rj_{f^{-1}(U),*} \circ f_U^\times$. Note in particular that applying $j_{f^{-1}(U)}^*$ to both sides gives

$$f_U^\times = j_{f^{-1}(U)}^* f^\times Rj_{U,*}.$$

This is also a special case of tor-independent base change, which we will prove later.

Let us begin to prove some results in addition to what we said in Lemma 2.4.56.

Proposition 2.5.45. *Let $f\colon X \to Y$ be a morphism of qcqs schemes. The following are equivalent:*

(a) f is quasi-perfect.
(b) For each quasicompact open $U \subset Y$, $f^{-1}(U) \to U$ is quasi-perfect.
(c) For $\{U_i\}$ a quasicompact open cover of Y, $f^{-1}(U_i) \to U_i$ is quasi-perfect for each i.

Proof. Of course $(b) \Rightarrow (c)$, and it is clear from the definitions that $(c) \Rightarrow (a)$ because perfectness is checked locally and Rf_* is computed target-locally. Assuming f is quasi-perfect, and keeping the notation above we have $f^\times Rj_{U,*} = Rj_{f^{-1}U,*} f_U^\times$. The LHS preserves coproducts, and $Rj_{f^{-1}U,*}$ is fully faithful and preserves coproducts, hence creates coproducts, so $f_U^\times$ preserves coproducts and we conclude by Lemma 2.4.56. □

With a bit more difficulty we can show something similar for quasiproperness.

Proposition 2.5.46. *Let $f\colon X \to Y$ be a morphism of qcqs schemes. The following are equivalent:*

(a) f is quasi-proper.
(b) Rf_*E is pseudocoherent for every $E \in \mathrm{Perf}(X)$.
(c) Rf_*E_i is pseudocoherent for all $\{E_i\}$ in a perfect generating set of $D_{\mathrm{qc}}(X)$.
(d) For each quasicompact open $U \subset Y$, $f^{-1}(U) \to U$ is quasi-proper.
(e) For $\{U_i\}$ a quasicompact open cover of Y, $f^{-1}(U_i) \to U_i$ is quasi-proper for each i.

PROOF. The obvious implications are $(a) \Rightarrow (b) \Rightarrow (c)$ and $(d) \Rightarrow (e) \Rightarrow (a)$. Assume (c), and observe that

$$\{A \in D_{\mathrm{qc}}(X) : Rf_*(A) \text{ is pseudocoherent}\}$$

is a thick subcategory of $D_{\mathrm{qc}}(X)$ since the pseudocoherent objects span a thick subcategory of $D_{\mathrm{qc}}(Y)$. Since $\mathrm{thick}(\{E_i\}) = \mathrm{Perf}(X)$ by perfect generation and Cor. 2.4.41, it follows that Rf_*E is pseudocoherent for all $E \in \mathrm{Perf}(X)$, proving (b).

Now assume (b) and let $A \in D_{\mathrm{qc}}(X)$ be m-pseudocoherent. By perfect approximation (Thm. 2.5.43), choose r and a triangle

$$E \to A \to C \to E[1]$$

with $E \in \mathrm{Perf}(X)$ and $C \in D_{\mathrm{qc}}^{<m+r}(X)$. Let d be the right-QCoh amplitude of Rf_*, then the triangle

$$Rf_*E \to Rf_*A \to Rf_*C \to Rf_*E[1]$$

in $D_{\mathrm{qc}}(Y)$ shows that $Rf_*C \in D^{<m+r+d}$, hence it is trivially $(m+r+d)$-pseudocoherent. Since Rf_*E is pseudocoherent, this implies Rf_*A is $(m+r+d)$-pseudocoherent. Finally, if A was in fact m-pseudocoherent, then this implies Rf_*A is $(m+r+d)$-pseudocoherent for all m, thus pseudocoherent. This proves (a) and shows $(a), (b), (c)$ are equivalent.

It remains to prove $(c) \Rightarrow (d)$. If $E \in \mathrm{Perf}(f^{-1}(U))$, then choose $P \in \mathrm{Perf}(X)$ such that $P|_U \cong E \oplus E[1]$. By assumption Rf_*P is pseudocoherent, so it follows that $Rf_{U,*}E \oplus Rf_{U,*}E[1]$ is pseudocoherent, and since pseudocoherent objects form a thick subcategory we deduce $Rf_{U,*}E$ is pseudocoherent. This implies the condition of (b) for U, so $f_U \colon f^{-1}(U) \to U$ is pseudocoherent. □

We note that the criterion of (b),(c), and even the locality of (d), rely in an essential way on the perfect approximation theorem. For instance it is a trivial consequence of this theorem that quasi-perfectness implies quasi-properness, but this is not at all clear from the definition.

Example 2.5.47. Let $f \colon X \to Y$ be an affine morphism of schemes. Then f is quasi-proper (resp. quasi-perfect) iff $f_*\mathcal{O}_X$ is pseudocoherent (resp. perfect). Indeed, one direction is trivial, and on the other hand these can be checked locally on the target so we may assume that X and Y are affine (and thus qcqs). Then $\mathcal{O}_X$ is a generator of $D_{\mathrm{qc}}(X)$, so $Rf_*\mathcal{O}_X = f_*\mathcal{O}_X$ being pseudocoherent (resp. perfect) implies the result by Prop. 2.5.46 and Lemma 2.4.56.

Our last big result of this section will be the following theorem of Lipman-Neeman:

THEOREM 2.5.48. *Let $f \colon X \to Y$ be a morphism of qcqs schemes. The following are equivalent:*

(a) f is quasi-perfect.
(b) f is quasi-proper and $f^\times$ is bounded.
(c) f is quasi-proper and has finite tor dimension.

First let us recap what we already know. We proved that $(c) \Rightarrow (a)$, and we just said that (a) implies f is quasi-proper. Let us first show the easier remaining claim, which is that if f is quasi-perfect then $f^\times$ is bounded. We have already shown that it has left finite QCoh-dimension (at most the dimension of Rf_*). On the other hand, let $E \in \mathrm{Perf}(X)$ be a *quantitative* perfect generator as in Thm. 2.5.41 for some $r \geq 0$. Since Rf_*E is perfect by assumption, let $(Rf_*E)^\vee$ have tor amplitude in $(-\infty, b]$. Assume $R\Gamma(Y, -)$ has right QCoh-dimension d.

Now, given $A \in D^{<a}$, this implies that

$$(Rf_*E)^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_Y} A = \mathbf{R}\mathcal{H}om_{\mathcal{O}_Y}(Rf_*E, A) \in D^{<a+b},$$

and hence by applying $R\Gamma(Y, -)$ we conclude

$$\mathrm{RHom}_{\mathcal{O}_Y}(Rf_*E, A) \cong \mathrm{RHom}_{\mathcal{O}_X}(E, f^\times A) \in D^{<a+b+d}.$$

By quantitative perfect generation, this implies $f^\times A \in D^{<a+b+d+r}$, hence $f^\times$ has right dimension at most $b + d + r$ as desired. The interesting part is the final remaining assertion.

Proposition 2.5.49. *Let $f\colon X \to Y$ be a morphism of qcqs schemes. If $f^\times$ is bounded, then f has finite tor dimension.*

To prove this, we will prove another technical lemma whose statement has a similar flavor to the quantitative statements we have been proving so far. The lemma will allow us to relate the question of boundedness of $f^\times$ to Ext on an affine scheme, and hence tor dimension.

Lemma 2.5.50. *Let X be a qcqs scheme. There exists $c \geq 0$ such that for every quasicompact open immersion $j\colon U \hookrightarrow X$, we have*

$$A \in D^{\leq m} \implies \mathrm{RHom}_{\mathcal{O}_X}(Rj_*\mathcal{O}_U, A) \in D^{\leq m+c}$$

for all $A \in D_{\mathrm{qc}}(X)$.

Proof. Let $X = V_1 \cup \cdots \cup V_n$ be an affine open cover of X, and as usual write

$$V_I = \bigcap_{i \in I} V_i \qquad I \subset \{1, \ldots, n\}.$$

Let c exceed the right QCoh-dimension of $R\Gamma(V_I, -)$ for all such I, as each V_I is qcqs. For fixed i, consider $(Rj_*\mathcal{O}_U)|V_i$, which corresponds to $R\Gamma(V_i, Rj_*\mathcal{O}_U) = R\Gamma(U \cap V_i, \mathcal{O}_{U \cap V_i})$. $U \cap V_i$ is a qcss scheme, so by choosing a cover $U \cap V_i = \bigcup_\alpha D(f_\alpha)$ for $f_\alpha \in \Gamma(V_i, \mathcal{O}_{V_i})$, this object is represented by the Čech complex $\check{C}^\bullet(\{D(f_\alpha)\}, \mathcal{O}_{U \cap V_i})$. If $V_i = \mathrm{Spec}\, R$, then

$$\check{C}^p(\{D(f_\alpha)\}, \mathcal{O}_{U \cap V_i}) = \bigoplus_{\alpha_0 < \cdots < \alpha_p} R_{f_{\alpha_0} \cdots f_{\alpha_p}}.$$

We now use the following trick; we have seen from Example 2.4.23 that for any $g \in R$ we have an exact sequence

$$0 \to \bigoplus_{k \geq 0} Re_k \to \bigoplus_{k \geq 0} Re_k \to R_g \to 0$$

where the maps are given by $e_k \mapsto e_k - ge_{k+1}$ and $e_k \mapsto g^{-k}$. We claim that each localized term of the Čech complex above can be replaced by a two-term free resolution assembling into a bicomplex. Once we do this, it will be clear from the bicomplex spectral sequence, taking vertical cohomology first, that the natural map from the bicomplex to the Čech complex is a quasi-isomorphism. This in particular gives $R\Gamma(U \cap V_i, \mathcal{O}_{U\cap V_i})$ a finite resolution by countably generated free modules concentrated in degree $[-1, \ell - 1]$, where ℓ is the length of the cover.

To prove the compatibility, we must show that the Čech differentials lift to the two copies of $\bigoplus_k Re_k$. Indeed, we first can lift a restriction map $R_g \to R_{gh}$ to the free resolution as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{k\geq 0} Re_k & \longrightarrow & \bigoplus_{k\geq 0} Re_k & \longrightarrow & R_g & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{k\geq 0} Re_k & \longrightarrow & \bigoplus_{k\geq 0} Re_k & \longrightarrow & R_{gh} & \longrightarrow & 0 \end{array}$$

Here, if we let the two vertical maps be given by $e_k \mapsto h^k e_k$, then both diagrams commute:

$$e_k \mapsto e_k - ge_{k+1} \mapsto h^k e_k - gh^{k+1}e_{k+1} \qquad e_k \mapsto h^k e_k \mapsto h^k e_k - gh^{k+1}e_{k+1}$$

and

$$e_k \mapsto g^{-k} \qquad e_k \mapsto h^k e_k \mapsto h^k (gh)^{-k} = g^{-k}.$$

Now each component of the Čech differentials is a linear combination of projections composed with restriction maps, so we can lift these too. To summarize, we have represented $R\Gamma(U \cap V_i, \mathcal{O}_{U\cap V_i})$ by a finite complex of free modules in degrees ≥ -1.

Next, apply $(\widetilde{-})$ to obtain a representative for $Rj_*\mathcal{O}_U|_{V_i}$, i.e. by a complex in degrees ≥ -1 with terms $\bigoplus \mathcal{O}_{V_i}$; restricting to V_I gives the same expression $C_I^\bullet$ with $\mathcal{O}_{V_I}$. We wish to estimate the functor $\mathrm{RHom}(Rj_*\mathcal{O}_U|_{V_I}, -)$. If $A \in D^{\leq m}$, then

$$\begin{aligned} \mathrm{RHom}\,\mathcal{O}_{V_I}(\bigoplus \mathcal{O}_{V_I}[-q], A) &= \prod \mathrm{RHom}_{\mathcal{O}_{V_I}}(\mathcal{O}_{V_I}, A[q]) \\ &= \prod R\Gamma(V_I, A)[q] \in D^{\leq m+c-q}. \end{aligned}$$

Then we can use the stupid filtration triangles

$$\sigma_{\leq q-1}C_I^\bullet \to \sigma_{\leq q}C_I^\bullet \to \bigoplus \mathcal{O}_{V_I}[-q] \to \sigma_{\leq q-1}C_I^\bullet$$

and the computation above to conclude $\mathrm{RHom}(Rj_*\mathcal{O}_U|_{V_I}, -)$ has right-QCoh dimension at most $c+1$, since the nonzero terms C_I^q have $q \geq -1$. At this stage, we are in good shape to apply the following general 'induction' principle. We note that it is trivial in the bounded case by the Čech spectral sequence, but this handles the unbounded case.

Lemma 2.5.51. *Let $X = V_1 \cup \cdots \cup V_n$ be a quasicompact open cover of a qcqs scheme, and $F\colon D_{\mathrm{qc}}(X) \to D(\mathcal{A})$ an exact functor for $\mathcal{A}$ an abelian category. For $I \subset \{1, \ldots, n\}$ if $F \circ Rj_{V_I,*}\colon D_{\mathrm{qc}}(V_I) \to D(\mathcal{A})$ has right QCoh-dimension $\leq c$ for all I, then F has right QCoh-dimension $\leq c + n - 1$.*

In our case, we apply this to $\mathrm{RHom}(Rj_*\mathcal{O}_U, -)$, where by adjunction its composition with $Rj_{V_I,*}$ equals $\mathrm{RHom}(Rj_*\mathcal{O}_U|_{V_I}, -)$. Thus by the lemma $\mathrm{RHom}(Rj_*\mathcal{O}_U, -)$ has right QCoh-dimension $\leq c+1+n-1 = c+n$, which is independent of U, so this completes the proof. □

Proof (of Lemma 2.5.51). We prove this by induction on n, where the $n = 1$ case is tautological. In the general case, write $X = W \cup V_n$ where $W = V_1 \cup \cdots \cup V_{n-1}$. Both W and the intersection $W \cap V_n = \bigcup_{i=1}^{n-1} V_i \cap V_n$ have quasicompact covers of length $n-1$ whose successive intersections are all successive intersections of the original cover $\{V_i\}$. Thus the inductive hypothesis applies to

$$F \circ Rj_{W,*} \qquad \text{and} \qquad F \circ Rj_{W\cap V_n,*}$$

both with constant c. Thus both functors have right QCoh-dimension $\leq c+n-2$. Now consider the Mayer-Vietoris triangle

$$A \to Rj_{W,*}A|_W \oplus Rj_{V_n,*}A|_{V_n} \to Rj_{W\cap V_n,*}A|_{W\cap V_n} \to A[1]$$

of Remark 2.4.26, and apply F, assuming $A \in D_{\mathrm{qc}}^{\leq m}(X)$. The summands of the middle portion lie in $D^{\leq m+c+n-2}$ and $D^{\leq m+c}$ respectively, and the right side lies in $D^{\leq m+c+n-2}$. Thus $A[1] \in D^{\leq m+c+n-2}$, i.e. $A \in D^{\leq m+c+n-1}$. This means that F has right QCoh-dimension $\leq c+n-1$. □

Finally, we can prove Prop. 2.5.49 and thus complete the proof of the Lipman-Neeman theorem (Thm. 2.5.48).

Proof (of Prop. 2.5.49). Having finite tor dimension is local on Y, and if $U \subset Y$ is a quasicompact open then we know that

$$f_U^\times = j^*_{f^{-1}(U)} f^\times Rj_{U,*},$$

and the three functors on the RHS are bounded so $f_U^\times$ is bounded. Thus we may assume $Y = \operatorname{Spec} R$ is affine. Let $\{U_i\}$ be a finite affine cover of X with inclusions $j_i \colon U_i \hookrightarrow X$, and by taking cohomology of Lemma 2.5.50 we may choose a single c so that $A \in D^{\leq -c}$ implies $\mathrm{Hom}(Rj_{i,*}\mathcal{O}_{U_i}, A) = 0$ for all i. Let b denote the right-QCoh dimension of $f^\times$.

Thus, for $n \geq b+c$, and an R-module M, we have $\widetilde{M}[n] \in D^{\leq -b-c}$, so $f^\times(\widetilde{M}[n]) \in D^{\leq -c}$, meaning

$$\begin{aligned} 0 &= \mathrm{Hom}_{D(X)}(Rj_{i,*}\mathcal{O}_{U_i}, f^\times \widetilde{M}[n]) \\ &= \mathrm{Hom}_{D(Y)}(Rf_*Rj_{i,*}\mathcal{O}_U, \widetilde{M}[n]) \\ &= \mathrm{Hom}_{D(R)}(S_i, M[n]) = \mathrm{Ext}^n_R(S_i, M) \end{aligned}$$

where we have written $U_i = \operatorname{Spec} S_i$ and used that $R(fj_i)_*\mathcal{O}_{U_i} = \widetilde{S_i}$ as an R-module because fj_i is affine. Since M was arbitrary, $\mathrm{pd}_R(S_i) < n$. This implies that $U_i \to Y$ has finite tor dimension, and thus $X \to Y$ has finite tor dimension. □

2.5.4. Appendix: Quasicoherent representatives. Let X be a scheme. In this section we will discuss when the natural functors

$$D^*(\mathrm{QCoh}(X)) \to D^*_{\mathrm{qc}}(X)$$

give equivalences of categories. We are independently interested in the notion of when, given an object $A \in D_{\mathrm{qc}}(X)$, we can represent it as a complex of quasicoherent sheaves. This is an especially difficult question because of the way we have set up our unbounded theory, namely without resorting to any general results about Grothendieck abelian categories. Here is our first major theorem:

THEOREM 2.5.52. *Let X be a quasicompact semiseparated scheme. Then the natural functors*

$$D^*(\mathrm{QCoh}(X)) \to D^*_{\mathrm{qc}}(X)$$

are equivalences for $* \in \{(\quad), +, -, b\}$.

First of all, it suffices to prove that $D(\mathrm{QCoh}(X)) \to D_{\mathrm{qc}}(X)$ is an equivalence. Indeed, if this holds, then the other functors are clearly fully faithful, and then they are essentially surjective because we can always apply smart truncation to a possibly unbounded representative of a bounded object and preserve quasicoherence.

The trick is to prove full faithfulness first. Indeed, once this is done, the essential image of the functor will be a triangulated subcategory $\mathcal{E} \subset D_{\mathrm{qc}}(X)$ isomorphic to $D(\mathrm{QCoh}(X))$. Then $\mathcal{E}$ is closed under coproducts in $D_{\mathrm{qc}}(X)$, because these are computed termwise on any representative, in particular complexes of quasicoherent sheaves. Now since $D_{\mathrm{qc}}(X)$ is perfectly generated, it suffices to show $\mathrm{Perf}(X) \subset \mathcal{E}$, and, by quasicompactness, it suffices to show $D^b_{\mathrm{qc}}(X) \subset \mathcal{E}$.

But this is clear by induction on the cohomological amplitude: indeed if $A \in D_{\mathrm{qc}}(X)$ is concentrated in one degree, then obviously $A \in \mathcal{E}$. If A is concentrated in $[a, b]$, then

$$A \to \tau_{\geq a+1} A \to H^a(A)[-a] \to A[1]$$

is exact, so by induction $A \in \mathcal{E}$. Thus it remains to show that the functor is fully faithful.

PROOF (OF THM. 2.5.52). Fix an object $A \in D(\mathrm{QCoh}(X))$ and consider the full replete subcategory $\mathcal{E} \subset D(\mathrm{QCoh}(X))$ such that

$$\mathrm{Hom}_{D(\mathrm{QCoh}(X))}(A, B[q]) \to \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(A, B[q])$$

is a bijection for all $q \in \mathbb{Z}$. Since Hom's are cohomological, $\mathcal{E}$ is triangulated.

First let $U \subset X$ be an affine open; by definition of qcss, $j\colon U \hookrightarrow X$ is an affine morphism. Therefore $j_*\colon \mathrm{QCoh}(U) \to \mathrm{QCoh}(X)$ is an exact functor, and moreover we can descend the adjunction $j^* \dashv j_*$ by Remark 1.3.21 to

$$j^*\colon D(\mathrm{QCoh}(X)) \rightleftarrows D(\mathrm{QCoh}(U)) : j_*$$

because j^* is exact and $\mathrm{QCoh}(U) \cong \mathrm{Mod}(\mathcal{O}_X(U))$ has enough injectives. Therefore for all $N \in D(\mathrm{QCoh}(X))$, we have

$$\begin{aligned}\mathrm{Hom}_{D(\mathrm{QCoh}(X))}(A, j_*N[q]) &= \mathrm{Hom}_{D(\mathrm{QCoh}(U))}(j^*A, N[q]) \\ &= \mathrm{Hom}_{D_{\mathrm{qc}}(U)}(j^*A, N[q]) \\ &= \mathrm{Hom}_{D_{\mathrm{qc}}(X)}(A, Rj_*N[q])\end{aligned}$$

by the affine equivalence. Now if we represent N by a complex $\mathcal{G}^\bullet$ of quasicoherent sheaves, then we know that $j_*\mathcal{G}^\bullet \xrightarrow{\sim} Rj_*\mathcal{G}^\bullet$ since j is affine (Example 2.3.45), so this implies that $j_*N \in \mathcal{E}$ for all N.

Now let $B \in D(\mathrm{QCoh}(X))$ and write $B = \mathcal{F}^\bullet$ for $\mathcal{F}^\bullet$ a complex of quasicoherent sheaves. Let $\mathcal{U}$ be a finite affine open cover of X, and consider the ordered Čech resolution:

$$0 \to \mathcal{F}^\bullet \to \prod_{i_0} j_{i_0,*}\mathcal{F}^\bullet|_{U_i} \to \cdots \to \prod_{i_0<\cdots<i_n} j_{i_0\cdots i_n,*}\mathcal{F}^\bullet|_{U_{i_0\cdots i_n}} \to 0$$

This is termwise an exact sequence, so we may view it as a finite exact sequence of complexes in $\mathrm{Ch}(\mathrm{QCoh}(X))$. Each $U_{i_0\cdots i_p}$ is an affine open, so each $j_{i_0\cdots i_p,*}\mathcal{F}^\bullet|_{U_{i_0\cdots i_p}}$ is in $\mathcal{E}$. Since the products are finite, that implies all terms above except the first are in $\mathcal{E}$. Since $\mathcal{E}$ is triangulated, this implies $\mathcal{F}^\bullet = B \in \mathcal{E}$. As B was arbitrary, this proves full faithfulness. □

Remark 2.5.53. There's a small missing step in the proof, which is that Remark 1.3.21 does not immediately give us an *unbounded* derived adjunction. But this can easily be fixed because we know that $\mathrm{QCoh}(U)$ has K-injective resolutions (with terms in $\mathrm{QCoh}(U)$ as U is affine, as in Remark 2.3.29), and j_* of such a K-injective resolution remains K-injective as j^* is exact, so we reduce to the case of the homotopy category just as in Remark 1.3.21.

Here is a sample application of this. If X is a scheme, we say X has the *resolution property* if every finite type sheaf $\mathcal{F}$ on X admits a surjection from a finite locally free sheaf. We will prove more basic facts about the resolution property in the coming sections. The intermediate result is:

Lemma 2.5.54. *Let X be a scheme with the resolution property and $\mathcal{F}^\bullet$ a bounded above complex of quasicoherent sheaves.*

(a) If $\mathcal{F}^\bullet$ is m-pseudocoherent, there is a quasi-isomorphism $\phi\colon \mathcal{E}^\bullet \to \mathcal{F}^\bullet$ in $\mathrm{Ch}(\mathrm{QCoh}(X))$ such that $\mathcal{E}^p$ is finite locally free for $p \geq m$.

(b) In the situation of (a), if there is $r \geq m$ for which $\mathcal{F}^p$ is finite locally free for $p \geq r$, then we may take $\mathcal{E}^p = \mathcal{F}^p$ and $\phi^p = \mathrm{Id}$ for all $p \geq r$.

We will show later that any Noetherian scheme with the resolution property is automatically semiseparated (and in fact this is true for any qcqs X by means of Noetherian approximation). To prove this we need a very standard lemma:

Lemma 2.5.55. *Let X be a qcqs scheme and $\mathcal{F} \in \mathrm{QCoh}(X)$.*

(a) If $U \subset X$ is a quasicompact open and $\mathcal{G} \subset \mathcal{F}|_U$ with $\mathcal{G} \in \mathrm{QCoh}(U)$, $\mathcal{G}$ extends to a quasicoherent subsheaf of $\mathcal{F}$ on X.

(b) In (a) if $\mathcal{G}$ is of finite type, then we may take the extension to X to be a finite type quasicoherent subsheaf of $\mathcal{F}$.
(c) $\mathcal{F}$ is the filtered colimit of its finite type quasicoherent subsheaves.
(d) If $\mathcal{F} \twoheadrightarrow \mathcal{G}$ is a surjection onto a finite type quasicoherent sheaf, there exists a finite type quasicoherent subsheaf $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}' \to \mathcal{G}$ is surjective.

PROOF. (a) Then $j\colon U \hookrightarrow X$ is qcqs, so $\mathcal{K} = \ker \mathcal{F} \to j_*(\mathcal{F}|_U/\mathcal{G})$ is quasicoherent. The exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to j_*(\mathcal{F}|_U/\mathcal{G})$$

makes it clear that $\mathcal{K}|_U = \mathcal{G}$.

(b) By (a), we may assume $\mathcal{G} = \mathcal{F}_U$. Since X can be covered by finitely many affine opens, it suffices (inductively) to show that for each $V = \operatorname{Spec} A$ we can extend $\mathcal{F}|_U$ to a quasicoherent finite type subsheaf of $\mathcal{F}|_{U\cup V}$. Indeed, write $U \cap V = D(f_1) \cup \cdots D(f_r)$ for $f_i \in A$ and $\mathcal{F}|_V = \widetilde{M}$ for M an A-module. By assumption M_{f_i} is finitely generated for each i, so by lifting generators we may choose $x_1, \ldots, x_m \in M$ whose images generate M_{f_i} over A_{f_i}. Let $N \subset M$ be the finite submodule generated by these elements, then the $\widetilde{N} \hookrightarrow \mathcal{F}|_V$ restricts to an isomorphism on each $D(f_i)$, hence $\widetilde{N}|_{U\cap V} \cong \mathcal{F}|_{U\cap V}$. By descent there is some $\mathcal{G} \subset \mathcal{F}_{U\cup V}$ such that $\mathcal{G}|_U \cong \mathcal{F}|_U$ and $\mathcal{G}|_V \cong \widetilde{N}$, as desired.

(c) Let $\{\mathcal{F}_\alpha\}$ be the filtered collection of finite type quasicoherent subsheaves with inclusions and consider the natural map $\varinjlim_\alpha \mathcal{F}_\alpha \to \mathcal{F}$ which is clearly injective because each $\mathcal{F}_\alpha \to \mathcal{F}$ was. For surjectivity, let $U \subset X$ be an affine open, $x \in \mathcal{F}(U)$, and $\widetilde{\langle x \rangle} \subset \mathcal{F}|_U$ the corresponding finite type subsheaf. This extends to a finite type quasicoherent subsheaf $\mathcal{F}_\beta \subset \mathcal{F}$ by (b), which shows that $\varinjlim_\alpha \mathcal{F}_\alpha(U) \to \mathcal{F}(U)$ is surjective. Thus $\varinjlim_\alpha \mathcal{F}_\alpha|_U \to \mathcal{F}|_U$ is surjective since U is affine, which implies the result.

(d) Write $\mathcal{F} = \varinjlim_\alpha \mathcal{F}_\alpha$ as in (c) and let $\mathcal{G}_\alpha = \operatorname{im}(\mathcal{F}_\alpha \to \mathcal{G})$ so that $\mathcal{G} = \varinjlim_\alpha \mathcal{G}_\alpha$. Since $\mathcal{G}$ is of finite type, we can choose a finite affine cover U_i with $\mathcal{G}(U_i)$ a finite $\mathcal{O}_X(U_i)$-module and $\mathcal{G}(U_i) = \bigcup_\alpha \mathcal{G}_\alpha(U_i)$. Thus $\mathcal{G}(U_i) = \mathcal{G}_{\beta_i}(U_i)$ for some β_i, and by finiteness there's a single β which works for all i. Therefore $\mathcal{G} = \mathcal{G}_\beta$, so $\mathcal{F}_\beta \to \mathcal{G}$ is surjective. □

We will use (d) in the proof of the lemma and again in the Noetherian case. There is also the following extension lemma which is useful for constructing explicit complexes.

Lemma 2.5.56. *Let $\mathcal{A}$ be an abelian category and $\phi\colon A^\bullet \to B^\bullet$ a chain map in $\operatorname{Ch}(\mathcal{A})$ with cone $C^\bullet = \operatorname{Cone}(\phi)$. Suppose that:*

(i.) $n \in \mathbb{Z}$ is such that $A^\bullet \in \operatorname{Ch}^{\geq n+1}(\mathcal{A})$ and $C^\bullet \in D^{\leq n}(\mathcal{A})$; and
(ii.) we are given a map $X \to Z^n(C^\bullet)$ such that $X \to Z^n(C^\bullet) \to H^n(C^\bullet)$ is surjective.

Then there is a chain map $\phi_0\colon A_0^\bullet \to B^\bullet$ from a complex $A_0^\bullet \in \operatorname{Ch}^{\geq n}(\mathcal{A})$ with the following properties:

(a) $\sigma_{\geq n+1}A_0^\bullet = A^\bullet$, $A_0^n = X$;
(b) and $\mathrm{Cone}(\phi_0) \in D^{\leq n-1}(\mathcal{A})$.

PROOF. Write the map $X \to C^n = A^{n+1} \oplus B^n$ in components (h, f). The fact that this factors through $Z^n(C^\bullet)$ means that

$$-d_A h = 0 \qquad \text{and} \qquad \phi h + d_B f = 0,$$

so if we define $A_0^\bullet$ to be

$$\cdots \to 0 \to 0 \to X \xrightarrow{-h} A_{n+1} \xrightarrow{d_A} A_{n+2} \to \cdots,$$

then this is indeed a complex. Now define $\phi_0 \colon A_0^\bullet \to B^\bullet$ to be ϕ^i for $i \geq n+1$, and $\phi_0^n \colon X \to B^n$ to be f. The second relation ensures that this is a chain map.

It remains to check property (b); consider the stupid truncation triangle $A \to A_0 \to X[-n] \to A[1]$. Since the composite $A \to A_0 \to B$ coincides with ϕ, the octahedral axiom gives the exact triangle

$$X[-n] \to C^\bullet \to \mathrm{Cone}(\phi_0) \to X[-n+1],$$

and it is easy to check that $H^n(X[-n]) = X \to H^n(C^\bullet)$ coincides with the composite $X \to Z^n(C^\bullet) \to H^n(C^\bullet)$, which is surjective. Thus the triangle shows $\mathrm{Cone}(\phi_0) \in D^{\leq n-1}(\mathcal{A})$. □

PROOF (OF LEMMA 2.5.54). Here's how we apply this lemma; let $\mathcal{F}^\bullet$ be m-pseudocoherent, and suppose that $\mathcal{F}^p$ is finite locally free for $p \geq r \geq m$. If $r = m$ then we're done, so assume $r - 1 \geq m$. Start with $\mathcal{E}^\bullet = \sigma_{\geq r}\mathcal{F}^\bullet$ which is a complex of finite locally frees with the natural inclusion $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$. The cone $C^\bullet$ is isomorphic in the derived category to $\sigma_{\leq r-1}\mathcal{F}^\bullet$ which is in $D^{\leq r-1}$. The triangle

$$\mathcal{E}^\bullet \to \mathcal{F}^\bullet \to C^\bullet \to \mathcal{E}^\bullet[1]$$

shows that $C^\bullet$ is m-pseudocoherent, with top cohomology in degree $r - 1 \geq m$, which we know implies $H^{r-1}(C^\bullet)$ is of finite type. By Lemma 2.5.55(d), we can find a finite type quasicoherent subsheaf of $Z^{r-1}(C^\bullet)$ which has full image in $H^{r-1}(C^\bullet)$; thus we may take a map from a finite locally free sheaf $\mathcal{E} \to Z^{r-1}(C^\bullet)$ which has full image in $H^{r-1}(C^\bullet)$. Since this factors through cycles, it lifts to a map $\mathcal{E}[-(r-1)] \to C^\bullet$ inducing a surjection on cohomology.

The construction above allows us to append $\mathcal{E}$ to $\mathcal{E}^\bullet$ to obtain a complex $\mathcal{E}_0^\bullet$ of finite locally frees concentrated in degrees $\geq r - 1$ with a map $\mathcal{E}_0^\bullet \to \mathcal{F}^\bullet$ that has cone in $D^{\leq r-2}$. We can now repeat this process until we obtain $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{E}^\bullet$ a complex of locally frees in degrees $\geq m$, with cone $C^\bullet$ in $D^{\leq m-1}$. Now the top cohomology of the cone may not be of finite type, so just apply the construction to the inclusion $Z^{m-1}(C^\bullet)[-(m-1)] \to C^\bullet$ which is just some quasicoherent sheaf.

Continue this inductively downwards, at each step reducing the top cohomology of the cone by 1 to obtain a final complex $\mathcal{E}^\bullet$ with a map $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$. Since each step only affects the bottom cohomology of $\mathcal{E}^\bullet$, this map is an isomorphism on all cohomology, and it has the desired properties. □

Proposition 2.5.57. *Let X be a qcss scheme with the resolution property, and let $\mathcal{R}_{vb} \subset \mathrm{Mod}(\mathcal{O}_X)$ denote the class of finite locally free sheaves.*

(a) *An object in $D^{\leq b}(X)$ is pseudocoherent iff it is represented by a complex of finite locally free sheaves concentrated in degrees $(-\infty, b]$.*
(b) *An object $E \in D(X)$ is perfect with tor amplitude in $[a, b]$ iff it is represented by a complex of finite locally free sheaves concentrated in $[a, b]$.*
(c) *The natural functors*

$$\mathrm{K}^-(\mathcal{R}_{vb}) \to \mathrm{PsCoh}(X) \qquad \mathrm{K}^b(\mathcal{R}_{vb}) \to \mathrm{Perf}(X)$$

are triangulated localizations by the quasi-isomorphisms.

Proof. For (b) the converse is clear because such a complex is locally a bounded complex of finite frees. For (a) the converse is clear because a bounded above complex of pseudocoherent sheaves is pseudocoherent (Lemma 2.2.52). For (a), first apply Thm. 2.5.52 to represent a pseudocoherent object by a complex $\mathcal{F}^\bullet$ of quasicoherent sheaves, and we can assume by smart truncation that $\mathcal{F}^\bullet$ is concentrated in $(\infty, b]$.

Now by Lemma 2.5.54(b), we can choose a quasi-isomorphism $\mathcal{F}_1^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{F}_1^\bullet$ in $(-\infty, b]$ a complex of quasicoherent sheaves with $\mathcal{F}_1^b$ a finite locally free sheaf, since $\mathcal{F}^\bullet$ is b-pseudocoherent. It is also $(b-1)$-pseudocoherent, we can repeat this construction inductively for all $i \leq b$ while modifying the complex only below degrees i. This is well-defined because termwise the complex is eventually constant, and thus we obtain a well-defined chain map $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ where $\mathcal{E}^\bullet$ is a complex of finite locally frees concentrated in $(-\infty, b]$. This is a quasi-isomorphism because in the corresponding inverse system the cohomologies are eventually constant.

Represent E by some $\mathcal{F}^\bullet \in \mathrm{Ch}(\mathrm{QCoh}(X))$ concentrated in $(-\infty, b]$ and take the quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ as in (a) where $\mathcal{E}^\bullet$ is a complex of finite locally frees in $(-\infty, b]$. Apply smart truncation $\mathcal{E}^\bullet \to \tau_{\geq a}\mathcal{E}^\bullet$ which is a quasi-isomorphism, and we claim $(\tau_{\geq a}\mathcal{E}^\bullet)^a$ is still finite locally free. Indeed, there is an exact sequence of the form

$$\cdots \to \mathcal{E}^{a-2} \to \mathcal{E}^{a-1} \to (\tau_{\geq a}\mathcal{E}^\bullet)^a \to 0$$

which implies $(\tau_{\geq a}\mathcal{E}^\bullet)^a$ is pseudocoherent (Lemma 2.2.43) hence finitely presented. On the other hand, by Prop. 2.2.53 this sheaf is flat. Thus it is a finite locally free sheaf.

(c) We use the triangulated equivalence of Thm. 2.5.52 without further mention. The natural map $\mathrm{K}^-(\mathcal{R}_{vb}) \to D(\mathrm{QCoh}(X))$ lands in $\mathrm{PsCoh}(X)$ by the easy direction of (a); as in Remark 2.4.9, if we let $\mathcal{D} \subset \mathrm{K}(\mathrm{QCoh}(X))$ be the pseudocoherent complexes, then we know that this is a localizing full triangulated subcategory (as it is saturated under quasi-isomorphism). Then the criterion of Remark 2.4.9(4) proves that the first functor of (c) is a triangulated localization by quasi-isomorphisms since all pseudocoherent complexes of quasicoherent sheaves admit a quasi-isomorphism from a bounded above complex of finite locally free sheaves, as shown in (a).

Likewise, it follows from (b) that $\mathrm{K}^b(\mathcal{R}_{vb}) \to D^b(\mathrm{QCoh}(X))$ has essential image $\mathrm{Perf}(X)$. Now let $\mathcal{D} \subset \mathrm{K}^b(\mathrm{QCoh}(X))$ be the full triangulated subcategory of bounded perfect complexes (of quasicoherent sheaves), which is again localizing because it is saturated under quasi-isomorphisms. Now if $\mathcal{F}^\bullet$ is a bounded perfect

complex, we can choose an interval $[a,b]$ so that $\mathcal{F}^\bullet$ has tor amplitude in $[a,b]$ and is concentrated in $[a,b]$. Then the argument of (b) gives a quasi-isomorphism $\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ with $\mathcal{E}^\bullet$ concentrated in $(\infty, b]$ and $\mathcal{E}^\bullet \to \tau_{\geq a}\mathcal{E}^\bullet$ a quasi-isomorphism. But since $\mathcal{F}^\bullet$ is concentrated in $[a,b]$, this map factors through a quasi-isomorphism $\tau_{\geq a}\mathcal{E}^\bullet \to \mathcal{F}^\bullet$ with the desired properties. The criterion of Remark 2.4.9(4) then shows that the functor is a triangulated localization. □

Now we return to the discussion of $D(\mathrm{QCoh}(X))$-$D_{\mathrm{qc}}(X)$ equivalence, and we move to the more subtle case which is when X is a Noetherian scheme. Once again it is true that the unbounded equivalence holds, but we will not prove it in this generality because it would require facts we don't have about Grothendieck abelian categories and the coherator. But the bounded below case is indeed accessible, which will also prove the practical case that $D^b \operatorname{Coh}(X) = D^b_{\mathrm{coh}}(X)$ for a Noetherian scheme X.

This case requires a non-obvious fact from commutative algebra, which is that if M is an injective R-module and R is Noetherian, then $\widetilde{M}$ is a flasque sheaf on $\operatorname{Spec} R$. To prove this, first if R is a ring and $I \subset R$ is an ideal, then we write

$$\Gamma_I(M) \coloneqq \{x \in M : I^n x = 0 \text{ for some } n > 0\}$$

for M an R-module. It is clear that this is again an R-module. The most basic observation is that for $f \in I$, $\Gamma_I(M)_f = 0$, meaning that $\widetilde{\Gamma_I(M)}|_{D(I)} = 0$. (If I add a section on real Noetherian local algebra, then this notion will be very important.)

Lemma 2.5.58. *Let R be a Noetherian ring and M an injective R-module. Then $\Gamma_I(M)$ is an injective R-module.*

Proof. Let $J \subset R$ be an ideal, and consider a map $\phi\colon J \to \Gamma_I(M)$. Since J and hence $\phi(J)$ are finite R-modules, there exists $n > 0$ such that $0 = I^n\phi(J) = \phi(I^n J)$. By the Artin-Rees lemma applied to $J \subset R$, there exists m such that $I^m \cap J \subset I^n J$, so $\phi(I^m \cap J) = 0$. Thus there is a well-defined map $I^m + J \to M$ given by $a + b \mapsto \phi(b)$ for $a \in I^m$ and $b \in J$. This extends to $R \to M$ by injectivity, and since $I^m \mapsto 0$ it follows that this factors through a map $R \to \Gamma_I(M)$ which extends ϕ. Thus $\Gamma_I(M)$ is injective (Lemma 1.4.12). □

Proposition 2.5.59. *Let R be a Noetherian ring and M an injective R-module. Then $\widetilde{M}$ is a flasque sheaf on* $\operatorname{Spec} R$.

Proof. Let $U \subset \operatorname{Spec} R$, then by Noetherianity write $U = D(f_1) \cup \cdots \cup D(f_r)$. Let $s \in \Gamma(U, \widetilde{M})$. We claim that we may choose N and $x_i \in M$ such that

$$s|_{D(f_i)} = x_i / f_i^N$$

and $f_j^N x_i = f_i^N x_j$ for all i, j. Indeed, we can clearly write $s|_{D(f_i)} = x_i/f_i^n$ for some n and $x_i \in M$, since these agree on overlaps $D(f_i f_j)$ we obtain one e such that $(f_i f_j)^e (f_j^n x_i - f_i^n x_j) = 0$ for all i, j. Then evidently we may replace x_i with $f_i x_i$ and $N = n + e$.

Next, let $I = (f_1^N, \dots, f_r^N)$, so $D(I) = U$. Let $\alpha\colon R^r \to I$ be the surjection associated to the generators, and $\beta\colon R^r \to M$ the map corresponding to the x_i. Then $f_j^N \beta(v) = \alpha(v) x_j$ for $v \in R^r$ as this relation holds on the basis. In particular, if $v \in \ker(\alpha)$, then $f_j^N \beta(v) = 0$, so β restricts to $\ker\alpha \to \Gamma_I(M)$.

By injectivity of $\Gamma_I(M)$, lift this to a map $\beta'\colon R^r \to \Gamma_I(M)$. β and β' agree on $\ker\alpha$, so $\beta - \beta'$ factors through $R^r/\ker\alpha = I$, inducing a map $\varphi\colon I \to M$. Then

$$\widetilde{\varphi}|_U\colon \widetilde{I}|_U = \mathcal{O}_U \to \widetilde{M}|_U$$

corresponds to a section of $\widetilde{M}|_U$. To compute it, note that $\varphi \circ \alpha = \beta - \beta'$, so $\varphi(f_i^N) = x_i - \beta'(e_i)$, so

$$\widetilde{\varphi}|_U(f_i^N) = x_i$$

as $\widetilde{\Gamma_I(M)}|_U = 0$ and β' has image in $\Gamma_I(M)$. On $D(f_i)$, this implies that $\widetilde{\varphi}(1) = x_i/f_i^N = s|_{D(f_i)}$, hence $\widetilde{\varphi}|_U(1) = s$. Finally, lift φ to a map $\mathcal{O}_{\operatorname{Spec} R} \to \widetilde{M}$, i.e. a section of $\widetilde{M}$. Its restriction to U corresponds to the map $\mathcal{O}_U \to \widetilde{M}|_U$, namely φ, so the section of $\widetilde{M}$ restricts to s. Since U and s were arbitrary, this proves that $\widetilde{M}$ is flasque. □

Here is the immediate corollary: if $f\colon X \to Y$ is a morphism of schemes with X a Noetherian affine scheme (so in particular f is automatically qcqs), then the right derived functor of f_* computed via the injective resolutions in $\mathrm{QCoh}(X)$ coincides with Rf_*, i.e. the diagram

$$\begin{array}{ccc} D^+(\mathrm{QCoh}(X)) & \longrightarrow & D^+_{\mathrm{qc}}(X) \\ {\scriptstyle R(f_*)}\big\downarrow & & \big\downarrow{\scriptstyle Rf_*} \\ D^+(\mathrm{QCoh}(Y)) & \longrightarrow & D^+_{\mathrm{qc}}(Y) \end{array}$$

commutes, as we know Rf_* can be computed on flasque resolutions.

Theorem 2.5.60. *Let X be a Noetherian scheme. Then the canonical functor*

$$i_X\colon D^*(\mathrm{QCoh}(X)) \to D^*_{\mathrm{qc}}(X)$$

is an equivalence of categories for $* \in \{+, b\}$.

Proof. As before, it suffices to prove the case $* = +$. Fix $A \in D^+(\mathrm{QCoh}(X))$, and let $\mathcal{E}$ be the full subcategory of $B \in D^+(\mathrm{QCoh}(X))$ for which the map

$$\mathrm{Hom}_{D^+(\mathrm{QCoh}(X))}(A, B[q]) \to \mathrm{Hom}_{D^+_{\mathrm{qc}}(X)}(i_X A, i_X B[q])$$

is a bijection for all $q \in \mathbb{Z}$. Then $\mathcal{E}$ is replete triangulated. First suppose $B = R(j_*)C$ for $C \in D^+(\mathrm{QCoh}(U))$ and $j\colon U \hookrightarrow X$ an affine open inclusion. Then $\mathrm{QCoh}(U)$ has enough injectives and j^* is exact, so by Remark 1.3.21 we have an

adjunction $j^* \dashv R(j_*)$, which implies

$$\begin{aligned}\mathrm{Hom}_{D^+(\mathrm{QCoh}(X))}(A, R(j_*)C[q]) &= \mathrm{Hom}_{D^+(\mathrm{QCoh}(U))}(j^*A, C[q]) \\ &= \mathrm{Hom}_{D^+_{\mathrm{qc}}(U)}(j^*i_XA, i_UC[q]) \\ &= \mathrm{Hom}_{D^+_{\mathrm{qc}}(X)}(i_XA, Rj_*i_UC[q]) \\ &= \mathrm{Hom}_{D^+_{\mathrm{qc}}(X)}(i_XA, i_XR(j_*)C[q]),\end{aligned}$$

where the second step is by the affine equivalence, and the third is the observation made above the proof. Thus $R(j_*)D^+(\mathrm{QCoh}(U)) \in \mathcal{E}$.

We will show by Noetherian induction that $\mathcal{E} = D^+(\mathrm{QCoh}(X))$. Suppose $Z \subset X$ is a closed subset such that $B \in \mathcal{E}$ whenever $\operatorname{Supp} B \subset Z'$ for $Z' \subsetneq Z$ a proper closed subset. Let $\operatorname{Supp} B \subset Z$; if Z is empty, then clearly $B \in \mathcal{E}$, so assume not and let U be an affine open with $U \cap Z \neq \varnothing$. Take the cone

$$B \to R(j_*)B|_U \to C \to B[1],$$

then it will suffice to prove that $\operatorname{Supp} C \subset Z \setminus U$, since then $C \in \mathcal{E}$ by induction, and $R(j_*)B|_U \in \mathcal{E}$ by the case above, so $B \in \mathcal{E}$. Since cohomology commutes with the exact functor i_X, we may apply i_X and show that $\operatorname{Supp} i_XC \subset Z \setminus U$ in the triangle

$$i_XB \to Rj_*(i_XB|_U) \to i_XC \to i_XB[1].$$

If we restrict to U, then the first map becomes an isomorphism, so $\operatorname{Supp} i_XC \subset X \setminus U$. On the other hand for each open $V \subset X \setminus Z$, we have $Rj_*(i_XB|_U)|_V = 0$, so the first two terms are both supported in Z, and hence the third is as well. Applying this to $Z = X$ gives full faithfulness.

Now let $\mathcal{E}$ be the essential image of i_X, which we now know is a triangulated subcategory equivalent to $D^+(\mathrm{QCoh}(X))$. If $j\colon U \hookrightarrow X$ is an affine open inclusion, then $Rj_*C \in \mathcal{E}$ for $C \in D^+_{\mathrm{qc}}(U)$, because the affine equivalence lets us replace C with i_UC, and then $Rj_*i_UC = i_XR(j_*)C$. We again proceed by Noetherian induction on a closed subset containing support. If $A \in D^+_{\mathrm{qc}}(X)$ is supported in Z, then for U an affine open with $U \cap Z \neq \varnothing$, we have the triangle

$$A \to Rj_*(A|_U) \to B \to A[1].$$

The argument above shows $\operatorname{Supp} B \subset Z \setminus U$, so by induction $B \in \mathcal{E}$. Thus $A \in \mathcal{E}$, proving essential surjectivity. □

In the Noetherian case, we want to be able to say something about $D^b\,\mathrm{Coh}(X)$ as well. We will deduce this from a more general lemma about Serre subcategories and quotients and apply the fact that $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ is a Serre subcategory when X is Noetherian.

Lemma 2.5.61. *Let $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory of an abelian category, and denote by $D^*_{\mathcal{B}}(\mathcal{A})$ the full subcategory of $D^*(\mathcal{A})$ with cohomology in $\mathcal{B}$.*

(a) $D^*_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory, and there is a natural functor

$$D^*(\mathcal{B}) \to D^*_{\mathcal{B}}(\mathcal{A})$$

induced by the universal property.

(b) Assume that for every epimorphism $M \twoheadrightarrow N$ for $N \in \mathcal{A}$, there is a subobject $M' \subset M$ with $M' \in \mathcal{B}$ such that $M' \to M \to N$ is an epimorphism. Then the natural functors

$$D^-(\mathcal{B}) \to D^-_{\mathcal{B}}(\mathcal{A}) \qquad D^b(\mathcal{B}) \to D^b_{\mathcal{B}}(\mathcal{A})$$

are triangulated equivalences.

(c) In the case of (b), every complex $A^\bullet \in \mathrm{K}^*(\mathcal{A})$ admits a quasi-isomorphism $B^\bullet \to A^\bullet$ with $B^\bullet \in \mathrm{K}^*(\mathcal{B})$.

Proof. (a) Closure under shifts is clear, and if $A \to B \to C \to A[1]$ is exact with $A, B \in D^-_{\mathcal{B}}(\mathcal{A})$, then the long exact sequence in cohomology includes

$$H^n(B) \to H^n(C) \to H^{n+1}(A)$$

with outer terms in $\mathcal{B}$, and thus $H^n(C) \in \mathcal{B}$ as $\mathcal{B}$ is Serre. Also since $\mathcal{B}$ is Serre, a complex with terms in $\mathcal{B}$ has cohomology in $\mathcal{B}$. Thus there is a natural functor $\mathrm{K}^*(\mathcal{B}) \to D^*_{\mathcal{B}}(\mathcal{A})$ sending quasi-isomorphisms to isomorphisms, and hence this descends to the derived category of $\mathcal{B}$.

(b) The bounded equivalence follows from the bounded above one: indeed, the latter implies that $D^b(\mathcal{B}) \to D^b_{\mathcal{B}}(\mathcal{A})$ is fully faithful, and that we can represent an object of $D^b_{\mathcal{B}}(\mathcal{A})$ by a bounded above complex $B^\bullet$ of objects of $\mathcal{B}$. But now $B^\bullet \to \tau_{\geq r} B^\bullet$ is a quasi-isomorphism for some $r \ll 0$, and $(\tau_{\geq r} B^\bullet)^r$ is a quotient of B^{r-1}, hence lies in $\mathcal{B}$.

The bounded above equivalence will follow from the fact that every complex $A^\bullet \in \mathrm{K}^-(\mathcal{A})$ with cohomology in $\mathcal{B}$ admits a quasi-isomorphism $B^\bullet \to A^\bullet$ with $B^\bullet \in \mathrm{K}^-(\mathcal{B})$. Indeed, this makes essential surjectivity clear; on the other hand note that the subcategory of $\mathrm{K}(\mathcal{A})$ with cohomology in $\mathcal{B}$ is full triangulated and localizing since it is saturated under quasi-isomorphisms in the sense of Remark 2.4.9. Moreover, by criterion (4) of that Remark, the functor is fully faithful.

So, let $A^\bullet \in \mathrm{K}^-(\mathcal{A})$ have cohomology in $\mathcal{B}$, and suppose $A^i = 0$ for $i \geq m$, and make the following descending inductive construction starting at m. The mth stage is the zero map $0 \to A^\bullet$. At the $n+1$th stage we will have a map $\phi_{n+1} \colon B^\bullet_{n+1} \to A^\bullet$ with $\mathrm{Cone}(\phi_{n+1}) \in D^{\leq n}$ and $B^\bullet_{n+1} \in \mathrm{Ch}^{\geq n+1}(\mathcal{B})$. Considering the surjection

$$Z^n(\mathrm{Cone}(\phi_{n+1})) \to H^n(\mathrm{Cone}(\phi_{n+1})),$$

our assumption says that we may choose a map $B^n \to Z^n(\mathrm{Cone}(\phi_{n+1}))$ whose composite with the above is surjective and $B^n \in \mathcal{B}$. By Lemma 2.5.56, we can glue B^n onto $B^\bullet_{n+1}$ to obtain a map $\phi_n \colon B^\bullet_n \to A^\bullet$ whose cone lies in $D^{\leq n-1}$, completing the inductive step.

The terms and differentials of $B^\bullet_n$ stabilize after the $(n+1)$th step, so we obtain a well-defined complex $B^\bullet \in \mathrm{Ch}(\mathcal{B})$ along with a map $B^\bullet \to A^\bullet$ such that ϕ_n equals the composite $\sigma_{\geq n} B^\bullet \to B^\bullet \to A^\bullet$. This composite induces an isomorphism on H^i for all $i \geq n+1$, and hence $H^i(B^\bullet) \to H^i(A^\bullet)$ is an isomorphism for all $i \geq n+1$. But n was arbitrary, so $B^\bullet \to A^\bullet$ is an isomorphism. □

Corollary 2.5.62. *Let X be a Noetherian scheme. The canonical functor*

$$D^b\mathrm{Coh}(X) \to D^b_{\mathrm{coh}}(X)$$

is a triangulated equivalence.

Proof. Since the equivalence $D^b\,\mathrm{QCoh}(X) \to D^b_{\mathrm{qc}}(X)$ of Thm. 2.5.60 commutes with cohomology functors, it follows that

$$D^b_{\mathrm{Coh}(X)}(\mathrm{QCoh}(X)) \to D^b_{\mathrm{coh}}(X)$$

is an equivalence. On the other hand since X is Noetherian, $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ is a Serre subcategory; since finite type quasicoherent sheaves coincide with coherent sheaves on X, Lemma 2.5.55(d) proves that $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ satisfies the hypothesis of Lemma 2.5.61(b). Thus we obtain an equivalence $D^b\,\mathrm{Coh}(X) \to D^b_{\mathrm{Coh}(X)}(\mathrm{QCoh}(X))$, and composing it with the equivalence above proves the result. □

For completeness, let us discuss the relation of the category $D^b_{\mathrm{coh}}(X)$ in the Noetherian case to pseudocoherence. We will write $D_{\mathrm{pc}}(X)$ for the full subcategory of pseudocoherent objects of $D(X)$. We have seen that this is a thick subcategory.

First, pseudocoherence behaves very well in the Noetherian setting. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. We claim that this is pseudocoherent in $D(X)$ iff $\mathcal{F}$ is a coherent sheaf. This is easy if we work locally, since we've shown that pseudocoherence on $\operatorname{Spec} A$ is equivalent to pseudocoherence on A. Thus we reduce to showing that if M is a module over a Noetherian ring A, then M is pseudocoherent iff M is finite. This is clear from Lemma 2.2.43 by the existence of finite-free resolutions. This makes it easy to prove the following:

Lemma 2.5.63. *Let X be a Noetherian scheme. Then $D_{\mathrm{pc}}(X) = D^-_{\mathrm{coh}}(X)$.*

Proof. ($\Leftarrow$) If A is bounded above with coherent cohomology, then by the above remark it is bounded above with pseudocoherent cohomology, which implies that A is pseudocoherent by Lemma 2.2.52. ($\Rightarrow$) We know A is bounded above, so let $H^n(A)$ be the top nonzero cohomology and consider the exact triangle

$$\tau_{\leq n-1}A \to A \to H^n(A)[-n] \to \tau_{\leq n-1}A[1].$$

Since A is pseudocoherent, $H^n(A)$ is quasicoherent, and it is finite type by Example 2.2.44. Thus it is coherent, hence pseudocoherent, hence $H^n(A)[-n]$ is pseudocoherent. This implies that $\tau_{\leq n-1}A$ is pseudocoherent by the 2-of-3 property (Lemma 2.2.42). Thus by the same argument $H^{n-1}(\tau_{\leq n-1}A) = H^{n-1}(A)$ is coherent, and so on. □

Note that if X is Noetherian, then $D^b_{\mathrm{coh}}(X) = \mathrm{tria}\{\mathcal{G} : \mathcal{G} \in \mathrm{Coh}(X)\}$. Indeed, this follows immediately from truncation triangles and induction on the cohomological amplitude.

Lemma 2.5.64. *Let $f\colon X \to Y$ be a morphism of Noetherian schemes. The following are equivalent:*

(a) $R^p f_*\mathcal{G}$ is coherent for all $p \geq 0$ and $\mathcal{G} \in \mathrm{Coh}(X)$.

(b) Rf_* *restricts to a functor* $D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$.
(c) f *is quasi-proper.*
(d) Rf_* *restricts to a functor* $D^+_{\mathrm{coh}}(X) \to D^+_{\mathrm{coh}}(Y)$.
(e) Rf_* *restricts to a functor* $D_{\mathrm{coh}}(X) \to D_{\mathrm{coh}}(Y)$.

Proof. $(a) \Rightarrow (b)$ The preimage $Rf_*^{-1}(D^b_{\mathrm{coh}}(Y))$ is a triangulated subcategory, so the observation above implies this (and that Rf_* is bounded). $(b) \Rightarrow (c)$ In particular all perfect objects map to pseudocoherent ones, so Prop. 2.5.46 implies this. (c) immediately implies (a), so the first three are equivalent.

Since Rf_* is bounded in both directions, $(e) \Rightarrow (d) \Rightarrow (b)$, and (d) follows immediately from the hypercohomology spectral sequence and (a), and then (e) follows from (d) and (c) and the canonical truncation triangles. □

It is in fact true that all of these conditions are equivalent to f being proper, and that (a) can be checked only for $p = 0, 1$, but we probably will not prove the direction that these conditions imply properness.

Example 2.5.65. Let $f\colon X \to Y$ be an affine morphism of Noetherian schemes. Then we know that the conditions above are all equivalent by Example 2.5.47 to $f_*\mathcal{O}_X$ being coherent, or in other words f being finite. This is also just clear from (a) and affine vanishing.

We also wish to relate our study of the category $D_Z(X)$ to coherence and pseudocoherence. We write $D^b_{\mathrm{coh},Z} = D^b_{\mathrm{coh}}(X) \cap D_Z(X)$ for a closed subset $Z \subset X$. Indeed, we have the following simple characterization:

Lemma 2.5.66. *Let* X *be a Noetherian scheme and* $i\colon Z \hookrightarrow X$ *a closed immersion. The following subcategories of* $D_{\mathrm{qc}}(X)$ *are equal:*

(a) $D^b_{\mathrm{coh},|Z|}(X)$.
(b) $\mathrm{tria}(i_*D^b_{\mathrm{coh}}(Z)) = \mathrm{thick}(i_*D^b_{\mathrm{coh}}(Z))$
(c) $\mathrm{tria}(\{i_*\mathcal{G} : \mathcal{G} \in \mathrm{Coh}(Z)\}) = \mathrm{thick}(i_*\mathcal{G} : \mathcal{G} \in \mathrm{Coh}(Z))$.
(d) $\mathrm{tria}(\{\mathcal{G} \in \mathrm{Coh}(X) : \mathrm{Supp}\, G \subset |Z|\})$

In particular, they depend only on $|Z|$.

Proof. We can ignore the 'thick' statements of (b) and (c) because (a) is thick. Since e.g. a closed immersion is finite, we have

$$\mathrm{tria}(\{i_*\mathcal{G} : \mathcal{G} \in \mathrm{Coh}(Z)\}) \subset \mathrm{tria}(i_*D^b_{\mathrm{coh}}(Z)) \subset D^b_{\mathrm{coh},|Z|}(X).$$

It is clear that $D^b_{\mathrm{coh},|Z|}(X) = \mathrm{tria}(\{\mathcal{G} \in \mathrm{Coh}(X) : \mathrm{Supp}\, G \subset |Z|\})$ by truncation triangles. So, let $Z = V(\mathcal{I})$ for a coherent ideal sheaf $\mathcal{I}$ and let $\mathcal{G}$ be coherent and supported on $|Z|$. Then we know that $\mathcal{I}^N\mathcal{G} = 0$ for $N \gg 0$, so we have a filtration

$$0 = \mathcal{I}^N\mathcal{G} \subset \mathcal{I}^{N-1}\mathcal{G} \subset \cdots \subset \mathcal{I}\mathcal{G} \subset \mathcal{G}$$

and hence exact triangles

$$\mathcal{I}^k\mathcal{G} \to \mathcal{I}^{k-1}\mathcal{G} \to \mathcal{F}_k \to \mathcal{I}^k\mathcal{F}[1]$$

where $\mathcal{F}_k$ is a sheaf with $\mathcal{I}\mathcal{F}_k = 0$, i.e. $\mathcal{F}_k = i_*\mathcal{F}'_k$ for some $\mathcal{F}'_k \in \mathrm{Coh}(Z)$. Thus descending induction on k shows that $\mathcal{I}^{k-1}\mathcal{G}$ is in $\mathrm{tria}(\{i_*\mathcal{F} : \mathcal{F} \in \mathrm{Coh}(Z)\})$, and thus we conclude. □

Let us use our formalism to give a more usable version of a statement we proved in the unbounded case. This will be a good example where unbounded/infinitely generated statements are easier because more derived functors exist.

Proposition 2.5.67. *Let X be a Noetherian scheme, $j\colon U \hookrightarrow X$ an open immersion, and $Z = X \setminus U$. The natural functor*

$$D^b_{\mathrm{coh}}(X)/D^b_{\mathrm{coh},Z}(X) \to D^b_{\mathrm{coh}}(U)$$

induced by restriction $j^*\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(U)$ is an equivalence.

Proof. By definition $\ker(j^*) = D^b_{\mathrm{coh},Z}(X)$. We claim the canonical map

$$D^b_{\mathrm{coh}}(X)/D^b_{\mathrm{coh},Z}(X) \to D_{\mathrm{qc}}(X)/D_Z(X)$$

is fully faithful. We will reduce this to the following lemma, and then prove the lemma afterwards:

Lemma 2.5.68. *Let X be a Noetherian scheme, $U \subset X$ an open subset, $\mathcal{F}^\bullet \in \mathrm{K}^b(\mathrm{Coh}(X))$, and $\mathcal{G}^\bullet \in \mathrm{K}^b(\mathrm{QCoh}(X))$. If $\mathcal{G}^\bullet|_U$ is acyclic, then every chain map $\mathcal{F}^\bullet \to \mathcal{G}^\bullet$ factors through a complex $\mathcal{K}^\bullet \in \mathrm{K}^b(\mathrm{Coh}(X))$ such that $\mathcal{K}^\bullet|_U$ is acyclic.*

Given the lemma, let's show that each of the maps in

$$\frac{D^b_{\mathrm{coh}}(X)}{D^b_{\mathrm{coh},Z}(X)} \to \frac{D^b_{\mathrm{qc}}(X)}{D^b_Z(X)} \to \frac{D^+_{\mathrm{qc}}(X)}{D^+_Z(X)} \to \frac{D_{\mathrm{qc}}(X)}{D_Z(X)}$$

is fully faithful, all using the criterion of Remark 2.5.15. For the latter two functors, we use the usual trick: let $A \to B$ be a map with $A \in D^b_{\mathrm{qc}}(X)$ and B in $D^+_Z(X)$. Then for $c \gg 0$, we know this factors as $A \to \tau_{\leq c}B \to B$. Likewise for the third functor, consider $A \to B$ where $A \in D_Z(X)$ and $B \in D^+_{\mathrm{qc}}(X)$. Then this factors through $A \to \tau_{\geq c}A \to b$ for $c \ll 0$.

For the first functor, let $A \in D^b_{\mathrm{coh},Z}(X)$ and $B \in D^b_Z(X)$, and consider a map $A \to B$. By Cor. 2.5.62 and Thm. 2.5.60, we can represent this by a roof of the form

$$\mathcal{F}^\bullet \leftarrow \mathcal{F}^\bullet_1 \to \mathcal{G}^\bullet$$

where $\mathcal{F}^\bullet \in \mathrm{K}^b(\mathrm{Coh}(X))$ and $\mathcal{G}^\bullet \in \mathrm{K}^b(\mathrm{QCoh}(X))$. By Lemma 2.5.61(c) we may replace $\mathcal{F}^\bullet_1$ with a complex in $\mathrm{K}^b(\mathrm{Coh}(X))$. By Lemma 2.5.68 the second map factors through some $\mathcal{K}^\bullet$ whose cohomology is supported on Z, and thus the roof gives such a factorization in the derived category. This proves the first functor is fully faithful as well.

Now consider the commutative diagram:

$$\begin{array}{ccc} D^b_{\mathrm{coh}}(X)/D^b_{\mathrm{coh},Z}(X) & \longrightarrow & D^b_{\mathrm{coh}}(U) \\ \downarrow & & \downarrow \\ D_{\mathrm{qc}}(X)/D_Z(X) & \xrightarrow{\cong} & D_{\mathrm{qc}}(U) \end{array}$$

since the composites are fully faithful, it follows that the top map is fully faithful, and thus has triangulated image. Therefore to see it is essentially surjective, it suffices to show that every $\mathcal{F} \in D^b_{\mathrm{coh}}(U)$ is in the image (as tria of these are the whole subcategory). Indeed, $j_*\mathcal{F}$ is quasicoherent and $\mathcal{F} = j_*\mathcal{F}|_U$, so by Lemma 2.5.55(b) $\mathcal{F}$ extends to a coherent sheaf on X. □

Proof (of Lemma 2.5.68). The image of $\mathcal{F}^\bullet$ in $\mathcal{G}^\bullet$ is a subcomplex of coherent sheaves of $\mathcal{G}^\bullet$, so we may assume $\mathcal{F}^\bullet \subset \mathcal{G}^\bullet$ is an inclusion. We factor the inclusion through a complex of coherent sheaves which is exact on U by descending induction on the top degree of $\mathcal{F}^\bullet$. Thus suppose $\mathcal{F}^\bullet|_U$ is exact in degrees $> n$.

Consider the coherent sheaf $Z^n(\mathcal{F}^\bullet) = \ker\left(d^n_{\mathcal{F}} \colon \mathcal{F}^n \to \mathcal{F}^{n+1}\right) \subset \mathcal{G}^n$, and take the preimage $\mathcal{P} \coloneqq d_{\mathcal{G}}^{-1}(Z^n(\mathcal{F}^\bullet)) \subset \mathcal{G}^{n-1}$, which comes with a map $d_{\mathcal{G}} \colon \mathcal{P} \to Z^n(\mathcal{F}^\bullet)$. We can also write this as in the pullback square

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & Z^n(\mathcal{F}^\bullet) \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{G}^{n-1} & \xrightarrow{d^{n-1}} & Z^n(\mathcal{G}^\bullet) \end{array}$$

which makes it clear that upon restriction to U, the bottom map becomes surjective and hence the top map $\mathcal{P}|_U \to Z^n(\mathcal{F}^\bullet)|_U$ is surjective too. By Lemma 2.5.55(d), we may find a coherent subsheaf $\mathcal{F}_0 \subset \mathcal{P}|_U$ such that

$$\mathcal{F}_0 \to \mathcal{P}|_U \to Z^n(\mathcal{F}^\bullet)|_U$$

remains surjective, and thus by Lemma 2.5.55(b) we may extend $\mathcal{F}_0$ to a coherent subsheaf $\mathcal{F}' \subset \mathcal{P}$ over all of X. Now define $\mathcal{K}^p = \mathcal{F}^p$ for $p \neq n-1$, and $\mathcal{K}^{n-1} = \mathcal{F}^{n-1} + \mathcal{F}' \subset \mathcal{P}$. We keep all the differentials the same except $\mathcal{K}^{n-1} \to \mathcal{K}^n$ is the restriction of $\mathcal{P} \to Z^n(\mathcal{F}^\bullet)$ which in particular makes $\mathcal{K}^\bullet$ a complex satisfying $\mathcal{F}^\bullet \subset \mathcal{K}^\bullet \subset \mathcal{G}^\bullet$. Then $\mathcal{K}^{n-1}$ is coherent, and the restriction $\mathcal{K}^{n-1}|_U$ contains $\mathcal{F}_0$ which surjects onto $Z^n(\mathcal{K}^\bullet)|_U$, thus $H^n(\mathcal{K}^\bullet|_U) = 0$. This completes the inductive step without modifying the complex in degrees $\geq n$, completing the proof. □

Remark 2.5.69. This is one of the many instances we have seen that full faithfulness allows us to trivially extend a statement to its triangulated closure. For instance, in the proof above we didn't have to worry about compatibly extending an object of $D^b_{\mathrm{coh}}(U)$ to $D^b_{\mathrm{coh}}(X)$, though the lemma proves that we can always do this. Note that the extension is certainly not given by Rj_* as it was in the unbounded quasicoherent case, because this is usually not coherent.

CHAPTER 3

Derived categories in Algebraic geometry

3.1. Classical regularity

3.1.1. Some local Noetherian algebra. We will now, for fun, discuss some classic results on local Noetherian rings, with the focus on giving homological criteria for concepts in scheme theory. The only reasons this is appearing here are that we just discussed the Koszul complex and that we will use these results in the next section.

In this section, we will continually work with $(R, \mathfrak{m})$ being a Noetherian local ring. Sometimes we also write $(R, \mathfrak{m}, k)$ to indicate the residue field $k = R/\mathfrak{m}$. We assume only standard concepts like Nakayama's lemma, associated primes, dimension, etc. We will build up to results like the Auslander-Buchsbaum-Serre theorem on regularity.

Throughout this section, we will write $\mathrm{Kos}_R^\bullet(\underline{f}; M) = \mathrm{Kos}_R^\bullet(\underline{f}) \otimes_R M$ for an R-module M, which is also the derived tensor product since the first term is K-flat. This retains many of the properties we discussed in the previous section (though it is not a dga). For instance, the induced endomorphism $f_i \colon \mathrm{Kos}_R^\bullet(\underline{f}, M) \to \mathrm{Kos}_R^\bullet(\underline{f}, M)$ is still nullhomotopic for each i since the tensor product of complexes descends to the homotopy category. Similarly, it is clear that

$$H^0(\mathrm{Kos}(\underline{f}), M) = M/(f_1, \ldots, f_r)M$$

as before. Here is another useful observation: we know that the cone of a map in the abelian category itself produces the corresponding complex in degrees $[-1, 0]$. In particular, we have the exact triangle

$$M \xrightarrow{f_r} M \to \mathrm{Kos}_R^\bullet(f_r, M) \to M[1],$$

so tensoring this with $\mathrm{Kos}_R^\bullet(f_1, \ldots, f_{r-1})$ produces the exact triangle

$$\mathrm{Kos}_R^\bullet(f_1, \ldots, f_{r-1}) \xrightarrow{f_r} \mathrm{Kos}_R^\bullet(f_1, \ldots, f_{r-1}) \to \mathrm{Kos}_R^\bullet(f_1, \ldots, f_r) \to$$

in $\mathrm{K}(R)$. This will be useful for inducting on the length of a sequence.

Let M be a finite R-module. An ordered sequence $(f_1, \ldots, f_r)$ of elements of $\mathfrak{m}$ is an *M-regular sequence* if the image of f_i is regular (i.e. a nonzerodivisor) over $M/(f_1, \ldots, f_{i-1})M$. Observe that if $M \neq 0$, then $M/(f_1, \ldots, f_r)M \neq 0$, because otherwise $M = 0$ by Nakayama's lemma. We can answer basic questions about this notion using the following homological characterization:

Lemma 3.1.1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finite R-module. The following are equivalent for a sequence $\underline{f} = (f_1, \ldots, f_r)$ in $\mathfrak{m}$:*

(a) $\underline{f}$ *is an* M*-regular sequence.*
(b) $H^p(\mathrm{Kos}_R(\underline{f}, M)) = 0$ *for* $p \neq 0$.
(c) $H^{-1}(\mathrm{Kos}_R(\underline{f}, M)) = 0$.

Proof. $(a) \Rightarrow (b)$ The relevant complex is

$$M \otimes^{\mathbf{L}}_R \mathrm{Kos}(f_1) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathrm{Kos}(f_r),$$

and we observe inductively that

$$M \otimes^{\mathbf{L}}_R \mathrm{Kos}(f_1) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R \mathrm{Kos}(f_i) = M/(f_1, \ldots, f_i)M.$$

Indeed for $i = 1$ this is $M \xrightarrow{f_1} M \simeq M/f_1M$ because f_1 is regular on M, and the inductive step is because

$$M/(f_1, \ldots, f_{i-1})M \otimes^{\mathbf{L}}_R \mathrm{Kos}(f_i) = \big[M/(f_1, \ldots, f_{i-1})M \to M/(f_1, \ldots, f_{i-1})M\big]$$

and f_i is regular on this quotient. Thus this is $M/(f_1, \ldots, f_i)M$, and taking $i = r$ gives the result.

$(c) \Rightarrow (a)$ This is also by induction on r, noting that f_1 is regular on M iff $M \xrightarrow{f_1} M$ is exact at the first spot. Let $\underline{f} = (f_1, \ldots, f_{r-1})$, and consider the exact triangle

$$\mathrm{Kos}(\underline{f}; M) \xrightarrow{f_r} \mathrm{Kos}(\underline{f}; M) \to \mathrm{Kos}(\underline{f}, f_r; M) \to \mathrm{Kos}(\underline{f}; M)[1].$$

Applying H^*, the assumption gives the exact sequence

$$H^{-1}(\mathrm{Kos}(\underline{f}; M)) \xrightarrow{f_r} H^{-1}(\mathrm{Kos}(\underline{f}; M)) \to 0,$$

and since the terms of the Koszul complex are finitely generated, the homology is too, so $H^{-1}(\mathrm{Kos}(\underline{f}; M)) = f_r H^{-1}(\mathrm{Kos}(\underline{f}; M))$ implies $H^{-1}(\mathrm{Kos}(\underline{f}; M)) = 0$ by Nakayama's lemma. Thus $(f_1, \ldots, f_{r-1})$ is M-regular by induction. The assumption also gives the exact sequence

$$0 \to H^0(\mathrm{Kos}(\underline{f}; M)) \xrightarrow{f_r} H^0(\mathrm{Kos}(\underline{f}; M)),$$

meaning f_r is injective on $M/(f_1, \ldots, f_{r-1})M$, completing the proof. □

This, for instance, shows that a sequence being M-regular (under these assumptions) is independent of the chosen order. Note that in the proof, we used Noetherianity to say that the cohomology of a complex with finitely generated terms is finitely generated. This is because the submodule of cycles is finitely generated. We used locality and finiteness to apply Nakayama's lemma.

If $(R, \mathfrak{m})$ is Noetherian local and M is finite, we define $\mathrm{depth}_R(M)$ to be the supremum of lengths of M-regular sequences. Once again, to prove basic properties about depth we will want a cohomological characterization of this. In the proof, we will use the following observation:

Remark 3.1.2. Consider $(R, \mathfrak{m}, k)$. Observe that $\mathfrak{m}$ acts trivially on the object $\mathrm{RHom}_R(k, B) \in D(R)$, and hence that it is canonically an object of $D(k)$. Indeed, the induced multiplication action by an element $f \in \mathfrak{m}$ is obtained either by

applying $\mathrm{RHom}_R(-, B)$ to $k \xrightarrow{f} k$ or $\mathrm{RHom}_R(k, -)$ to $B \xrightarrow{f} B$, yet f acts by zero on k.

We will use finiteness of depth in the proof, and then prove this afterwards.

Lemma 3.1.3. *Let $(R, \mathfrak{m}, k)$ be Noetherian local and M finite. Suppose $f_1, \dots, f_r$ is a maximal M-regular sequence. Then*

$$r = \mathrm{depth}_R(M) = \inf\{i : \mathrm{Ext}^i_R(k, M) \neq 0\}.$$

In particular, all maximal M-regular sequences have the same length, and any M-regular sequence can be extended to one of length $\mathrm{depth}_R(M)$.

Proof. For an R-module N, write $s(N) = \inf\{i : \mathrm{Ext}^i_R(k, N) \neq 0\}$. Suppose $f \in \mathfrak{m}$ is an N-regular element; then we have the exact sequence

$$0 \to N \xrightarrow{f} N \to N/fN \to 0,$$

and thus $0 \to \mathrm{Ext}^i(k, N) \to \mathrm{Ext}^i(k, N/fN) \to \mathrm{Ext}^{i+1}(k, N) \to 0$, where the zeroes are due to the remark above the proof. In particular, it is clear that

$$s(N/fN) = s(N) - 1.$$

Therefore $s(M/(f_1, \dots, f_r)M) = s(M) - r$ by induction and M-regularity, thus to prove $r = s(M)$ it suffices to show that

$$\mathrm{Hom}_R(k, M/(f_1, \dots, f_r)M) \neq 0.$$

Indeed, by maximality, every element of $\mathfrak{m}$ acts as a zero-divisor on $Q = M/(f_1, \dots, f_r)M$, which means that $\mathfrak{m} \subset \bigcup \mathrm{Ass}_R(Q)$. By finiteness and prime avoidance, $\mathfrak{m}$ is contained (hence equal) to an associated prime, so there exists $x \neq 0 \in Q$ such that $\mathfrak{m}x = 0$. This gives a nonzero map $R \to Q$ factoring through k.

Since $\mathrm{depth}_R(M) = d < \infty$, there exists a regular sequence of length d. Of course, it must be maximal by definition, and hence $d = s(M)$, so $d = r$. The claim about extending a regular sequence follows from the fact that we can always extend a regular sequence to a maximal one (e.g. by finiteness of depth), and then it has length equal to $\mathrm{depth}_R(M)$. □

Example 3.1.4. If $(R, \mathfrak{m}, k)$ is Noetherian local, then $\mathrm{depth}_R(k) = 0$. Indeed, there are no k-regular sequences because there are no k-regular elements of $\mathfrak{m}$.

Let us now review some basic dimension theory over Noetherian local rings. Recall that $\mathrm{ht}(\mathfrak{p}) = \dim(R_\mathfrak{p})$ denotes the height of a prime, and in particular $\mathrm{ht}(\mathfrak{m}) = \dim(R_\mathfrak{m}) = \dim(R)$. We use the somewhat misleading notation $\dim_R(M)$ for $\dim \mathrm{Supp}\, M$, only for this section. Recall that $\mathrm{Supp}\, M = V(\mathrm{Ann}_R(M))$ for M a finite module, and that the spaces in question are Noetherian and thus have finitely many minimal primes.

Let $(R, \mathfrak{m})$ be Noetherian local. Then $\mathfrak{m}$ is finitely generated, so by Krull's height theorem $\mathrm{ht}(\mathfrak{m}) = \dim(R) < \infty$. Hence for all modules M, $\dim_R M \leq$

$\dim(R) < \infty$. If $\dim(R) = d$, we define a *system of parameters for* R to be a sequence $f_1, \ldots, f_d \in \mathfrak{m}$ of length d such that

$$\sqrt{(f_1, \ldots, f_d)} = \mathfrak{m},$$

i.e. $V(f_1, \ldots, f_d) = \{\mathfrak{m}\}$.

Since $\{\mathfrak{m}\} \subset V(f_1, \ldots, f_d)$ always holds, and $\{\mathfrak{m}\}$ is a closed irreducible subset, this is equivalent to asking

$$\dim V(f_1, \ldots, f_d) = \dim R/(f_1, \ldots, f_d) = 0.$$

We can use this characterization to easily show existence:

Lemma 3.1.5. *Let* $(R, \mathfrak{m})$ *be Noetherian local with* $\dim(R) > 0$. *Then* R *has a system of parameters.*

Proof. Since $\operatorname{Spec} R$ has finitely many components, there are finitely many (but > 0) minimal primes $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = d$. $\mathfrak{m}$ can't be in the union of all such primes, since otherwise by prime avoidance we would have $0 = \dim R/\mathfrak{m} = d$. Thus choose $f_1 \in \mathfrak{m}$ avoiding all such minimal primes. Then $\dim(R/f_1) \leq d - 1$, so by induction choose $\overline{f}_2, \ldots, \overline{f}_d \in R/(f_1)$ such that

$$0 = \dim(R/(f_1))/(\overline{f}_2, \ldots, \overline{f}_d),$$

then the RHS is equal to $R/(f_1, \ldots, f_d)$. □

Lemma 3.1.6. *Let* $(R, \mathfrak{m})$ *be Noetherian local. If* $I \subset R$ *is generated by* r *elements, then* $\dim(R/I) \geq \dim R - r$.

Proof. Suppose $\dim(R/I) = e$ and choose a system of parameters $\overline{y}_1, \ldots, \overline{y}_e$ for R/I such that $\sqrt{(\overline{y}_1, \ldots, \overline{y}_e)} = \mathfrak{m}/I$, which is allowed to be empty if $\dim(R/I) = 0$. Then $\sqrt{(I, y_1, \ldots, y_e)} = \mathfrak{m}$ in R, so $\mathfrak{m}$ is minimal over the ideal $(I, y_1, \ldots, y_e)$ which has $e + r$ generators. Thus $\operatorname{ht}(m) = \dim(R) \leq e + r$ by Krull's height theorem. □

Corollary 3.1.7. *Let* $(R, \mathfrak{m})$ *be Noetherian local and* M *finite and consider a sequence* $f_1, \ldots, f_r \in \mathfrak{m}$.

(a) Then $\dim_R M/(f_1, \ldots, f_r)M \geq \dim_R M - r$.

(b) If $\underline{f} = (f_1, \ldots, f_r)$ *is* M*-regular, then in fact* $\dim_R M/(f_1, \ldots, f_r)M = \dim_R M - r$.

(c) We have $\operatorname{depth}_R(M) \leq \dim_R(M) \leq \dim(R) \leq \dim_k \mathfrak{m}/\mathfrak{m}^2 < \infty$.

Proof. For (a) and (b), it is clear that it suffices to prove the case $r = 1$. For (c), note that by (b) the length of a regular sequence is bounded above by $\dim_R(M) - \dim_R(M/(f_1, \ldots, f_r)M) \leq \dim_R(M)$. We already know the other inequalities.

Let $f \in \mathfrak{m}$, and note that $\operatorname{Supp} M/fM = \operatorname{Supp} M \cap V(f)$. The $\subseteq$ direction is clear, and if $M_\mathfrak{p} \neq 0$ yet $f \in \mathfrak{p}$, then $M_\mathfrak{p} \neq fM_\mathfrak{p}$ by Nakayama's lemma.

(a) Since $\operatorname{Supp} M = V(\operatorname{Ann}(M))$, $\dim_R M$ is the maximum of dimensions of $A/\mathfrak{p}$ for minimal $\mathfrak{p} \in V(\operatorname{Ann}(M))$. Choose such a minimal prime $\mathfrak{p}$ with $\dim R/\mathfrak{p} =$

$\dim_R(M)$, and suppose $f \in \mathfrak{m}$. $V(\mathfrak{p}) \subset \operatorname{Supp} M$ implies $V(\mathfrak{p}, f) \subset \operatorname{Supp} M/fM$, i.e. $\dim R/(\mathfrak{p}, f) \leq \dim_R(M/fM)$. However

$$\dim(R/(\mathfrak{p}, f)) \geq \dim(R/\mathfrak{p}) - 1 = \dim_R(M) - 1$$

by the previous lemma.

(b) Now assume f is M-regular, then f avoids all the associated primes of M. In particular, this includes the minimal primes of $\operatorname{Supp} M$ (which are always associated). Take $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_e$ a chain of primes in $\operatorname{Supp} M/fM$ with $e = \dim_R M/fM$. In particular this is also a chain in $\operatorname{Supp} M$, yet $\mathfrak{q}_0$ cannot be minimal there since $f \in \mathfrak{q}_0$. Thus $\dim_R M \geq \dim_R M/fM + 1$. □

A finite R-module M is *Cohen-Macaulay* if $\operatorname{depth}_R(M) = \dim_R(M)$. We also say the ring R is Cohen-Macaulay if it is Cohen-Macaulay over itself. If M is Cohen-Macaulay, let $f_1, \ldots, f_d$ be a regular sequence with $d = \dim_R(M)$, then $\dim_R M/(f_1, \ldots, f_r)M = 0$. Conversely, suppose there is an M-regular sequence $f_1, \ldots, f_r$ with $\dim_R M/(f_1, \ldots, f_r)M = 0$. Then $r = \dim_R M$, yet also $\operatorname{depth}_R(M) \geq r$, hence $\dim_R(M) = \operatorname{depth}_R(M)$ by (c), so M is Cohen-Macaulay.

Let A be a general ring and M be a nonzero A-module. Recall that we say M has *finite projective dimension* if the functors $\operatorname{Ext}^p(M, -)$ are zero for $p \gg 0$, and in this case we write $\operatorname{pd}_A(M)$ for the largest p such that $\operatorname{Ext}^p(M, -)$ is nonzero. This makes sense because $\operatorname{Hom}_A(M, -)$ is nonzero if M is nonzero.

Lemma 3.1.8. *Let A be a ring and $M \neq 0$ an A-module. The following are equivalent.*

(a) $\operatorname{pd}_A(M) \leq d$.
(b) M has a length d projective resolution.
(c) For any resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

by projective A-modules, $\ker(P_d \to P_{d-1})$ *is projective. In particular,*

$$0 \to \ker(P_d \to P_{d-1}) \to P_{d-1} \to \cdots P_0 \to M \to 0$$

is a length d projective resolution.

Proof. It is clear that $(c) \Rightarrow (b)$ by applying it to any projective resolution, and $(b) \Rightarrow (a)$ by computing Ext's on the given length d resolution. Assume (a), and split into short exact sequences

$$0 \to K_{i+1} \to P_i \to K_i \to 0$$

where $K_0 = M$. Then $\ker(P_d \to P_{d-1}) = K_d$. We claim that $\operatorname{pd}_A(K_i) = d - i$, which will show K_d has all higher vanishing Ext's and thus is projective. Indeed, our assumption is the base case $i = 0$, and otherwise the long exact sequence from the sequence above proves the induction hypothesis. □

Now return to our assumption that $(R, \mathfrak{m})$ is Noetherian local. Assume M is a finite R-module. Then we know that we can choose a free resolution of M by finite frees; indeed, once $F_r \to \cdots F_0 \to M \to 0$ is chosen with F_i finite free, the

kernel $\ker(F_r \to F_{r-1})$ is finite since R is Noetherian, and hence we can choose a surjection from another finite free module.

In light of this, the previous lemma, and the fact that finite projective modules over a local ring are free, this shows that $\mathrm{pd}_R(M) < \infty$ iff M has a finite resolution by finite frees, and $\mathrm{pd}_R(M)$ is the minimal length of one. A good question to ask is whether, and in what sense, we can choose these resolutions to be minimal in other ways (e.g. if the ranks in the resolution can be taken minimal).

Let M be a finite module and $F \twoheadrightarrow M$ a surjection from a finite free module. It is called a *minimal free cover* if $\ker(F \to M) \subset \mathfrak{m}F$. The following are equivalent for a surjection $\phi\colon F \to M$ with F finite free:

(a) ϕ is a minimal free cover.
(b) ϕ induces an isomorphism $F/\mathfrak{m}F \to M/\mathfrak{m}M$ of k-modules.
(c) $\operatorname{rank} F = \mu(M)$, i.e. the size of a minimal generating set of M.

Proof: $(a) \Leftrightarrow (b)$ Consider the surjection $F/\mathfrak{m}F \to M/\mathfrak{m}M$, then its kernel is $(\ker(\phi) + \mathfrak{m}F)/\mathfrak{m}F$, which is zero iff $\ker\phi \subset \mathfrak{m}F$. $(b) \Leftrightarrow (c)$ A surjection of finite dimensional vector spaces is an isomorphism iff the dimensions of both sides are the same.

We remark that for any finite module M, a minimal free cover always exists, because we can just choose minimal generators and take a surjection from a free module of rank $\mu(M)$ mapping onto those generators. Next, if M is a finite module, a finite-free resolution

$$\cdots F_2 \xrightarrow{d} F_1 \xrightarrow{d} F_0 \to M \to 0$$

is *a minimal free resolution* if $d(F_i) \subset \mathfrak{m}F_{i-1}$ for all $i \geq 0$, treating $M = F_{-1}$. A minimal free resolution always exists: once $F_i \to \cdots \to F_0 \to M \to 0$ is constructed, consider the kernel in $0 \to K_i \to F_i \to F_{i-1}$ and take a minimal free cover $F_{i+1} \to K_i \to 0$. Then in the resulting free resolution, at each step we have

$$\operatorname{im}(F_{i+1} \to F_i) = \ker(F_i \to F_{i-1}) = \ker(F_i \to K_{i-1}) \subset \mathfrak{m}F_i$$

as desired. The first observation is that the resolution $F^\bullet \to M[0]$ is minimal iff when we tensor to k the resulting augmented sequence has no differentials. In particular, this means that

$$\operatorname{rank}_k F_i = \dim_k \operatorname{Tor}_i^R(M,k) \qquad i \geq 0,$$

where we are viewing $\operatorname{Tor}_i^R(M,k)$ as a k-module because $M \otimes^{\mathbf{L}}_R k$ can be viewed before or after restriction of scalars. We define $\beta_i^R(M) \coloneqq \dim_k \operatorname{Tor}_i^R(M,k)$ to be the *Betti numbers* of the module M, which coincide with the ranks of the free modules in any minimal free resolution.

For example, this means that $\beta_0^R(M) = \mu(M)$ is the size of a minimal generating set. Moreover, suppose $\mathrm{pd}_R(M) = d$. Then any minimal free resolution has length at least d. Next, clearly M has tor-dimension $\leq d$ because we can use a free resolution of length d to compute tors. Thus, fixing $\mathcal{F}^\bullet \to M$ a minimal free resolution, the length of $\mathcal{F}^\bullet$ is $\geq d \geq$ the largest nonzero Betti number. But the length of $\mathcal{F}^\bullet$ equals the largest nonzero Betti number. Therefore every minimal free resolution has length $\mathrm{pd}_R(M)$, coinciding with the top Betti number. In

particular, the tor amplitude of M equals $[-\operatorname{pd}_R(M), 0]$, and this is detected by simply tensoring with the single module k.

An interesting consequence of this is that to figure out the projective dimension and Betti numbers of any finite module M, we can just find a free resolution of k and tensor M with this. This is an indication that the R-module k controls the homological invariants of all finite R-modules, as we will see shortly.

Remark 3.1.9. It is easy to prove that any free resolution contains a minimal free resolution as a summand (by working inductively and noting that if a matrix coefficient of a differential contains a unit, then it can be split off in an $R \xrightarrow{1} R$ summand). It is also easy to show that any two minimal free resolutions are isomorphic as complexes, namely by taking a homotopy equivalence and noting that it induces an isomorphism after tensoring with k. But we won't worry about these details because we don't need them.

Remark 3.1.10. Note that M need not be finite to do most of the above; namely, if M is any R-module then we can construct a minimal free resolution with the same property, except that the free modules may not have finite rank. The argument above still says that the top cohomology of $M \otimes^{\mathbf{L}}_R k$ still determines the projective dimension = tor-dimension.

Our next goal is to prove the Auslander-Buchsbaum formula, which is quite easy based on what we have said. We just need to write down some more homological characterizations of depth and the result will immediately follow based on two of our spectral sequences.

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring and suppose $\mathfrak{m} = (f_1, \dots, f_e)$ is a minimal generating set of the maximal ideal, i.e. a basis for $\mathfrak{m}/\mathfrak{m}^2$ over k. For a finite module M, fix the notation

$$K(M)^\bullet \coloneqq \operatorname{Kos}^\bullet_R(f_1, \dots, f_e; M)$$

for the corresponding Koszul complex. So, $H^0(K(M)) = M/\mathfrak{m}M$, and since $H^p(K(M))$ is annihilated by $\mathfrak{m}$, we can view it as a finite-dimensional k-vector space, and it is isomorphic as an R-module to k^ℓ for some ℓ. Note that $K(M) = K(R) \otimes_R M = K(R) \otimes^{\mathbf{L}}_R M$. Likewise, by Koszul self-duality, we have

$$\operatorname{RHom}_R(K(R), M) = \underline{\operatorname{Hom}}_R(K(R), M) = K(R)^\vee \otimes_R M = K(M)[-e].$$

The idea is to apply the hyper-ext and hyper-tor spectral sequences to these two descriptions of $K(M)$, which we proved in Example 2.2.11 and Example 2.2.12. They say that

$$E_2^{p,q} = \operatorname{Ext}^p(H^{-q}(K(R)), M) \implies H^{p+q}(K(M)[e]) = H^{p+q-e}(K(M))$$

and

$$E_2^{p,q} = \operatorname{Tor}^p(H^q(K(R)), M) \implies H^{p+q}(K(M)).$$

Let $n_q \coloneqq \dim_k H^{-q}(K(R)) \cong k^{n_q}$. Then the first sequence is

$$\operatorname{Ext}^p(k, M)^{n_q} \implies H^{p+q-e}(K(M)).$$

We showed that $d = \operatorname{depth}_R(M)$ is the smallest p for which $\operatorname{Ext}^p(k, M) \neq 0$, and we know that $n_0 = 1$. Therefore, the first nonzero cohomology group $H^{p+q-e}(K(M))$, i.e. the one where $p+q$ is smallest, is the diagonal where $p+q = d+0 = d$, and we have precisely $H^{-(e-d)}(K(M)) = \operatorname{Ext}^d(k, M)$. Since $K(M)$ is indexed in negative degrees, this is the left-most nonzero cohomology group of $K(M)$. In other words:

Lemma 3.1.11. *Let $(R, \mathfrak{m}, k)$ be Noetherian local and M a finite module. Then*

$$\mu(\mathfrak{m}) - \operatorname{depth}_R(M) = \sup\{q : H^{-q}(\operatorname{Kos}_R^\bullet(\underline{f}, M)) \neq 0\},$$

where $\underline{f}$ is any minimal generating set for $\mathfrak{m}$. Specifically, we have

$$H^{-q}(\operatorname{Kos}_R^\bullet(\underline{f}, M)) = \operatorname{Ext}^d(k, M)$$

for $q = \mu(\mathfrak{m}) - \operatorname{depth}_R(M)$.

In other words, this computes the depth of a module in terms of the top nonzero Koszul homology with respect to generators of the maximal ideal.

THEOREM 3.1.12. *Let $(R, \mathfrak{m})$ be Noetherian local and M a finite R-module. Then*

$$\operatorname{pd}_R(M) + \operatorname{depth}_R(M) = \operatorname{depth}(R)$$

whenever $\operatorname{pd}_R(M) < \infty$.

PROOF. Keep $e = \mu(\mathfrak{m})$, $n_q = \dim_k H^{-q}(K(R))$, and set $d_M = \operatorname{depth}_R(M)$, $d_R = \operatorname{depth}(R)$, and $r = \operatorname{pd}_R(M)$. The lemma shows that $n_q = 0$ for $q > e - d_R$, and the hyper-tor spectral sequence says that

$$E_2^{-p,-q} = \operatorname{Tor}_p^R(k, M)^{n_q} \implies H^{-p-q}(K(M)).$$

Thus $\dim_k E_2^{-p,-q} = n_q \beta_p^R(M)$, and thus the bottom-left corner of this spectral sequence looks like:

$$\begin{array}{cccc} 0 & 0 & n_{e-d_R+1}\beta_r^R(M) & n_{e-d_R+1}\beta_{r-1}^R(M) \\ 0 & 0 & n_{e-d_R}\beta_r^R(M) & n_{e-d_R}\beta_{r-1}^R(M) \\ 0 & 0 & 0 & 0 \end{array}$$

In particular the term $n_{e-d_R}\beta_r^R(M)$ survives to E_∞ and coincides with the left-most nonzero homology of $K(M)$. In other words,

$$e - d_R + r = e - d_M \implies d_R = d_M + r,$$

as desired. □

Remark 3.1.13. We can also extract the following numerical identity from the proof. Since the number n_{e-d_R} coincides with $\dim_k \operatorname{Ext}_R^{d_R}(k, R)$, we have the formula

$$\beta^R_{\operatorname{pd}_R(M)}(M) \cdot \left(\dim_k \operatorname{Ext}_R^{\operatorname{depth}(R)}(k, R)\right) = \dim_k \operatorname{Ext}_R^{\operatorname{depth}_R(M)}(k, M)$$

which allows us to compute the top nonzero Koszul homology in terms of the same invariant for R itself and the Betti numbers of M.

Example 3.1.14. In the above result, the Auslander-Buchsbaum formula, it is crucial that we assume M has finite projective dimension, since $\operatorname{depth}(R)$ is always finite and thus it cannot hold in the infinite case. E.g., we saw in the example of $R = k[\![x]\!]/(x^2)$ that $\operatorname{depth}(R) = \dim(R) = 0$, yet $\operatorname{pd}_R(k) = \infty$.

We will spend the rest of this section discussing regular local rings. We will focus on the homological algebra characterizations of regularity since they are a bit easier to work with abstractly. At the end we will prove that this is equivalent to the standard definition of regularity.

Remark 3.1.15. Let $(R, \mathfrak{m})$ be Noetherian local and suppose $f_1, \dots, f_e$ and $g_1, \dots, g_e$ are two minimal generating sets for $\mathfrak{m}$, i.e. two bases for $\mathfrak{m}/\mathfrak{m}^2$. Then we can choose a change-of-basis matrix between the two bases, and lift it to R to get a map $R^e \to R^e$. Note that this is an isomorphism after tensoring with $R/\mathfrak{m}$, hence its determinant is nonzero mod $\mathfrak{m}$, i.e. a unit, so it is invertible. This matrix (a_{ij}) satisfies $\sum_i a_{ij} f_i = g_j$, and thus

$$\operatorname{Kos}^\bullet_R(f_1, \dots, f_r) \cong \operatorname{Kos}^\bullet_R(g_1, \dots, g_r)$$

by a result two sections ago. In particular, $\operatorname{Kos}^\bullet_R(\underline{f}; M)$ doesn't depend on the choice of minimal generators. So for M finite, we write

$$h^R_q(M) \coloneqq \dim_k H^{-q}(\operatorname{Kos}^\bullet_R(\underline{f}; M)) < \infty$$

for the corresponding dimension of the Koszul homology, independent of $\underline{f}$.

The key step in proving the main characterization of regular local rings is the following lemma, which is a simple consequence of the formula we just proved. We will use the following fact: If $I \subset R$ is an ideal, and M is an R/I-module, then $\operatorname{depth}_{R/I}(M) = \operatorname{depth}_R(M)$. Indeed, it is easy to see that a sequence $f_1, \dots, f_r \in \mathfrak{m}$ is M-regular iff $\overline{f}_1, \dots, \overline{f}_r \in \mathfrak{m}/I$ is M-regular.

Lemma 3.1.16. *Let $(R, \mathfrak{m})$ be Noetherian local and let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a regular element. If $M \neq 0$ is a finite $R/(f)$-module with $\operatorname{pd}_R(M) < \infty$, then*

$$\operatorname{pd}_{R/(f)}(M) = \operatorname{pd}_R(M) - 1.$$

In particular, $\operatorname{pd}_{R/(f)}(M) < \infty$ as well. Necessarily $\operatorname{pd}_R(M) \geq 1$.

Proof. Note that $f \neq 0$, and thus M cannot be free over R since it has nontrivial annihilator, so $\operatorname{pd}_R(M) \geq 1$. Note that M is also finite over R. Take a minimal free cover

$$(*) \qquad 0 \to K \to F \to M \to 0$$

over R, so F is finite free and $K \subset \mathfrak{m}F$. Since $fF \mapsto fM = 0$, $fF \subset K$. Letting $L = K/fF$ (as an $R/(f)$-module), this gives an exact sequence

$$0 \to fF \to K \to L \to 0$$

of R-modules, and an exact sequence

$$0 \to L \to F/fF \to M \to 0$$

of $R/(f)$-modules. We prove the claim by induction on $\mathrm{pd}_R(M) \geq 1$.

First assume $\mathrm{pd}_R(M) = n > 1$. Since F is free, $\mathrm{pd}_R(K) = n - 1$ by LES in Ext's, and since $fF \cong F$ over R (as f is R-regular), $\mathrm{pd}_R(L) \leq n - 1$. Thus the inductive hypothesis gives in particular $\mathrm{pd}_{R/(f)}(L) < \infty$, and then since $F/fF = F \otimes_R R/(f)$ is free over $R/(f)$, this means $\mathrm{pd}_{R/(f)}(M) < \infty$.

Now we may apply Auslander-Buchsbaum to M over both R and $R/(f)$. By the remark preceding the proof,

$$\mathrm{depth}_R(M) = \mathrm{depth}_{R/(f)}(M) \qquad \mathrm{depth}(R) = \mathrm{depth}(R/(f)) + 1,$$

where the second claim may be checked as R-modules, where it is clear because (f) is R-regular. Therefore the formula shows $\mathrm{pd}_{R/(f)}(M) = \mathrm{pd}_R(M) - 1$.

Now suppose $\mathrm{pd}_R(M) = 1$. Then K is finite free. If we localize $(*)$ at f, then $M_f = 0$ implies that $K_f \xrightarrow{\sim} F_f$, thus K, F are free of the same rank. Let $e_1, \ldots, e_r \in F$ be a basis, and consider the elements $\{fe_1, \ldots, fe_r\} \subset fF \subset K$. We claim that these are linearly independent in $K/\mathfrak{m}K$. This will finish the proof, because then this is a basis, and hence a minimal generating set of K. But then $fF = K$, so $L = 0$, and $M = F/fF$ is already free over $R/(f)$.

To prove the claim, suppose $c_i \in R$ are such that $\sum_i c_i f e_i \in \mathfrak{m}K \subset \mathfrak{m}^2 F$. Since $F = \bigoplus_i Re_i$, this means that $c_i f \in \mathfrak{m}^2$ for all i. If $c_i \notin \mathfrak{m}$ for any i, then it would be invertible in R and thus $f \in \mathfrak{m}^2$, a contradiction. Thus $c_i \equiv 0 \bmod \mathfrak{m}$ proving linear independence. □

We use this to prove the main characterization of regular local rings, called the Auslander-Buchsbaum-Serre (ABS) theorem. Recall that if A is a Noetherian ring, we write

$$\mathrm{gl.dim}(A) \coloneqq \sup\{\mathrm{pd}_A(M) : M \text{ is an } A\text{-module}\}$$

for the *global dimension* of the ring A. The condition in the lemma above, namely that $f \in \mathfrak{m} \setminus \mathfrak{m}^2$, is not hard to satisfy. Indeed, suppose $\mathrm{depth}(R) > 0$; then $\mathfrak{m} \neq \mathfrak{m}^2$ (otherwise $\mathfrak{m} = 0$), and $\mathfrak{m}$ is not an associated prime, as otherwise all elements would be zerodivisors. Therefore

$$\mathfrak{m} \nsubseteq \mathfrak{m}^2 \cup \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R)} \mathfrak{p}$$

by prime avoidance (recall that we can take up to two of the ideals in the avoidance lemma to be non-prime). In particular, there is an element $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ which avoids the associated primes, i.e. which is a regular element.

THEOREM 3.1.17. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. The following are equivalent:*

(i) $\mathfrak{m}$ *is generated by a regular sequence.*
(ii) Every minimal generating set of $\mathfrak{m}$ *forms a regular sequence.*
(iii) $\mathrm{depth}(R) = \mu(\mathfrak{m})$.
(iv) $h_1^R(R) = 0$

(v) $h_i^R(R) = 0$ *for* $i > 0$.
(vi) $\mathrm{pd}_R(k) < \infty$.
(vii) $\mathrm{gl.dim}(R) < \infty$.
(viii) $\mathrm{gl.dim}(R_\mathfrak{p}) < \infty$ *for each* $\mathfrak{p} \in \operatorname{Spec} R$.

Moreover, in this case, the equalities

$$\mu(\mathfrak{m}) = \mathrm{pd}_R(k) = \mathrm{depth}(R) = \mathrm{gl.dim}(R) = \dim(R)$$

all hold.

Definition 3.1.18. A Noetherian local ring R satisfying the equivalent conditions of Thm. 3.1.17 is called a *regular local ring*. A locally Noetherian scheme X is *regular* if $\mathcal{O}_{X,x}$ is a regular local ring for each $x \in X$. A Noetherian ring A is a *regular ring* if $\operatorname{Spec} A$ is a regular scheme.

Proof (of Thm. 3.1.17). Clearly $(viii) \Rightarrow (vii)$ (take $\mathfrak{p} = \mathfrak{m}$) and $(vii) \Rightarrow (vi)$, so we show $(vi) \Rightarrow (viii)$. Note that $n = \mathrm{pd}_R(k)$ equals the tor-dimension of k, hence the tor dimension of $k_\mathfrak{p} = k$ over $R_\mathfrak{p}$ is $\leq n$ for all $\mathfrak{p}$. Thus $\mathrm{Tor}_p^{R_\mathfrak{p}}(M, k) = 0$ for $p \geq n$ and all $A_\mathfrak{p}$-modules M, so $\mathrm{pd}_{R_\mathfrak{p}}(M) \leq n$. In particular $\mathrm{gl.dim}(R_\mathfrak{p}) \leq n$. As a special case $\mathrm{gl.dim}(R) \leq n$, but obviously the other direction holds so $\mathrm{gl.dim}(R) = n$, which is one of the equalities at the end.

It is clear from Remark 3.1.15 that $(v) \Rightarrow (ii) \Rightarrow (i)$. If $\mathfrak{m}$ is generated by a regular sequence, then $\mathrm{depth}(R) \geq \mu(m)$, but we always have $\mathrm{depth}(R) = \mu(m)$ so these two coincide. Since $\dim(R)$ is in between these two numbers, this gives the equality of the three quantities. In particular, a regular sequence generating $\mathfrak{m}$ is a minimal generating set and thus (v) holds. Note that this implies (vi), because the Koszul complex on these generators gives a finite free resolution of k by (v).

Also, $(v) = (iv)$ because to check regularity of a sequence it suffices to show that H^{-1} of the Koszul complex is 0. Moreover, clearly $(ii) \Rightarrow (iii)$, and since the top nonzero Koszul homology of R equals $\mu(\mathfrak{m}) - \mathrm{depth}(R)$, it follows that $(iii) \Rightarrow (v)$.

Finally, assume (vi). Since $\mathrm{depth}_R(k) = 0$, Auslander-Buchsbaum implies that $\mathrm{depth}(R) = \mathrm{pd}_R(k)$, which completes the proof of all equalities at the end of the statement. We prove (i) by induction on $n = \mathrm{pd}_R(k)$. If $n = 0$, then k is free and hence has trivial annihilator, which forces $\mathfrak{m} = 0$. If $n > 0$, then $\mathrm{depth}(R) > 0$ and we may choose $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ a regular element. Since k is restricted from $R/(f)$, the previous lemma shows

$$\mathrm{pd}_{R/(f)}(k) = n - 1,$$

so by induction $\mathfrak{m}/(f) \subset R/(f)$ is generated by a regular sequence $\overline{f}_2, \ldots, \overline{f}_n$. Thus $(f, f_2, \ldots, f_n)$ is a regular sequence generating $\mathfrak{m}$. □

We will now use this criterion to develop some basic theory about regular local rings, with a focus on using the homological criterion we developed above. First we start with some basic applications to schemes.

Example 3.1.19. Let k be a field, and consider the Noetherian local ring

$$R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}.$$

Since $R/(x_n) = k[x_1, \ldots, x_{n-1}]_{(x_1,\ldots,x_{n-1})}$ is a domain, it follows inductively that $(x_1, \ldots, x_n)$ is a regular sequence in R. Thus R is a regular local ring. The same proof works if we take

$$R = k[\![x_1, \ldots, x_n]\!],$$

the corresponding completion.

Let X be a locally Noetherian scheme, and define

$$X_{\text{reg}} \coloneqq \{x \in X : \mathcal{O}_{X,x} \text{ is a regular local ring}\}.$$

In good cases, X_{reg} will be an open subset, but this is not true in general. But what is always true is the following:

Lemma 3.1.20. *Let X be a locally Noetherian scheme. Then X is regular iff $\mathcal{O}_{X,x}$ is regular for each closed point x.*

Proof. This follows because in a locally Noetherian scheme, every point specializes to a closed point, and a prime localization of a regular local ring is regular local by Thm. 3.1.17. □

We remark that the proof of points specializing to closed points in a locally Noetherian scheme is slightly nontrivial (but classical), however in the Noetherian case this is obvious because X is quasicompact.

Example 3.1.21. Let k be an algebraically closed field and $A = k[x_1, \ldots, x_n]$ the polynomial ring on n variables. Then A is a regular ring. Indeed, by the Nullstellensatz every closed point is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for $a_i \in k$, and the localization at this point is the Noetherian local ring

$$R = k[x_1, \ldots, x_n]_{(x_1-a_1,\ldots,x_n-a_n)} \cong k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}.$$

We saw in the previous example that this was a regular local ring. Thus by the lemma, A is regular.

Example 3.1.22. Let A be a regular ring and $S \subset A$ a multiplicative subset. Then $A[S^{-1}]$ is a regular ring. Indeed, all primes of $A[S^{-1}]$ are of the form $\mathfrak{p}A[S^{-1}]$ for some $\mathfrak{p} \subset A$ with $\mathfrak{p} \cap S = \varnothing$, and the corresponding localization coincides with $A_{\mathfrak{p}}$.

3.1.2. Regularity and smoothness. Our next goal is to give a criterion for regularity in terms of perfect objects of the derived category of a scheme. To do this, we want to understand pseudocoherent objects and finite tor dimension objects in $D_{\text{qc}}(X)$. For simplicity, let's assume X is Noetherian, so that all pseudocoherent objects are bounded above, and all perfect objects have finite tor dimension. It is easy to formulate a corresponding statement to anything we say for the locally Noetherian case.

Now, let $(R, \mathfrak{m})$ be a regular local ring of dimension d, so that $d = \text{gl.dim}(R)$. Since an individual module M consequently has tor amplitude in $[-d, 0]$, it follows that if $E \in D^{[a,b]}(R)$ with b possibly infinite, then E has tor amplitude in $[a - d, b]$.

To formulate the next result, for $E \in D(X)$ we define

$$\operatorname{Supp}(E) = \bigcup_n \operatorname{Supp} H^n(E).$$

In particular if $E \in D^b(X)$ has finite type cohomology (e.g. $E \in D^b_{\mathrm{coh}}(X)$), then $\operatorname{Supp}(E) \subset X$ is closed.

Lemma 3.1.23. *Let X be a Noetherian scheme and $E \in D^+_{\mathrm{qc}}(X)$. If $\operatorname{Supp}(E) \subset X_{\mathrm{reg}}$, then E has bounded-below tor amplitude. If $E \in D^b_{\mathrm{qc}}(X)$, then E has finite tor-dimension. More precisely, if $\dim(X) = n$, $\operatorname{Supp}(E) \subset X_{\mathrm{reg}}$, and $E \in D^{[a,b]}_{\mathrm{qc}}(X)$, then E has tor amplitude in $[a-n, b]$, where b is possibly ∞.*

Proof. We know that the tor dimension of E can be computed on the stalks. The assumption is that E_x is either 0 or $E_x \in D^{[a,b]}(\mathcal{O}_{X,x})$ where $\mathcal{O}_{X,x}$ is a regular local ring. In both cases, if $\dim(\mathcal{O}_{X,x}) = n_x$, we know that E_x has tor amplitude in $[a-n_x, b]$. If $n = \dim(X)$, then $n \geq n_x$ for all x, so the result follows. □

Now we can state the categorical criterion for perfectness. We note that this is especially interesting because we have seen that $\operatorname{Perf}(X) \subset D_{\mathrm{qc}}(X)$ coincides with the full triangulated subcategory of compact objects.

Proposition 3.1.24. *Let X be a Noetherian scheme. Then $\operatorname{Perf}(X) \subset D^b_{\mathrm{coh}}(X)$, and this inclusion is an equality iff X is regular.*

Proof. Since X is quasicompact, $\operatorname{Perf}(X) = \operatorname{Perf}^b(X)$, and by pseudocoherence it is contained in $D^b_{\mathrm{coh}}(X)$. The previous lemma supplies the ($\Leftarrow$) case. Conversely, let $\{x\}$ be a closed point of X with the reduced subscheme structure, and consider the coherent sheaf $\mathcal{O}_{\{x\}}$ on X. By assumption this has finite tor dimension, and since $\mathcal{I}_{\{x\},x} = \mathfrak{m}_x$, its stalk at x is

$$(i_*\mathcal{O}_{\{x\}})_x = \mathcal{O}_{X,x}/\mathcal{I}_{\{x\},x} = \kappa(x),$$

which thus has finite tor dimension (hence pd) as an $\mathcal{O}_{X,x}$-module. This implies $\mathcal{O}_{X,x}$ is a regular local ring, so we conclude by Lemma 3.1.20. □

In the remainder of this section, we prove some more classical results. To illustrate the power of the homological techniques, we will try to delay using any nontrivial commutative algebra until the final result on regular local rings. For instance, we will even avoid the Krull intersection theorem.

In the next two results we will use the following trivial case: If $f \in \mathfrak{m}$ is a regular element, then $\bigcap_n (f^n) = 0$. Indeed, let J be the intersection ideal, which is finitely generated by Noetherianity. Moreover, if $fz \in J$, then for each $n > 0$ we can write $fz = yf^n$, thus $z = yf^{n-1}$ for all n, i.e. $z \in J$. In other words $J = fJ$, so $J = 0$. Note this implies that for every $a \neq 0$, there is a unique largest $n \geq 0$ so that $a \in (f^n)$.

Lemma 3.1.25. *Let $(R, \mathfrak{m})$ be a regular local ring. Then R is a domain.*

Proof. Induct on $d = \dim(R) = \operatorname{pd}_R(k)$. We know that if $d = 0$ then $k = R$ is a field, so assume $d \geq 1$, then we have seen how to use Lemma 3.1.16 to choose

$f \in \mathfrak{m} \setminus \mathfrak{m}^2$ regular so that $R/(f)$ is a regular local ring with $\mathrm{pd}_{R/(f)}(k) = d-1$, hence $R/(f)$ is a domain.

Now let $x, y \neq 0 \in R$ such that $xy = 0$. Let $i, j \geq 0$ be maximal so that $x \in (f^i)$ and $y \in (f^j)$, so $x = f^i x_0$ and $y = f^j y_0$ with $x_0, y_0 \notin (f)$. Thus $f^{i+j} x_0 y_0 = 0$, so by regularity $x_0 y_0 = 0$. But reducing mod (f) implies that x_0 or y_0 is in (f), a contradiction. □

Remark 3.1.26. Recall that (e.g. proven in other notes) if X is locally Noetherian, connected, and pointwise a domain, then X is integral. Therefore a connected regular scheme is integral.

In the next proposition, we are using the convention that a DVR is a local PID which isn't a field.

Proposition 3.1.27. *Let $(R, \mathfrak{m})$ be Noetherian local. The following are equivalent:*

(a) R is a DVR.
(b) $\mathfrak{m}$ is generated by one nonzerodivisor.
(c) R is a regular local ring of dimension 1.

In this case, all nonzero ideals of R are of the form (π^n) for $n \geq 0$, where π is any element of $\mathfrak{m} \setminus \mathfrak{m}^2$.

Proof. By our definition, $(a) \Rightarrow (b)$. (b) implies that R is regular local, of dimension exactly one since the nonzerodivisor is in particular not 0. If (c) holds, then we know that R is a domain with $\mathfrak{m} = (\pi)$ for any $\pi \in \mathfrak{m} \setminus \mathfrak{m}^2$. We will show the final claim, which implies that R is a PID and hence (a).

As in the previous proof, for each x let $i \geq 0$ be largest so that $x \in (\pi^i)$, i.e. $x = \pi^i x_0$ with $x_0 \notin (\pi) = \mathfrak{m}$, which means $x_0 \in R^\times$, thus all elements are a unit times π^i for some i. Now let I be a nonzero ideal and let $x \in I$ be a nonzero element which minimizes the i for which $x \sim \pi^i$. Then $(x) = (\pi^i)$, but on the other hand every other $y \in I$ is a unit times π^k for $k \geq i$, so $y \in (\pi^i)$ as well. Thus $I = (\pi^i)$. □

Example 3.1.28. It follows that a PID, or more generally a Dedekind domain, is a regular ring.

Recall that the injective dimension $\mathrm{id}_A(M)$ of an A-module M is defined as the largest p for which $\mathrm{Ext}^p_A(-, M)$ is a nonzero functor. The same proof as the one we gave for projective dimension shows that $\mathrm{id}_A(M)$ coincides with the smallest length of an injective resolution of M. Since this only relies on $\mathrm{Ext}^{n+1}(-, M)$ being zero, this shows that (just as in the projective case) the first n for which $\mathrm{Ext}^{n+1}(-, M) = 0$ is the injective dimension.

Lemma 3.1.29. *Let A be a ring and $n \geq 0$. The following are equivalent:*

(a) $\mathrm{gl.dim}(A) \leq n$.
(b) $\mathrm{pd}_A(M) \leq n$ for all finitely generated modules M.
(c) $\mathrm{pd}_A(M) \leq n$ for all cyclic modules M.
(d) $\mathrm{id}_A(M) \leq n$ for all modules M.
(e) $\mathrm{Ext}^i_A(M, N) = 0$ for all $i > n$ and modules M, N.

(f) $\operatorname{Ext}_A^{n+1}(M,N) = 0$ *for all modules* M *and* N.

Proof. It is clear from our results on projective dimension that $(a) = (e) = (f)$, and it is clear from the description above that $(d) = (e)$. Obviously $(a) \Rightarrow (b) \Rightarrow (c)$, so the only interesting thing to prove is $(c) \Rightarrow (a)$. For this, we will prove a slightly more general statement and deduce the result from that:

Lemma 3.1.30. *Let* M *be an* A*-module equipped with a well-ordered filtration* $0 = M_0 \subset M_1 \subset \cdots \subset M_\alpha \subset \cdots \subset M_\lambda = M$ *satisfying the following properties:*

(a) For every limit ordinal δ, $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$.

(b) For each α, $\operatorname{pd}_A(M_{\alpha+1}/M_\alpha) \leq n$.

Then $\operatorname{pd}_A(M) \leq n$.

Proof. Write $Q_\alpha = M_{\alpha+1}/M_\alpha$; we induct on n. If $n = 0$, then each sequence $0 \to M_\alpha \to M_{\alpha+1} \to Q_\alpha \to 0$ splits. Letting $s_\alpha \colon Q_\alpha \to M_{\alpha+1}$ be a section, we will construct an isomorphism

$$\phi_\alpha \colon \bigoplus_{\beta<\alpha} Q_\beta \to M_\alpha$$

for all $\alpha \leq \lambda$ by transfinite recursion, then taking $\alpha = \lambda$ will show M is projective. At a successor stage, define

$$\phi_{\alpha+1} \colon (\bigoplus_{\beta<\alpha} Q_\beta) \oplus Q_\alpha \to M_\alpha \qquad (x,q) \mapsto \phi_\alpha(x) + s_\alpha(q),$$

then by construction of the s_α's, this is still an isomorphism. If α is a limit ordinal, then ϕ_α is defined using ϕ_β for all $\beta < \alpha$, which is well-defined since an element of the direct sum has only finitely many terms. It is clear that this remains injective, and surjectivity is by property (a).

Now let $n > 0$. Let $F_\alpha = A[M_\alpha]$, i.e. it is free with basis all elements of M_α. Then the filtration $(M_\alpha)_\alpha$ induces a filtration (F_α) which also satisfies $\bigcup_{\alpha<\delta} M_\alpha$ for limit ordinals. Then take the kernel $0 \to K_\alpha \to F_\alpha \to M_\alpha \to 0$, so it is clear that $(K_\alpha)_\alpha$ is a filtration with the same property. For each α, consider the quotient sequence

$$0 \to K_{\alpha+1}/K_\alpha \to F_{\alpha+1}/F_\alpha \to M_{\alpha+1}/M_\alpha \to 0.$$

By construction, the middle term is the free module with basis the set of elements in $M_{\alpha+1} \setminus M_\alpha$. Therefore $\operatorname{pd}_A(K_{\alpha+1}/K_\alpha) \leq n-1$. By induction, $\operatorname{pd}_A(K_\lambda) \leq n-1$. Then the exact sequence $0 \to K_\lambda \to F_\lambda \to M \to 0$ shows that $\operatorname{pd}_A(M) \leq n$. □

To apply this in the situation at hand, let M be an arbitrary module and assume that all cyclic modules have projective dimension $\leq n$. Choose a well-ordered generating set $\{x_\alpha : \alpha < \lambda\}$ of M by the axiom of choice, and define

$$M_\beta \coloneqq \sum_{\alpha<\beta} Ax_\alpha.$$

Then $M_{\beta+1}/M_\beta$ is cyclic generated by the image of x_β, and clearly $M_\delta = \bigcup_{\beta<\delta} M_\beta$ for limit ordinals δ. Thus applying the lemma to this filtration shows $\operatorname{pd}_A(M) \leq n$. □

Proposition 3.1.31. *Let A be a Noetherian ring of dimension n. Then A is a regular ring if and only if* $\operatorname{gl.dim}(A) < \infty$*, in which case* $\operatorname{gl.dim}(A) = n$.

Proof. By assumption $\dim(A_{\mathfrak{p}}) \leq n$ for $\mathfrak{p} \in \operatorname{Spec} A$. ($\Rightarrow$) It suffices to show that $\operatorname{pd}_A(M) \leq n$ for each finite module M. Since A is Noetherian, M is pseudocoherent, and therefore

$$\operatorname{Ext}^p_A(M,N)_{\mathfrak{p}} = \operatorname{Ext}^p_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}).$$

Since $A_{\mathfrak{p}}$ is regular local with $\operatorname{gl.dim}(A_{\mathfrak{p}}) \leq n$, it follows that the RHS above is 0 for all $p > n$ and all p. Thus $\operatorname{Ext}^p_A(M,N) = 0$ for all $p > n$ and N, as desired. Note that if we choose a prime $\mathfrak{m} \in \operatorname{Spec}(A)$ with $\dim(A_{\mathfrak{m}}) = n$ (necessarily maximal), then $\kappa(\mathfrak{m})$ as an $A_{\mathfrak{m}}$-module satisfies $\operatorname{Ext}^n_{A_{\mathfrak{m}}}(\kappa(\mathfrak{m}), A_{\mathfrak{m}}) \neq 0$ because $\operatorname{depth}(A_{\mathfrak{m}}) = n$, so viewing $\kappa(\mathfrak{m}) = A/\mathfrak{m}$ as a finite A-module we deduce that

$$\operatorname{Ext}^n_A(A/\mathfrak{m}, A) \neq 0$$

by the stalk formula above. Thus $\operatorname{gl.dim}(A) = n$.

($\Leftarrow$) Then every module has finite tor dimension, in particular $\kappa(\mathfrak{p})$ has finite tor dimension over A. Thus we know $\kappa(\mathfrak{p})_{\mathfrak{p}} = \kappa(\mathfrak{p})$ has finite tor (= projective) dimension over $A_{\mathfrak{p}}$, which implies $A_{\mathfrak{p}}$ is regular local. □

Example 3.1.32. We showed that if k is algebraically closed, then $k[x_1, \ldots, x_n]$ is a regular ring (of dimension n). Thus it has global dimension n. The proof shows that to achieve the supremum over projective dimension, we just take the residue field of a height n point. Thus, k with the trivial action has projective dimension n. We will generalize this shortly.

Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. To say more advanced things about R especially in the non-regular case, we can consider the collection of invariants that we have defined so far. Namely, we have $h^R_i(M)$ for the Koszul homology, $\beta^R_i(M)$ for the Betti numbers, and we have also been considering the Ext dimension, which we will now write as

$$\operatorname{ext}^i_R(k,M) \coloneqq \dim_k \operatorname{Ext}^i_R(k,M).$$

We have already seen various relations between these numbers, all of which came from the two spectral sequences discussed earlier. Let us go back and write down precisely what these two sequences say. Since we only care about these as k-vector spaces, we write the terms in the sequence as numbers. Writing $e = \mu(\mathfrak{m})$ for the embedding dimension, the two sequences had E_2-pages

$$E_2^{p,q} = \operatorname{ext}^p_R(k,M) h^R_q(R) \implies h^R_{e-p-q}(M)$$

and

$$E_2^{-p,-q} = \beta^R_p(M) h^R_q(R) \implies h^R_{p+q}(M)$$

for a finite module M. It is helpful to have the pictures in mind. We omit the letter R from the notation.

$$\begin{array}{cccc} \operatorname{ext}^d(k,M)h_m & \operatorname{ext}^{d+1}(k,M)h_m & \operatorname{ext}^{d+2}(k,M)h_m & \cdots \\ \vdots & \vdots & \vdots & \cdots \\ \operatorname{ext}^d(k,M)h_1 & \operatorname{ext}^{d+1}(k,M)h_1 & \operatorname{ext}^{d+2}(k,M)h_1 & \cdots \\ \operatorname{ext}^d(k,M)h_0 & \operatorname{ext}^{d+1}(k,M)h_0 & \operatorname{ext}^{d+2}(k,M)h_0 & \cdots \end{array}$$

Here, $d = \operatorname{depth}(M)$, since we know that d is the first nonzero Ext group, and $m = e - \operatorname{depth}(R)$ which is the top nonzero h_p. Note that $h_0 = 1$. Similarly, the other sequence is:

$$\begin{array}{cccc} \cdots & \beta_2(M)h_0 & \beta_1(M)h_0 & \beta_0(M)h_0 \\ \cdots & \beta_2(M)h_1 & \beta_1(M)h_1 & \beta_0(M)h_1 \\ \cdots & \vdots & \vdots & \vdots \\ \cdots & \beta_2(M)h_m & \beta_1(M)h_m & \beta_0(M)h_m \end{array}$$

Here, the top-right is $(0,0)$. The sequence may or may not terminate towards the left side if M has finite projective dimension. This is an extremely large amount of numerical information. We remark that in case R is regular local, both spectral sequences are trivial because $h_i = 0$ for $i > 0$, and so they just recover what we would already know from computing using the Koszul complex of the maximal ideal as a projective resolution for k.

Example 3.1.33. Suppose R satisfies the property that $\mu(\mathfrak{m}) = \operatorname{depth}(R) + 1$, which is not unusual; most codim one singularities will have this property, such as a singularity on a curve. Then both spectral sequences only have two rows because $m = 1$, so we can get long exact sequences in the classical way. Let's start with the top sequence. First of all, any differentials on E_3 will be zero, so $E_3 = E_\infty$. Based on this, e.g. in the first sequence, let

$$r_p \coloneqq \operatorname{rank}\left(\operatorname{ext}^p(k,M)h_1 \to \operatorname{ext}^{p+2}(k,M)\right)$$

be the rank of the only nontrivial differentials. Since $h_i(M)$ is zero after $i = e$, there will be a stable range after which we obtain a periodicity, and an unstable range. The definition immediately gives the following recurrence relation:

$$h_{e-p}(M) = \operatorname{ext}^p(k,M) + h_1 \operatorname{ext}^{p-1}(k,M) - r_{p-2} - r_{p-1},$$

where at the lower edge $h_{e-d}(M) = \operatorname{ext}^d(k,M)$. Here we are using that $r_{d-1} = r_{d-2} = 0$. The stable range is where $p \geq e$, in which case we obtain

$$\operatorname{ext}^{p+2}(k,M) = h_1 \operatorname{ext}^p(k,M),$$

i.e. there is eventually a 2-periodicity. Likewise, doing the same thing with the second spectral sequence, let s_p denote the corresponding nontrivial differentials. Then

$$h_p(M) = \beta_p(M) + h_1\beta_{p-1}(M) - s_{-p} - s_{-p-1},$$

with edge case $h_0(M) = \beta_0(M)$ as is already known. The stable range is $p \geq e - d$, where $\beta_{p+2}(M) = h_1\beta_p(M)$.

Example 3.1.34. Let Q be a regular local ring of dimension n, and $f \in Q$ an element in the maximal ideal. Then the Noetherian local ring $R = Q/(f)$ is called a *hypersurface* ring. Since Q is a domain, f is regular, thus $\operatorname{depth}(R) = n - 1$. We also know by our dimension result that $\dim(R) = n - 1$, so R is Cohen-Macaulay. Since $\mu(\mathfrak{m}_Q) = n$, either $\mu(\mathfrak{m}_R) = n$ or $n - 1$. In the latter case, this would imply R is regular local, so let's assume R is singular. Then $\mu(\mathfrak{m}_R) = \operatorname{depth}(R) + 1$, and we are in the situation of the example above.

Now since Q is regular, $\operatorname{Kos}^\bullet_Q(y_1, \ldots, y_n)$ is a free resolution for k over Q. Therefore the homology of

$$\operatorname{Kos}^\bullet_Q(y_1, \ldots, y_n) \otimes_Q R = \operatorname{Kos}^\bullet_R(\overline{y}_1, \ldots, \overline{y}_n)$$

computes $\operatorname{Tor}^Q_i(k, R)$, but by construction $\overline{y}_1, \ldots, \overline{y}_n$ is a minimal generating set for $\mathfrak{m}_R$, so it also computes $h^R_i(R)$. But now we have a length 1 free resolution $0 \to Q \xrightarrow{f} Q \to R \to 0$ of R over Q, and tensoring with k gives $k \xrightarrow{0} k$. In particular

$$h^R_0(R) = 1 \qquad h^R_1(R) = 1 \qquad h^R_i(R) = 0 \quad i \geq 0.$$

Thus in our previous computation we may take $h_1 = 1$, and we get literal 2-periodicity $\beta_{p+2}(M) = \beta_p(M)$ for all $p \geq n - \operatorname{depth}(M)$. We also obtain this in the ext groups $\operatorname{ext}^{p+2}(k, M) = \operatorname{ext}^p(k, M)$ for $p \geq n$. Let's consider two special cases.

1. $M = R$. In this case the β relations are trivial, but we can consider Ext's. We know that $\operatorname{depth}(R) = n - 1$,

$$\operatorname{ext}^0(k, R) = \cdots = \operatorname{ext}^{n-2}(k, R) = 0,$$

and $\operatorname{ext}^{n-1}(k, R) = h_1(R) = 1$. We also know that

$$\operatorname{ext}^n(k, R) = \operatorname{ext}^{n+2}(k, R) = \cdots \qquad \operatorname{ext}^{n+1}(k, R) = \operatorname{ext}^{n+3}(k, R) = \cdots$$

and $\operatorname{ext}^n(k, R) \leq 1$, but it is difficult to say any more from what we have written. It is however true that R is a Gorenstein singularity since it's a complete intersection, which shows for abstract reasons that in fact ext^n and ext^{n+1} are zero, as are all the higher exts.

2. $M = k$. Now the Koszul numbers $h_i(k)$ are n choose i, because all differentials in the Koszul complex are killed by tensoring with k. We have 2-periodicity in the $\beta_i(k)$ for $i \geq n$, so at least one of $\beta_n(k)$ and $\beta_{n+1}(k)$ has to be nonzero, otherwise k would have finite projective dimension. Beyond this, there is not much to say except in a specific example.

Example 3.1.35. For concreteness, let $R = k[\![x,y]\!]/(y^2 - x^3) \cong k[\![t^2,t^3]\!]$ be the cuspidal cubic. Then $\dim(R) = \operatorname{depth}(R) = 1$ by the previous example. Let's compute the Betti numbers of k by building a minimal resolution. It is helpful to start by writing the augmented Koszul complex and seeing what goes wrong:

$$0 \to R \xrightarrow{(-y,x)^T} R^2 \xrightarrow{(x,y)} R \twoheadrightarrow k \to 0$$

To test exactness at R^2, suppose $(f,g) \in R^2$ such that $fx + gy = 0$. We claim that the other map doesn't have full image: indeed, take $(f,g) = (-x^2, y)$ and then the relation is satisfied, however this is clearly not in the image. Thus we replace this with

$$R^2 \xrightarrow{A} R^2 \xrightarrow{(x,y)} R \to k \to 0,$$

where

$$A = \begin{pmatrix} -y & -x^2 \\ x & y \end{pmatrix}.$$

Then by construction the composite is zero, so now suppose $fx + gy = 0$. Using the t coordinate, this means $ft^2 + gt^3 = t^2(f + tg) = 0$, so $f = -tg$. Since $tg \in k[t^2,t^3]$, this means $g \in (t^2,t^3) = (x,y)$, otherwise tg would have a t^1 term. Writing $g = xu + yv$, we get $f = -yu - x^2v$. Thus $(f,g) = A(-u,v)$, showing exactness. Since A and (x,y) have coefficients in the maximal ideal, this also shows that $\mu(\mathfrak{m}) = 2$, and hence R is a singular hypersurface as above. One can now iterate this computation and show similarly that

$$\cdots \to R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{B} R^2 \xrightarrow{A} R^2 \xrightarrow{(x,y)} R \to k \to 0$$

is exact, where

$$B = \begin{pmatrix} y & x^2 \\ x & y \end{pmatrix}$$

which also has coefficients in $\mathfrak{m}$. This confirms that $\beta_i^R(k) = 2$ for $i \geq 2$, and also explicitly explains the 2-periodicity we were seeing in the proof of the example.

Finally, in the rest of this section we discuss more criteria for regularity. As a warmup, here is the easiest one.

Lemma 3.1.36. *Let $f\colon X \to Y$ be a faithfully flat map of locally Noetherian schemes. If X is regular, then Y is regular.*

Proof. Since f is surjective, for each $y \in Y$ we may choose $x \mapsto y$ and consider the corresponding faithfully flat local map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ with the latter a regular local ring. By commutativity of tor with flat base change,

$$\operatorname{Tor}_*^{\mathcal{O}_{Y,y}}(\kappa(y), N) \underset{\mathcal{O}_{Y,y}}{\otimes} \mathcal{O}_{X,x} \cong \operatorname{Tor}_*^{\mathcal{O}_{X,x}}(\kappa(y) \underset{\mathcal{O}_{Y,y}}{\otimes} \mathcal{O}_{X,x}, N \underset{\mathcal{O}_{Y,y}}{\otimes} \mathcal{O}_{X,x}),$$

and since $\operatorname{gl.dim}(\mathcal{O}_{X,x}) = n < \infty$, the RHS is zero in degrees $> n$. Since $\mathcal{O}_{X,x}$ is faithfully flat, this implies $\operatorname{Tor}_*^{\mathcal{O}_{Y,y}}(\kappa(y), N)$ is zero in degrees $> n$, hence $\operatorname{pd}_{\mathcal{O}_{Y,y}}(\kappa(y)) < \infty$. □

For instance, a field extension $k \hookrightarrow K$ is faithfully flat as it is free, thus if $X \to \operatorname{Spec}(k)$ is a k-scheme then writing $X_K = X \times_k \operatorname{Spec}(K)$, $X_K \to X$ is faithfully flat by stability under base change. In particular if X_K is regular then X is regular.

Lemma 3.1.37. *Let $\phi\colon (R, \mathfrak{m}_R, k_R) \to (S, \mathfrak{m}_S, k_S)$ be a flat local map of Noetherian local rings. If R and $F = S/\mathfrak{m}_R S$ are both regular, then S is regular, and moreover*

$$\dim S = \dim R + \dim F$$

holds.

Proof. By assumption we may choose regular sequences

$$\mathfrak{m}_R = (x_1, \dots, x_r) \qquad \mathfrak{m}_F = \mathfrak{m}_S/\mathfrak{m}_R S = (\overline{y}_1, \dots, \overline{y}_s)$$

generating the respective maximal ideals. Choose lifts of $\overline{y}_i$ to $y_i \in \mathfrak{m}_S$, so that

$$\mathfrak{m}_S = (\phi(x_1), \dots, \phi(x_r), y_1, \dots, y_s) = (\phi(\underline{x}), \underline{y}).$$

We claim this is a regular sequence (which will also prove the dimension equality). Indeed, first since S is flat over R,

$$\operatorname{Kos}^\bullet_S(\phi(\underline{x})) = \operatorname{Kos}^\bullet_R(\underline{x}) \otimes^{\mathbf{L}}_R S = R/\mathfrak{m} \otimes_R S = F.$$

Therefore,

$$\begin{aligned} \operatorname{Kos}_S(\phi(\underline{x}), \underline{y}) &= \operatorname{Kos}_S(\phi(\underline{x})) \otimes^{\mathbf{L}}_S \operatorname{Kos}_S(\underline{y}) \\ &= F \otimes^{\mathbf{L}}_S \operatorname{Kos}_S(\underline{y}) = \operatorname{Kos}_F(\overline{\underline{y}}) \\ &= k_F = k, \end{aligned}$$

in $D(S)$, proving the claim. □

Corollary 3.1.38. *Let $f\colon X \to Y$ be a flat morphism of locally Noetherian schemes. If Y is regular, and the fiber X_y is regular for all $y \in Y$, then X is regular.*

Proof. For $x \in X$, let $y = f(x)$, then the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is a flat local map with regular domain. Then $\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x} = \mathcal{O}_{X_y,x}$ is regular, by assumption, therefore $\mathcal{O}_{X,x}$ is regular by the flat-local criterion. □

In practice, we want to use a stronger condition for morphisms which is actually stable under base change, since the one above is not; thus we make the following definition. First, recall that if $X \to Y$ is locally of finite type and Y' is a locally Noetherian Y-scheme, then $X \times_Y Y'$ is locally Noetherian. So, if $X \to \operatorname{Spec}(k)$ is locally of finite type over a field, we say that X is *geometrically regular* if X_K is regular for all field extensions $k \subset K$.

Suppose $X \to Y$ is locally of finite type. Then it has *geometrically regular fibers* if $X_y \to \operatorname{Spec}(\kappa(y))$ is geometrically regular for all $y \in Y$. In other words, $X_y \times_{\kappa(y)} \operatorname{Spec}(K)$ is regular for all extensions $\kappa(y) \subset K$. It is also equivalent to ask for all geometric fibers to be regular because we proved that regularity descends under extension of the base field.

Finally, a morphism $X \to Y$ is *smooth* if it is locally of finite type, flat, and has geometrically regular fibers. (Note: the usual definition of smooth requires locally of finite presentation, but since we will not need this assumption we omit it.) Here are the basic properties of smoothness.

Lemma 3.1.39.

(a) A composition of smooth morphisms is smooth.
(b) An open immersion is smooth.
(c) An arbitrary base change of a smooth morphism is smooth.
(d) If $X \to Y$ is smooth and Y is regular, then X is regular.
(e) $f\colon X \to Y$ is smooth iff $f^{-1}(U) \to U$ is smooth for all U in an open cover of Y.
(f) $X \to Y$ is smooth iff $U \hookrightarrow X \to Y$ is smooth for all U in an open cover of X.
(g) If we have a cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $X' \to Y'$ is smooth and $Y' \to Y$ is fpqc, then $X \to Y$ is smooth.
(h) A morphism $X \to Y$ is smooth iff it is flat or finite type, and the fibers $X_y \to \operatorname{Spec} \kappa(y)$ are smooth for all $y \in Y$.

Proof. (c) Locally of finite type and flat are both stable under base change. Then for $y' \in Y$ lying over y, we have

$$\left(X \times_Y Y'\right)_{y'} = X_y \times_{\kappa(y)} \operatorname{Spec} \kappa(y'),$$

hence for any extension $K \supset \kappa(y')$, $\left(X \times_Y Y'\right)_{y',K} = (X_y)_K$ is regular.

(d) This is by Cor. 3.1.38.

(a) This holds for locally finite type/presentation and flatness, and if $X \to Y \to Z$ is a composition with $z \in Z$, then $X_z = X \times_Y Y_z$. For all $K \supset \kappa(z)$, $(X_z)_K = X \times_Y (Y_z)_K$, so $(X_z)_K \to \operatorname{Spec} Y_{zK}$ is smooth by (c), hence $(X_z)_K$ is regular by (d).

(b) An open immersion is locally of finite presentation and flat, and its fibers are either empty or $\operatorname{Spec}(\kappa(x))$, which is clearly geometrically regular.

(e) One direction follows from (c), and conversely, finite type/presentation are local on the target, while $f^{-1}(U) \to U$ has the same fibers as $X \to Y$ for $y \in U$, so geometric regularity is too.

(f) One direction follows from (a) and (b). Conversely, locally of finite type/presentation and flatness are local on the source. If $U \to X \to Y$ is smooth for all U in an open cover, then for each $y \in Y$, consider $X_y \times_{\kappa(y)} K$ for some extension $K \supset \kappa(y)$. Then $U_y \times_{\kappa(y)} K \subset X_y \times_{\kappa(y)} K$ is regular for all U in a cover of X, which thus forms a cover of X_y, and hence X_y is geometrically regular.

(g) Locally of finite type/presentation and flatness are fpqc local on the target. If $y' \in Y'$ lies over $y \in Y$, then $X'_{y'} = X_y \times_{\kappa(y)} \operatorname{Spec} \kappa(y')$, so $(X'_{y'})_K = (X_y)_K$ for

an extension $K \supset \kappa(y') \supset \kappa(y)$. Since flatness descends along field extensions, it is clear that X_y is geometrically regular. Since $Y' \to Y$ is surjective, this covers all fibers.

(h) This is clear, because if $X \to \operatorname{Spec} k$ is smooth, then $X_K \to \operatorname{Spec}(K)$ is smooth by (c), so X_K is regular by (d). □

Example 3.1.40. We show by induction on n that $\operatorname{Spec} k[x_1, \ldots, x_n] \to \operatorname{Spec} k$ is a smooth morphism. Everything is flat over a field, and polynomial rings are finitely presented k-algebras by definition. When $n = 1$, we have to show $k[x]$ is geometrically regular over k, but indeed $K[x]$ is a PID for every K, and hence regular (Example 3.1.28). For $n > 1$, take the smooth morphism $\operatorname{Spec} k[x_1, \ldots, x_{n-1}] \to \operatorname{Spec} k$, base change to obtain a smooth morphism

$$\operatorname{Spec} k[x_1, \ldots, x_n] \to \operatorname{Spec} k[x_n],$$

and finally compose with the smooth morphism $\operatorname{Spec} k[x_n] \to \operatorname{Spec} k$ by $n = 1$. This also gives a proof that $k[x_1, \ldots, x_n]$ is a regular ring which avoids the Nullstellensatz. Since it has dimension n, it has global dimension n, which is known as Hilbert's syzygy theorem.

Example 3.1.41. More generally, let S be a scheme. Then $\mathbb{A}^n_S \to S$ is a smooth morphism. Indeed, it suffices to show that $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$ is smooth. It is of finite presentation, and $\mathbb{Z}[x_1, \ldots, x_n]$ is free over $\mathbb{Z}$ hence flat. The fibers are $\mathbb{A}^n_{\mathbb{F}_p} \to \operatorname{Spec} \mathbb{F}_p$ and $\mathbb{A}^n_{\mathbb{Q}} \to \operatorname{Spec} \mathbb{Q}$, which are smooth by Example 3.1.40. In particular, if A is a regular ring, then $A[x_1, \ldots, x_n]$ is regular. If A has dimension r, then $A[x_1, \ldots, x_n]$ has global dimension $r + n$. Another corollary is that $\mathbb{P}^n_S \to S$ is smooth for every S by locality on the source.

In the rest of this subsection we will give some of the classical criteria for regularity. Notably, we will still not rely on the characterization of regularity as $\dim(R) = \mu(\mathfrak{m})$. We will see that in general, the place where we can most easily detect regularity is in the case of a regular immersion. The most important regularity criterion is how to tell whether $k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}/I$ is a regular local ring for some ideal I. This is because if we are given an fppf morphism $f\colon X \to S$ and want to test for smoothness, then we have to check whether $X_s \times_{\kappa(s)} \operatorname{Spec} K$ is regular for K an algebraically closed field. This is locally of the form $\operatorname{Spec} K[x_1, \ldots, x_n]/I$ for some ideal I, and by the Nullstellensatz we only have to check regularity at points of the form

$$(x_1 - a_1, \ldots, x_n - a_n) \qquad a_i \in K,$$

whose local rings reduce to those of the form above.

Let's start with a Noetherian local ring $(Q, \mathfrak{m}_Q)$. If $I \subset \mathfrak{m}_Q$ is an ideal, define the Noetherian local ring $R \coloneqq (Q/I, \mathfrak{m}_R)$ with $\mathfrak{m}_R = \mathfrak{m}_Q/I$. Then R and Q have the same residue field $k = Q/\mathfrak{m}_Q$. Note that by definition, we have an exact sequence

$$0 \to \frac{I}{I \cap \mathfrak{m}_Q^2} \to \frac{\mathfrak{m}_Q}{\mathfrak{m}_Q^2} \to \frac{\mathfrak{m}_R}{\mathfrak{m}_R^2} \to 0$$

of k-vector spaces, giving the cotangent space of R as a quotient of that of Q. Thus this natural quotient map is an isomorphism iff $I \subset \mathfrak{m}_Q^2$. If this condition holds and Q is a regular local ring, then we say that Q/I is a *minimal regular presentation* of R.

More generally, let Q be a regular local ring; by choosing a splitting of the sequence above, we can lift a basis for $\mathfrak{m}_R/\mathfrak{m}_R^2$ to a linearly independent set in $\mathfrak{m}_Q/\mathfrak{m}_Q^2$. The fact that this can be completed to a basis (hence a regular sequence of length $\dim(Q)$) implies that whenever we choose $\overline{f}_1, \ldots, \overline{f}_r$ generating $\mathfrak{m}_R = \mathfrak{m}_Q/I$ with lifts $f_1, \ldots, f_r \in \mathfrak{m}_Q$, then these will form a regular sequence in Q. The ideal $J \coloneqq (f_1, \ldots, f_r)$ is called a transverse parameter ideal, and can give information about R; for instance, what we have said implies that $\mathrm{Kos}_Q^\bullet(f_1, \ldots, f_r)$ gives a resolution of Q/J, and on the other hand if we tensor this with R we will obtain $\mathrm{Kos}_R^\bullet(\overline{f}_1, \ldots, \overline{f}_r)$ which we know controls a lot of interesting regularity information about R. Thus $R \otimes_Q^{\mathbf{L}} Q/J$ is the key to detecting things like regularity of the subvariety R.

Before continuing any generalities, let's discuss the specific and most-used example, which is that of a hypersurface. Again assume Q is regular local and $I = (f)$ for some $f \in \mathfrak{m}_Q$. We have already seen that if $f \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$, then $R = Q/(f)$ is again a regular local ring of one dimension lower. Conversely, if $f \in \mathfrak{m}_Q^2$, then $\mu(\mathfrak{m}_R) = \mu(\mathfrak{m}_Q)$, yet $\dim(R) = \dim(Q) - 1$ so the ring must be singular. Thus R is regular iff $f \in \mathfrak{m}_Q \setminus \mathfrak{m}_Q^2$.

Now specialize to $Q = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ and let $f \in \mathfrak{m}_Q$. By clearing denominators, we may assume f is represented by a polynomial in $k[\underline{x}]$. But we know that

$$f \equiv \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0) x_i \pmod{\mathfrak{m}_Q^2},$$

and since $\{x_1, \ldots, x_n\}$ is a basis for $\mathfrak{m}_Q/\mathfrak{m}_Q^2$, we conclude that $Q/(f)$ is singular iff $\partial f/\partial x_i|_{\underline{x}=0} = 0$. This provides the following criterion:

Lemma 3.1.42. *Let k be a field and $f \in A \coloneqq k[x_1, \ldots, x_n]$ be a nonzero polynomial, and let $\overline{k}$ be an algebraic closure. The following are equivalent:*

(a) $\operatorname{Spec} A/(f)$ *is smooth over* $\operatorname{Spec} k$.
(b) For any algebraically closed field $K \supset k$ and $\underline{a} = (a_1, \ldots, a_n) \in K^n$ with $f(\underline{a}) = 0$, $\partial f/\partial x_i|_{\underline{x}=\underline{a}} \neq 0$ for all i.
(c) For each $\underline{a} = (a_1, \ldots, a_n) \in \overline{k}^n$, with $f(\underline{a}) = 0$, $\partial f/\partial x_i|_{\underline{x}=\underline{a}} \neq 0$ for all i.
(d) $(f, \partial_{x_1} f, \ldots, \partial_{x_n} f) = (1)$ over $\overline{k}[x_1, \ldots, x_n]$.
(e) $(f, \partial_{x_1} f, \ldots, \partial_{x_n} f) = (1)$ over A.

Proof. $A/(f)$ is a smooth k-algebra precisely when over all algebraically closed fields K containing k, the base change $K[x_1, \ldots, x_n]/(f)$ is regular. If we take $K = \overline{k}$, then it is clear from the Nullstellensatz, Lemma 3.1.20, and the local ring description above (after shifting points) that $(a) \Leftrightarrow (b)$. Clearly $(b) \Rightarrow (c)$. It follows again from the Nullstellensatz that $(c) = (d)$.

Likewise, (b) is equivalent to the version of (d) where $\overline{k}$ is replaced with any such K. But now this is equivalent to (e), because $A \to K[x_1, \ldots, x_n]$ is faithfully flat (since $k \to K$ is), and thus $(A/J) \otimes_k K = 0$ implies $A/J = 0$, i.e. $J = A$, where $J = (f, \partial_{x_1}, \ldots, \partial_{x_n})$. □

Remark 3.1.43. Let $A = R[x_1, \ldots, x_n]$ for some ring R and $f \in A$, and assume that $A/(f)$ is flat over R. Then again $J = (f, \partial_{x_1} f, \ldots, \partial_{x_n} f) = A$ implies that $A/(f)$ is a smooth R-algebra. Indeed, this implies that for each $\mathfrak{p} \in \mathrm{Spec}(R)$, we have $J\kappa(\mathfrak{p})[x_1, \ldots, x_n] = \kappa(\mathfrak{p})[x_1, \ldots, x_n]$, which by the lemma implies that the fiber is smooth over $\kappa(\mathfrak{p})$. The converse holds if f is a nonzerodivisor. In this case, we have a sequence $0 \to A \xrightarrow{f} A \to A/(f) \to 0$, and since $A/(f)$ is flat the sequence remains exact upon tensoring with $\kappa(\mathfrak{p})$. This ensures that f does not become zero over any of the fibers, and hence the criterion above applies fiberwise again.

Example 3.1.44. Let $A = \mathbb{Z}[x_1, \ldots, x_n]$ and $f \in A$ a nonzero polynomial. Consider the sequence $0 \to A \xrightarrow{f} A \to A/(f) \to 0$, which shows that $\mathrm{Tor}_1(A/(f), \mathbb{F}_p)$, i.e. the module of p-torsion, is zero iff multiplication by f remains injective upon base changing to $\mathbb{F}_p$. In particular, $A/(f)$ is torsion-free (= flat) iff f is not 0 modulo p for any p, i.e. if the coefficients of f have gcd 1 (namely f is 'primitive'). Then $A/(f)$ is smooth over $\mathbb{Z}$ exactly when $(f, \partial_{x_1} f, \ldots, \partial_{x_n} f) = (1)$. This shows that a hypersurface defined by such an f over any ring R is smooth over R.

Now let's return to a more general setting and discuss regular immersions. Let $(Q, \mathfrak{m}_Q, k)$ be Noetherian local and $I \subset \mathfrak{m}$ an ideal. Let $(x_1, \ldots, x_r) = I$ be a minimal generating set. Then we have seen that $\mathrm{Kos}^\bullet_Q(x_1, \ldots, x_r)$ is independent (up to isomorphism) of the choice of minimal generating set of I. Thus, we say I is a *complete intersection ideal* if a (or, equivalently, any) minimal generating set of I forms a regular sequence in Q.

Let's see that this notion is well-behaved in a number of ways. First of all, we define $\mathrm{grade}_Q(I, M)$ to be the supremum over lengths of an M-regular sequence contained in I. We write $\mathrm{grade}_Q(I) = \mathrm{grade}_E(I, Q)$ for the maximal length of a regular sequence inside I.

Lemma 3.1.45. *Let $(Q, \mathfrak{m}, k)$ be Noetherian local. Then we have*

$$\mathrm{grade}_Q(I, M) = \inf_{\mathfrak{p} \supset I} \mathrm{depth}_{Q_\mathfrak{p}}(M_\mathfrak{p}) = \inf\{j : \mathrm{Ext}^j_Q(Q/I, M)\}$$

for any ideal $I \subset \mathfrak{m}$ and finite module M, and this coincides with the length of any maximal M-regular sequence in I.

Proof. The Ext-characterization and its equality to the length of any maximal M-regular sequence is proved identically to the case where $I = \mathfrak{m}$ which we showed earlier; the case of interest is the initial equality. First, if $f_1, \ldots, f_r$ is an M-regular sequence contained in I, then localizing the corresponding Koszul complex at any $\mathfrak{p} \supset I$ gives an $M_\mathfrak{p}$-regular sequence of the same length.

Conversely, let $f_1, \ldots, f_r$ be a maximal M-regular sequence contained in I, so $r = \mathrm{grade}_Q(I, M)$. Let $N = M/(f_1, \ldots, f_r)$, then by assumption, I is contained

in the union of associated primes of N. By prime avoidance, let $\mathfrak{q} \supset I$ be an associated prime. Now upon localization at $\mathfrak{q}$, the sequence remains $M_\mathfrak{q}$-regular by the argument above, with quotient $N_\mathfrak{q}$. But by construction, $\mathfrak{q}$ contains no $N_\mathfrak{q}$-regular elements, hence the given sequence is maximal in $\mathfrak{q}Q_\mathfrak{q}$. This implies $r = \operatorname{depth}_{Q_\mathfrak{q}}(M_\mathfrak{q})$, giving the other direction. □

Note that as a consequence, $\operatorname{grade}_Q(I) \leq \mu(I)$. Indeed, any minimal prime $\mathfrak{p}$ over I has $\operatorname{ht}(\mathfrak{p}) \leq \mu(I)$, and

$$\operatorname{depth}(Q_\mathfrak{p}) \leq \dim(Q_\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) \leq \mu(I),$$

therefore the claim follows. We also write $\operatorname{ht}(I) = \inf_{\mathfrak{p} \supset I} \operatorname{ht}(\mathfrak{p})$, so this shows that in fact $\operatorname{grade}_Q(I) \leq \operatorname{ht}(I)$.

As a corollary, the following are equivalent for $(Q, \mathfrak{m}, k)$ and $I \subset Q$:

(a) I is a complete intersection ideal.
(b) I is generated by a regular sequence $f_1, \ldots, f_r$, in which case $r = \operatorname{grade}_Q(I) = \operatorname{ht}(I) = \mu(I)$.
(c) $\operatorname{grade}_Q(I) = \mu(I)$.

The description above makes it clear that (a) and (b) are equivalent. To show that (b) implies (a), we can copy the spectral sequence proof from the earlier case when $I = \mathfrak{m}$ to show that $\mu(I) - \operatorname{grade}_Q(I, M)$ equals the top homology degree (i.e. lowest cohomology degree) of $H^{-*}(\operatorname{Kos}_Q(f_1, \ldots, f_r))$, where $f_1, \ldots, f_r$ is any minimal generating set of I. In particular if the two numbers coincide then any such minimal generating set is a regular sequence.

Remark 3.1.46. We usually consider complete intersections when the ring Q is a regular local ring, or at least when Q is Cohen-Macaulay. In that case, we observe that since $\operatorname{depth}(Q_\mathfrak{p}) = \dim(Q_\mathfrak{p})$, we have $\operatorname{grade}_Q(I) = \operatorname{ht}(I)$.

Definition 3.1.47. Let $i\colon Z \to X$ be a closed immersion of locally Noetherian schemes. i is a *regular immersion* if the quasicoherent ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ corresponding to i has the property that $\mathcal{I}_z \subset \mathcal{O}_{X,z}$ is a complete intersection ideal for all $z \in Z$.

Given a closed immersion, one writes $\operatorname{Reg}(i) \subset Z$ for the set of points of Z for which $\mathcal{I}_{i(z)} \subset \mathfrak{m}_{i(z)} \subset \mathcal{O}_{X,i(z)}$ is a complete intersection ideal. We will now see that this locus has good properties.

First, suppose $i\colon Z \hookrightarrow X$ is a closed immersion and $z \in \operatorname{Reg}(i)$. Thus we may find an open nbhd $U \subset X$ and sections $f_1, \ldots, f_r \in H^0(U, \mathcal{I}_Z)$ such that $(f_{1,z}, \ldots, f_{r,z}) = \mathcal{I}_{Z,z}$ form a regular sequence generating $\mathcal{I}_{Z,z}$ in $\mathcal{O}_{X,z}$. Since $\mathcal{I}_Z$ is finite type, we may shrink U and assume that $f_1, \ldots, f_r$ generate $\mathcal{I}_Z|_U$. Then consider the kernel in

$$0 \to \mathcal{K}_i \to \mathcal{O}_U/(f_1, \ldots, f_{i-1}) \xrightarrow{f_i} \mathcal{O}_U/(f_1, \ldots, f_{i-1}).$$

Since X is locally Noetherian, each $\mathcal{K}_i$ has closed support, and thus by shrinking U again r times, we may assume that $(f_1, \ldots, f_r)$ forms a regular sequence generating $\mathcal{I}_Z$ over some open nbhd $U \subset X$, with the property that $\mathcal{O}_U/(f_1, \ldots, f_r) \neq 0$. Now for $x \in U \setminus Z$, the stalk of this sequence generates the unit ideal in $\mathcal{O}_X$; but

for each $z' \in U \cap Z$, we obtain a regular sequence in $\mathcal{O}_{X,z}$ generating $\mathcal{I}_{Z,z}$. In particular, $U \cap Z \subset \mathrm{Reg}(i)$ and thus $\mathrm{Reg}(i)$ is open.

Lemma 3.1.48. *Let $i\colon Z \to X$ be a regular immersion of Noetherian schemes. Then $i_*\mathcal{O}_Z$ is perfect on X, so in particular i is quasi-perfect.*

Proof. The first statement implies the latter because i is affine, and this claim is local on the target. By the above we may shrink X and assume there is a regular sequence in $H^0(X, \mathcal{O}_X)$ cutting out $i_*\mathcal{O}_Z$, so the Koszul complex provides a finite free resolution. □

The next result requires a bit more buildup, which should be added here.

Lemma 3.1.49. *A section of a smooth morphism of locally Noetherian schemes is a regular immersion.*

3.1.3. Classical applications of regularity. Our next goal is to prove that regular local rings are UFDs, which forms the basis for various results in scheme theory. We will give an interesting application of this, which is to prove that all coherent sheaves on a Noetherian regular semiseparated scheme of dimension n have a length n resolution by vector bundles.

Lemma 3.1.50. *Let R be a regular local ring. Then* $\mathrm{Pic}(\mathrm{Spec}(R_f)) = 0$ *for any $f \in R$.*

Proof. Let $\widetilde{N}$ be a line bundle on $\operatorname{Spec} R_f$, so in particular N is finitely generated by some $\{x_1, \dots, x_r\}$. Let M be the R-submodule $\sum_i Rx_i \subset N$, then by flatness of localization we have $M_f \subset N$, yet clearly this is an equality since M_f contains $\sum_i R_f x_i = N$.

In particular, since M is finite over a regular local ring, it has a finite resolution by finite free modules. By localizing the corresponding exact sequence at f, we obtain an exact sequence

$$0 \to F_n \to \cdots \to F_0 \to N \to 0$$

with F_i finite free over R_f. Then this is a sequence of constant rank vector bundles on $\operatorname{Spec} R_f$, so by the determinant formula (e.g. Prop 2.2.5 of the exterior power notes) we obtain

$$\cdots \otimes \det(\widetilde{F_1})^\vee \otimes \det(\widetilde{F_0}) \otimes \det(\widetilde{N})^\vee = \mathcal{O}_{\operatorname{Spec} A_f},$$

and since $\widetilde{N}$ has rank 1 this means:

$$\widetilde{N} \cong \det(\widetilde{F_0}) \otimes \det(\widetilde{F_1})^\vee \otimes \cdots$$

The RHS is finite free, hence $\widetilde{N}$ is free. □

With this, the main result is relatively easy to prove. The only remaining input is the following simple observation: Let A be a domain and $a \in A$ an irreducible element. If $p \in A$ is prime and $a/1 \in A_p$ is prime, then in fact a is prime in A.

Indeed, suppose $b, c \in A$ such that $a \mid bc$, then this holds in A_p, so we may assume $a/1 \mid b/1$. Thus we may write $p^n b = ad$ in A for some d, and assume $n \geq 1$ otherwise we are done; if $p \mid a$, then since a is irreducible we must have $(a) = (p)$ is prime. Otherwise, since $p \mid ad$, this means $p \mid d$, so we deduce $p^{n-1}b = ad'$ by cancellation. We may continue reducing the exponent of p until it is zero, thus eventually $b = ad''$ for some d'', i.e. $a \mid b$.

Theorem 3.1.51. *Let $(R, \mathfrak{m})$ be a regular local ring. Then R is a UFD.*

Proof. If $\dim(R) = 0$ then R is a field, so assume $\dim(R) > 0$. Then we know that we can take any $p \in \mathfrak{m} \setminus \mathfrak{m}^2$ and $R/(p)$ will be a regular local ring of dimension $\dim(R) - 1$, so in particular p is prime. Let $\mathfrak{p} \subset R$ be a height 1 prime, then we have to show $\mathfrak{p}$ is principal. If $p \in \mathfrak{p}$, then $0 \subsetneq (p) \subset \mathfrak{p}$ shows $(p) = \mathfrak{p}$ is principal, so assume $p \notin \mathfrak{p}$. Then $\mathfrak{p}R_p \subset R_p$ is a prime, and we claim it is a line bundle.

Each prime of R_p is of the form $\mathfrak{q}R_p$ for $p \notin \mathfrak{q}$, thus $\mathrm{ht}_R(\mathfrak{q}) \leq \dim(R) - 1$ (since $p \in \mathfrak{m}$ so $\mathfrak{q} \neq \mathfrak{m}$). Therefore $R_\mathfrak{q}$ is a regular local ring of dimension $\leq \dim(R) - 1$ and

hence a UFD. But now $(\mathfrak{p}R_p)_{\mathfrak{q}R_p} = \mathfrak{p}R_{\mathfrak{q}}$, and $\mathrm{ht}_{R_{\mathfrak{q}}}(\mathfrak{p}R_{\mathfrak{q}}) = \dim(R_{\mathfrak{q}}) = \mathrm{ht}_R(\mathfrak{q}) = 1$, so $\mathfrak{p}R_{\mathfrak{q}}$ is a principal ideal in $R_{\mathfrak{q}}$. Thus all prime localizations of $\mathfrak{p}R_p$ are rank 1 free modules, proving the claim.

By the lemma, $\mathfrak{p}R_p \subset R_p$ is free of rank 1, hence $\mathfrak{p}R_p = (f/p^m)$ for some m and $f \in p$. Since R is Noetherian, we can write $f = a_1 \cdots a_r$ as a product of irreducible elements, so $a_i \in \mathfrak{p}$ for some i. Then $a_i/1 \in \mathfrak{p}R_p$ and divides the prime element f/p^m in R_p, thus $a_i/1 \sim f/p^m$, so $\mathfrak{p}R_p = (a_i/1)$, i.e. $a_i/1$ is prime. By the observation preceding the proof, it follows that a_i is prime in R, and $(0) \subsetneq (a_i) \subset \mathfrak{p}$ shows $(a_i) = \mathfrak{p}$ is principal. □

This result is sometimes also called the Auslander-Buchsbaum theorem. Recall that a scheme X for which $\mathcal{O}_{X,x}$ is a UFD for all x is called *locally factorial*. So the result shows that a regular scheme is locally factorial. Since UFDs are integrally closed domains, this also shows that regular schemes are all normal. There is an important converse in dimension 1, which we can now show.

Proposition 3.1.52. *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension 1. Then R is regular iff R is a DVR iff R is an integrally closed domain.*

Proof. We have already proven the equivalence of the first two conditions, and we have just seen that the first two imply the third. Suppose the third: note the only two primes are $(0) \subsetneq \mathfrak{m}$. Let $0 \neq x \in \mathfrak{m}$, and observe that $\sqrt{(x)}$ is the intersection of primes containing x, i.e. $\sqrt{(x)} = \mathfrak{m}$. Since $\mathfrak{m}$ is finitely generated, take the smallest n for which $\mathfrak{m}^n \subset (x)$. If $n = 1$ we are done by Prop. 3.1.27, and if $n > 1$ then $\mathfrak{m}^{n-1} \subsetneq (x)$.

Thus we may choose $y \in \mathfrak{m}^{n-1} \subset \mathfrak{m}$ with $y \notin (x)$, so by assumption y/x is in $\operatorname{Frac} R$ and not R; but $(y/x)\mathfrak{m} \subset \mathfrak{m}^n/x(x) \subset R$. We claim $(y/x)\mathfrak{m} \nsubseteq \mathfrak{m}$. If this were false, then we have an R-linear endomorphism $(y/x)\colon \mathfrak{m} \to \mathfrak{m}$, and since $\mathfrak{m}$ is finitely generated this implies (y/x) is integral over R (see Lemma 1.3.3 in the exterior power notes). This would contradict integral closure.

Thus take $z \in \mathfrak{m}$ such that $(y/x)z \notin \mathfrak{m}$, i.e. yz/x is a unit in R, so letting $\pi = z(yz/x)^{-1} \in \mathfrak{m}$ gives an element of $\mathfrak{m}$ such that $\pi(y/x) = 1$. But for every $w \in \mathfrak{m}$, $(y/x)w \in R$, so $w = \pi(y/x)w \in (\pi)$, which shows that $\mathfrak{m} = (\pi)$. Hence by Prop. 3.1.27, R is a DVR. □

This is nice because it says that for any (locally Noetherian) normal scheme X, a codimension 1 point $\eta_Z \in X$ has $\mathcal{O}_{X,\eta_Z}$ a regular local ring. We sometimes say that normal schemes are 'regular in codimension 1'; e.g. any reduced curve admits a smooth model.

Next we'll discuss global resolutions by vector bundles. Let X be a locally Noetherian scheme. X has the *resolution property* if every coherent sheaf is a quotient of a vector bundle (a locally free sheaf of finite rank). Most things that we work with have the resolution property by virtue of the following example:

Example 3.1.53. Let X be a locally Noetherian scheme with an ample line bundle $\mathcal{L}$. Then X has the resolution property. Indeed, for each coherent sheaf $\mathcal{F}$, there exists $n \gg 0$ such that $\mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{F}$ is globally generated, i.e. there is a

surjection $\mathcal{O}_X^m \to \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} \mathcal{F} \to 0$. Thus $(\mathcal{L}^{\otimes -n})^m \to \mathcal{F} \to 0$ is a surjection from a vector bundle. In particular, this applies whenever X is quasiprojective over a Noetherian ring.

It is an interesting and difficult question to answer whether or not a scheme has the resolution property. For instance, it is unknown whether there is a Noetherian proper scheme without the resolution property, or even a Noetherian separated scheme without it. It is a fun result to prove that if X is Noetherian with the resolution property then X is semiseparated, but this uses the fact that $D_{\mathrm{qc}}(X) = D(\mathrm{QCoh}(X))$ in the Noetherian case, which we haven't shown.

Proposition 3.1.54. *Let X be a regular scheme of dimension n. If X has the resolution property, then every coherent sheaf admits a resolution by vector bundles of length n.*

Proof. This is the usual trick: by Noetherianity, we can inductively build a (finite) locally free resolution of a coherent sheaf $\mathcal{F}$, where at each step the kernel remains coherent, so we may take a vector bundle surjecting onto it. Do this and take the kernel at the nth step:

$$0 \to \mathcal{K} \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{F} \to 0.$$

Since $\mathcal{F}$ is coherent, it suffices to show $\mathcal{K}_x$ is a free $\mathcal{O}_{X,x}$-module for all x. But indeed $\mathcal{O}_{X,x}$ is a regular local ring of dimension $\leq n$, so by dimension shifting we see $\operatorname{pd} \mathcal{K}_x = 0$, i.e. it is free. □

Example 3.1.55. Here is a slightly more general statement. Let X have the resolution property and suppose $E \in \mathrm{Perf}^b(X)$ has perfect amplitude (i.e. tor amplitude) in $[a, b]$. Then E is represented by a complex $\mathcal{E}^\bullet$ of vector bundles concentrated in $[a, b]$. Using the collection of vector bundles as an adapted class on $\mathrm{Coh}(X)$, it follows that E is isomorphic to a complex $\mathcal{F}^\bullet$ of vector bundles concentrated in $(-\infty, b]$. By Prop. 2.2.53, $\tau_{\geq a}\mathcal{F}^\bullet$ is a complex of flat sheaves, so the bottom cohomology is flat. It is also the cokernel of a map of coherent sheaves, hence coherent. A finitely presented flat sheaf is locally free, which completes the proof.

Proposition 3.1.56. *Let X be a Noetherian separated locally factorial scheme. Then X has the resolution property.*

The hypothesis arises because the equivalence between Weil divisors and Cartier divisors works over locally factorial schemes. We will use this to show that X has an 'ample family of line bundles', and then that X has the resolution property.

Indeed, first since X is Noetherian it has finitely many components $\{X_i\}$, and since each local ring is a domain we know that they must be disjoint (and clopen). Then if the result is known in the integral case, then if $\mathcal{F}$ is a coherent sheaf on X we may take vector bundles $\mathcal{E}_i \twoheadrightarrow \mathcal{F}_{X_i}$, then the direct sum $\bigoplus_i j_{i,*}\mathcal{E}_i$ of their pushforwards still surjects onto $\mathcal{F}$, because in this case $j_{i,*}$ coincides with $j_{i,!}$. Thus we may assume X is integral. We will now show that the collection of opens $\{X_s : s \in \mathcal{L}(X), \mathcal{L} \in \mathrm{Pic}(X)\}$ forms a basis for X.

Indeed, choose $x \in X$ and U an open nbhd of x. First suppose that $Z \coloneqq X \setminus U$ is irreducible, and thus view it as a closed integral subscheme. If $\eta_Z \in Z$ is the generic point, then we claim that $\mathcal{O}_{X,x} \not\subseteq \mathcal{O}_{X,\eta_x}$ as subsets of $K(X)$. If this were false, then the canonical map $\operatorname{Spec} \mathcal{O}_{X,\eta_x} \to X$ and the composition

$$\operatorname{Spec} \mathcal{O}_{X,\eta_x} \to \operatorname{Spec} \mathcal{O}_{X,x} \to X$$

would be two maps agreeing at the closed point of the source, so by separatedness they must agree. This factorization would imply (by the correspondence between specializations and maps from local rings to X) that $x \in \overline{\{\eta_Z\}} = Z$, which is a contradiction. Therefore, we may choose $f \in K(X)$ such that $f \in \mathcal{O}_{X,x}$ but $f \notin \mathcal{O}_{X,\eta_Z}$.

Now, let $D = (f)_\infty$ be the pole divisor of f, which is effective. By construction $x \notin D$, but $Z \subset D$. Since X is locally factorial, this is an effective Cartier divisor, and $\mathcal{L} \coloneqq \mathcal{O}_X(D)$ comes with a canonical section $s_D \in \mathcal{L}$ such that $V(s_D) = D$. In particular, $x \in X_s$ and $X \setminus X_s \supset Z$, i.e. $X_s \subset U$. This proves our claim in the case that Z is irreducible. If it isn't, then let $Z = Z_1 \cup \cdots \cup Z_r$ be the components and apply the case above to $X \setminus Z_i$ for each i, which gives line bundles $\mathcal{L}_i$ with sections s_i such that $x \in X_{s_i} \subset X \setminus Z_i$. Then write $s = s_1 \cdots s_r \in \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r$, which satisfies

$$x \in X_s = X_{s_1} \cap \cdots \cap X_{s_r} \subset X \setminus \bigcup_i Z_i = U$$

as desired.

Now let's axiomatize this condition: Let X be a quasicompact scheme equipped with a collection S of line bundles such that the open sets $\{X_s : s \in \mathcal{L}, \mathcal{L} \in S\}$ form a basis for the topology on X. If moreover the collection S is closed under multiplication, i.e. is a multiplicative submonoid of $\operatorname{Pic}(X)$, then we will call S an *ample family of line bundles*. The basic property is the following:

Lemma 3.1.57. *Let X be a qcqs scheme with an ample family S of line bundles. Then for each finite type sheaf $\mathcal{F}$ over X there exists $\mathcal{L} \in S$ such that $\mathcal{L} \otimes \mathcal{F}$ is finitely globally generated.*

Proof. This has the same proof as in the ample case, which is namely the qcqs lemma for line bundles. Namely, since X is quasicompact we may find finitely many line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ with sections $s_i \in \mathcal{L}_i(X)$ such that:

(a) $X = X_{s_1} \cup \cdots \cup X_{s_n}$.
(b) There is a surjection $\mathcal{O}_{X_{s_i}}^{k_i} \twoheadrightarrow \mathcal{F}|_{X_{s_i}}$ for each i.

By the qcqs lemma, for each i, given the sections $a_1, \ldots, a_{k_i} \in \mathcal{F}(X_{s_i})$, and for each j we can lift $s^m a_j$ to a global section of $\mathcal{L}_i^{\otimes m} \otimes \mathcal{F}$. By choosing m uniformly across the a_j (and multiplying by s more times), we may lift all the $s^m a_j$ to sections of $\mathcal{L}_i^{\otimes m} \otimes \mathcal{F}$. By replacing $\mathcal{L}_i$ with $\mathcal{L}_i^{\otimes m} \in S$ and our sections with $s_i^m a_j$, we may assume there is a global map $\mathcal{O}_X^{k_i} \to \mathcal{L}_i \otimes \mathcal{F}$ which is surjective on X_{s_i}.

By doing this for all i, we can multiply our chosen global sections of $\mathcal{L}_i \otimes \mathcal{F}$ by $s_1 \cdots \widehat{s_i} \cdots s_n$ to obtain global maps $\mathcal{O}_X^{k_i} \to \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n \otimes \mathcal{F}$ which are surjective

on X_{s_i}. Finally, letting $n = \sum_i k_i$ this induces a map $\mathcal{O}_X^n \to \mathcal{L} \otimes \mathcal{F}$ where $\mathcal{L} = \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_n$, which is surjective on all X_{s_i}, hence surjective. □

Note that this completes the proof of Prop. 3.1.56, since in that case we showed that the collection of all line bundles is an ample family for X; thus for a coherent sheaf we can find a surjection $(\mathcal{L}^\vee)^{\oplus n} \to \mathcal{F} \to 0$ for some n and $\mathcal{L}$.

Thus far, we have relied on essentially no hard facts in commutative algebra. The point of this was to emphasize that regularity is an easy notion as long as we define it homologically instead of dimension-theoretically. The issue with this is that we still have not related our results to the classical definitions. Therefore, we will now prove two results that require more difficult commutative algebra.

The simpler one is about perfect fields and smoothness. It means that we only have to check regularity over one perfect (e.g. algebraically closed) field, instead of having to check over all fields. The somewhat nontrivial fact we have to use is that extensions of perfect fields are separably generated, meaning if $K \supset k$ is a finitely generated extension and k is perfect with $\mathrm{tr.deg}_k(K) = r$, there are $x_1, \ldots, x_r$ such that $K \supset k(x_1, \ldots, x_r)$ is finite separable. (Note however that this is trivial in characteristic 0, so we at least have a self-contained proof in that case.)

Lemma 3.1.58. *Let $q\colon X \to \mathrm{Spec}(k)$ be locally of finite type with X regular and k perfect. Then q is smooth. In particular, an fppf morphism $f\colon X \to Y$ is smooth iff for each $y \in Y$, there exists a perfect field $K \supset \kappa(y)$ such that $X_y \times_{\kappa(y)} \mathrm{Spec}\, K$ is regular.*

Proof. The conclusion follows immediately from the first part and the definition. X is automatically flat over k and of finite presentation (as k is Noetherian), so we just have to show geometric regularity. Let $K \supset k$ be a field extension, and consider the base change $X_K \to X$ which is flat. We have to show that the fibers are regular by the flat local criterion: The fiber over $x \in X$ is

$$(X_K)_x = \mathrm{Spec}(K) \times_k \mathrm{Spec}\, \kappa(x) = \mathrm{Spec}(K \otimes_k \kappa(x)),$$

so we have to show $K \otimes_k \kappa(x)$ is a regular ring. Now, since X is locally of finite type over k, $\kappa(x) \supset k$ is a finitely generated extension, and since k is perfect we may find

$$F = k(t_1, \ldots, t_r) \subset \kappa(x)$$

where $\kappa(x) \supset F$ is a finite separable extension. By the primitive element theorem, F is generated by one element, and hence we may write $\kappa(x) = K[u]/(f)$ for some separable irreducible polynomial $f(u) \in \kappa(x)[u]$. Now consider the ring $A \coloneqq K \otimes_k F$; note that

$$A = K \otimes_k k[t_1, \ldots, t_r][S^{-1}] = K[t_1, \ldots, t_r][S^{-1}]$$

where S is the multiplicative subset $k[t_1, \ldots, t_r] \setminus \{0\}$, and therefore A is a regular ring. On the other hand

$$K \otimes_k \kappa(x) = K \otimes_k F[u]/(f) = A[u]/(f).$$

Observe that if $d = \deg f$, then $\{1, u, \ldots, u^{d-1}\}$ forms a basis for $A[u]/(f)$ over A, because f is monic and both linear independence and generation follow from the

division algorithm. Therefore $A[u]/(f)$ is flat over A; we are going to apply the flat local criterion once more to $A \to A[u]/(f)$ to conclude regularity.

For $\mathfrak{p} \in \operatorname{Spec} A$ a prime, it remains to show that the fiber

$$A[u]/(f) \otimes_A \kappa(\mathfrak{p}) = \kappa(\mathfrak{p})[u]/(f)$$

over $\mathfrak{p}$ is regular. Since f is separable (over any extension field), we may write it as a product $f = f_1 \cdots f_s \in \kappa(\mathfrak{p})[u]$ of distinct irreducible polynomials. By the Chinese remainder theorem,

$$\kappa(\mathfrak{p})[u]/(f) = \prod_i \kappa(\mathfrak{p})[u]/(f_i)$$

is a finite product of fields, and hence its spectrum is a disjoint union of regular local rings, hence it is regular. This completes the proof that $A[u]/(f) = K \otimes_k \kappa(x)$ is regular, hence that X_K is regular, hence that X is geometrically regular over k. □

Remark 3.1.59. Suppose k is perfect and $\overline{k}$ is an algebraic closure. Then a (finite type) k-scheme X is regular iff $X \to \operatorname{Spec} k$ is smooth iff $X_{\overline{k}} \to \operatorname{Spec} \overline{k}$ is smooth iff $X_{\overline{k}}$ is regular. So, for example, if we are checking regularity of a subvariety of $\operatorname{Spec} k[x_1, \ldots, x_n]$, it certainly does not suffice to check only at the k-points; we have to check at $\overline{k}$-rational points, since this shows that a singularity existing at an $\overline{k}$-point will show up as a closed singular (but non-rational) point over k.

The next is the standard definition/characterization of being a regular local ring, which we have omitted in favor of the other definitions that are easier to work with homologically. The nontrivial result we have to rely on is the following: If $(R, \mathfrak{m})$ is Noetherian local, then we define the associated graded ring by

$$\operatorname{gr}_{\mathfrak{m}}(R) \coloneqq \bigoplus_{n \geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}.$$

It is a result using Hilbert-Samuel functions that $\dim \operatorname{gr}_{\mathfrak{m}}(R) = \dim(R)$.

Proposition 3.1.60. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring. Then R is a regular local ring iff $\dim(R) = \mu(\mathfrak{m})$.*

Proof. We have already proven the ($\Rightarrow$) direction. Let $n = \dim(R)$ and consider a minimal generating set $(x_1, \ldots, x_n) = \mathfrak{m}$ of the maximal ideal. There is a natural map

$$k[T_1, \ldots, T_n] \to \operatorname{gr}_{\mathfrak{m}}(R)$$

given by sending $T_i \mapsto x_i$ viewed as elements of degree one in $\mathfrak{m}/\mathfrak{m}^2$. Since $\operatorname{gr}_{\mathfrak{m}}(R)$ is generated (by definition) in degree 1, this map is surjective. Since $\dim \operatorname{gr}_{\mathfrak{m}}(R) = n$, this map is an isomorphism, as $k[T_1, \ldots, T_n]$ is a Noetherian domain of dimension n.

Based on this we claim that x_n is a non-zerodivisor in R. Indeed, if $ax_n = 0$ for $a \in \mathfrak{m}$, then by the Krull intersection theorem there is a unique $r > 0$ such that $a \in \mathfrak{m}^r$ but $a \notin \mathfrak{m}^{r+1}$. Treating a as a nonzero element of $\operatorname{gr}_{\mathfrak{m}}(R)$ in degree r, this shows ax_n is a nonzero element of $\operatorname{gr}_{\mathfrak{m}}(R)$ in degree $r+1$, i.e. $ax_n \in \mathfrak{m}^{n+1} \setminus \mathfrak{m}^{n+2}$, hence $ax_n \neq 0$ in R.

Now the Noetherian local ring $R/(x_n)$ has embedding dimension $n-1$ with generators $x_1, \dots, x_{n-1}$ of the maximal ideal, and since x_n is regular, it has dimension $n-1$. By induction, $R/(x_n)$ is a regular local ring, so $x_1, \dots, x_{n-1}$ is a regular sequence in $R/(x_n)$, thus $x_1, \dots, x_n$ is a regular sequence in R. □

3.1.4. Appendix: Hilbert's Syzygy Theorems.

Corollary 3.1.61. *Let k be a field and $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^n_k$. There is an exact sequence*

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0$$

of $\mathcal{O}_{\mathbb{P}^n_k}$-modules where each $\mathcal{E}_i$ is a finite direct sum of line bundles $\mathcal{O}_{\mathbb{P}^n_k}(d)$ for various $d \in \mathbb{Z}$.

3.2. Applications of cohomology

3.2.1. Classical applications. We have thus far managed to avoid computing any non-trivial cohomology of any schemes. In this section we will compute the cohomology of projective space $\mathbb{P}^n_R \to \operatorname{Spec}(R)$ over a ring R, and prove many of the classical results, such as Serre's criterion for affineness/ampleness, Chevalley's theorem, Grothendieck coherence, and so on.

Let R be a ring, and $\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$ with the standard grading. Without further qualification, by $\mathbb{P}^n$ we mean $\mathbb{P}^n_R$. We also use $S = R[x_0, \dots, x_n]$ for the graded ring. Recall that $\mathbb{P}^n_R$ is a quasicompact separated scheme with standard affine open cover $U_i \coloneqq D_+(x_i)$ by $n+1$ opens. In particular, we know immediately from Čech cohomology that all quasicoherent sheaves $\mathcal{F}$ on $\mathbb{P}^n$ have $H^q(X, \mathcal{F}) = 0$ for $q \geq n+1$.

Let us first establish the Serre vanishing theorem for projective morphisms, which we get from the following seemingly unrelated observation. On S, multiplication by x_0 gives a map of graded modules $S(i) \to S(i+1)$, so we can consider the directed system

$$S \xrightarrow{x_0} S(1) \xrightarrow{x_0} S(2) \to \cdots$$

of graded modules. Each term admits a map $S(r) \to S_{x_0}$ to the graded localization sending $a \mapsto a/x_0^r$, which is graded since if $a \in S(r)_d = S_{d+r}$, then a/x_0^r has degree d. Therefore we obtain a map $\varinjlim_r S(r) \to S_{x_0}$, which is surjective since all homogeneous elements are in the image by construction. If some $a \in S(r)$ maps to 0, then $x_0^\ell a = 0$ for some ℓ, i.e. a becomes zero in the colimit. Thus the map is an isomorphism. By sheafifying, we can phrase this fact by saying

$$\varinjlim \left(\mathcal{O}_{\mathbb{P}^n} \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^n}(1) \xrightarrow{x_0} \cdots\right) = j_* \mathcal{O}_{U_0}$$

where $j \colon U_0 \hookrightarrow \mathbb{P}^n$ is the inclusion of $U_0 = D_+(x_0)$.

Example 3.2.1. We'll show by induction on n that $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for all $q > 0$ and $d \geq -n$. We will use the projection formula without further mention. This is clear when $n = 0$ because $\mathbb{P}^0$ is affine. For $n > 0$, let $H = V_+(x_0) \cong \mathbb{P}^{n-1}$ and consider the hyperplane sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-1) \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_H(d) \to 0$$

twisted by $\mathcal{O}_{\mathbb{P}^n}(d)$. By induction, when $d \geq -n+1$, the right-hand sheaf has cohomology concentrated in degree 0, so certainly

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) \to H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

is an isomorphism for $q \geq 2$. It is also an isomorphism for $q = 1$, because either $d < 0$ where this is clear, or the map $\mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_H(d)$ is the surjection

$$R[x_0, \ldots, x_n] \to R[x_1, \ldots, x_n] \qquad x_0 \mapsto 0$$

on global sections. Therefore each morphism in the directed system

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n)) \xrightarrow{x_0} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n+1)) \xrightarrow{x_0} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n+2)) \to \cdots$$

is an isomorphism. Since $\mathbb{P}^n$ is qcqs, cohomology commutes with filtered colimits and we have an isomorphism

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \xrightarrow{\sim} H^q(\mathbb{P}^n, \varinjlim_{d \geq -n} \mathcal{O}_{\mathbb{P}^n}(d)) = H^q(\mathbb{P}^n, \varinjlim_{d \geq 0} \mathcal{O}_{\mathbb{P}^n}(d))$$

for all $d \geq -n$ (of course, by the isomorphism we are free to start the colimit at $d = 0$). As we just showed, the RHS is

$$H^q(\mathbb{P}^n, \varinjlim_{d \geq 0} \mathcal{O}_{\mathbb{P}^n}(d)) = H^q(\mathbb{P}^n, j_* \mathcal{O}_{U_0}) \cong H^q(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n}) = 0,$$

using the fact that $j \colon U_0 \hookrightarrow \mathbb{P}^n$ is an affine morphism as $\mathbb{P}^n$ is separated.

Corollary 3.2.2. *Let Y be a locally Noetherian scheme, $f \colon X \to Y$ be a projective morphism, and $\mathcal{L}$ an f-ample line bundle on X. For each $A \in D^b_{\mathrm{coh}}(X)$, there exists $n_0 \in \mathbb{Z}$ such that the natural maps*

$$H^p(Rf_*(A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d})) \to f_* H^p(A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d})$$

are isomorphisms for all $d \geq n_0$ and all $p \in \mathbb{Z}$. In particular, if $\mathcal{F}$ is a coherent sheaf then for $d \gg 0$,

$$R^p f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes d}) = 0$$

for all $p > 0$.

Proof. We have seen that this map being an isomorphism is controlled by the edge maps of the spectral sequence

$$R^p f_*(H^q(A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d})) \implies H^{p+q}(Rf_*(A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d})).$$

Since $\mathcal{L}^{\otimes d}$ is flat, the LHS is $R^p f_*(H^q(A) \otimes \mathcal{L}^{\otimes d})$. Since $H^q(A)$ is nonzero for only finitely many q, we can immediately reduce to proving the special case in the theorem statement since it will imply that for $d \gg 0$, the sequence will only have a $p = 0$ row.

Now this claim is local on the target, so by choosing an affine open nbhd in Y we may assume $Y = \operatorname{Spec} R$ for R a Noetherian ring, f factors through a closed immersion as $X \hookrightarrow \mathbb{P}^m_R \to \operatorname{Spec}(R)$, and $\mathcal{L}$ is an R-ample line bundle on X. Then $R^i f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes d})$ corresponds to the R-module $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d})$. We can replace

$\mathcal{L}$ with a finite tensor power and assume $\mathcal{L}$ is very ample, and thus it gives a projective embedding

$$i\colon X \hookrightarrow \mathbb{P}^n_R \qquad i^*(\mathcal{O}_{\mathbb{P}^n_R}(1)) = \mathcal{L}.$$

Therefore $\mathcal{F} \otimes \mathcal{L}^{\otimes d} = \mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^n_R}(d)$, so

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes d}) = H^i(\mathbb{P}^n_R, (i_*\mathcal{F})(d))$$

by closed projection, with $i_*\mathcal{F}$ still coherent. Thus we have reduced to the case where f is $\mathbb{P}^n_R \to \operatorname{Spec} R$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^n_R}(1)$.

We prove this for all coherent sheaves on $\mathbb{P}^n_R$ by descending induction on a number r for which, for each $\mathcal{F}$, whenever $m \gg 0$, $H^q(\mathbb{P}^n, \mathcal{F}(m)) = 0$ for all $q > r$. This is clear when $r = n$ since $\mathcal{F}$ has no cohomology in higher degrees. Assume this has been shown for r and let $\mathcal{F}$ be a coherent sheaf.

$$0 \to \mathcal{K} \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n_R}(-a_i) \to \mathcal{F} \to 0.$$

By Example 3.2.1, we can twist by $\mathcal{O}_{\mathbb{P}^n}(m)$ for $m \gg 0$ so that the middle term has no higher cohomology, and by our hypothesis we can increase m until $\mathcal{K}(m)$ has no cohomology in degrees $> r$:

$$0 \to \mathcal{K}(m) \to \bigoplus_i \mathcal{O}_{\mathbb{P}^n}(m - a_i) \to \mathcal{F}(m) \to 0$$

The sequence shows $H^i(\mathbb{P}^n, \mathcal{F}(m)) \xrightarrow{\sim} H^{i+1}(\mathbb{P}^n, \mathcal{K}(m))$ for $i \geq 1$, which means that for $m \gg 0$ (depending only on $\mathcal{F}$), $\mathcal{F}(m)$ has no cohomology in degrees $> r - 1$ as long as $r \geq 1$. This proves the inductive step. □

This result is called the *Serre vanishing* theorem, and gives a good example of the flavor of proofs in foundations of sheaf cohomology.

Example 3.2.3. Let X be a projective scheme over a Noetherian ring R and $\mathcal{L}$ an ample line bundle. Then the family $\{\mathcal{L}^{\otimes m} : m \in \mathbb{Z}\}$ forms a (weak) generating set for $D^b_{\mathrm{coh}}(X)$. To see this, choose $A \in D^b_{\mathrm{coh}}(X)$ and let $d \gg 0$ so that

$$\mathbb{H}^p(X, A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d}) = H^0(X, H^p(A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d})) = H^0(X, H^p(A) \otimes \mathcal{L}^{\otimes d}).$$

Assuming $A \neq 0$, choose p such that $H^p(A) \neq 0$, then by ampleness we may increase d until $H^p(A) \otimes \mathcal{L}^{\otimes d}$ is generated by global sections. In particular the RHS above is nonzero, which implies that

$$\mathbb{H}^p(X, A \otimes^{\mathbf{L}} \mathcal{L}^{\otimes d}) = \operatorname{Hom}_{D^b_{\mathrm{coh}}(X)}(\mathcal{L}^{\otimes -d}[-p], A) \neq 0$$

as desired. However a warning is that $D^b_{\mathrm{coh}}(X)$ doesn't have arbitrary sums, so this generating set doesn't immediately enjoy the nice properties we proved in the previous section.

We will next again apply Example 3.2.1 to compute the cohomology of twisting sheaves on $\mathbb{P}^n$.

THEOREM 3.2.4. *Let R be a ring and $n > 0$. Then $H^p(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d))$ is a finite free R-module for all $d \in \mathbb{Z}$ and $p \geq 0$, with rank given by*

$$\operatorname{rk}_R H^p(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) = \begin{cases} \binom{n+d}{d} & p = 0, d \geq 0 \\ \binom{-d-1}{-n-d-1} & p = n, d \leq -n-1 \\ 0 & \text{otherwise} \end{cases}$$

for all $d \in \mathbb{Z}$ and $p \geq 0$. More precisely, there is a canonical isomorphism $\operatorname{tr}\colon H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-n-1)) \xrightarrow{\sim} R$, and the natural maps

$$H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-n-1-d)) \times H^0(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-n-1))$$

form a perfect pairing of free R-modules for $d \in \mathbb{Z}$.

PROOF. Everything is already known to us about $\mathcal{O}_{\mathbb{P}^n}(d)$ for $d \geq -n$ by Example 3.2.1 and the standard identification of H^0 with $R[x_0, \ldots, x_n]_d$ in the case that $d \geq 0$. Otherwise all we know is $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for $d \leq 0$.

The remainder of this proof is about the standard Koszul exact sequence associated to $x_0, \ldots, x_n \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, which we denote by $K^\bullet$, and write $K^\bullet(d)$ for its twist by $\mathcal{O}_{\mathbb{P}^n}(d)$. This is in degrees $[-n-1, 0]$ and looks like

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1+d) \to \mathcal{O}_{\mathbb{P}^n}(-n+d)^{n+1} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(d) \to 0.$$

Now consider the hypercohomology spectral sequence; since the complex is exact, the spectral sequence will converge to 0. First, if we take $d = 0$, then the entire sequence is zeroes except for $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = R$ at $E_1^{0,0}$ and the column $E_1^{-n-1,*}$. The only chance of a nonzero differential is the one on the nth page which will be

$$H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) = E_n^{-n-1,n} \to E_n^{0,0} = R,$$

and therefore this must be an isomorphism since $E_n = E_\infty$. This also shows that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) = 0$ for $i < n$. This gives the map tr. By composing the canonical pairing mentioned in the statement with the trace map, we obtain a canonical map

$$\Phi_d\colon H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-n-1-d)) \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^\vee$$

with RHS finite free, which sends a class α to the dual element $f \mapsto \operatorname{tr}(\alpha f)$ where αf is the cup product class in $H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n}(-n-1))$. When $d = 0$, this is the trace isomorphism by construction.

Now take $d = -1$. Now the only possibly nontrivial terms in the spectral sequence are the left-most row, i.e. the cohomology of $\mathcal{O}_{\mathbb{P}^n}(-n-2)$, and the single term $E^{-n,n} = H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1))^{n+1} = R^{n+1}$ from what we just computed. Thus it is clear that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-2)) = 0$ for $i < n$, and the top row becomes an exact sequence. Repeating this argument and shifting d by -1 each time gives

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1+d)) = 0 \qquad d \leq 0, i < n,$$

and proves that the complex $H^n(\mathbb{P}^n, K^\bullet(-d))$ is exact for all $d > 0$, since this complex is the top row of the spectral sequence.

On the other hand, the complex $H^0(\mathbb{P}^n, K^\bullet(d))$ is also exact for $d > 0$, because under this hypothesis the only line bundles involved in $K^\bullet(d)$ are $\mathcal{O}(m)$ for $m \geq -n$,

which have no higher cohomology. Since this is a complex of finite free modules (i.e. it is strictly perfect), $H^0(\mathbb{P}^n, K^\bullet(d))^\vee$ is also exact. We can construct a morphism between these complexes using the natural maps Φ_d as follows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-d)) & \longrightarrow & H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-2-d))^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \Phi_d} & & \downarrow{\scriptstyle \Phi_{d-1}^{\oplus(n+1)}} & & \\ 0 & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^\vee & \longrightarrow & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d+1))^{\vee,n+1} & \longrightarrow & \cdots \end{array}$$

Each square commutes by definition of the Koszul differentials and the naturality properties of the cup product. We claim Φ_d is an isomorphism; for $d < 0$ both sides are 0, we have already seen $d = 0$, and then $d > 0$ follows by induction and the morphism of exact sequences above. The explicit ranks follow from the ranks of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ for $d \geq 0$. □

The following result, called the *Grothendieck coherence theorem*, is an important consequence. Its proof will be similar to Serre vanishing, with a couple of extra steps. We use the fact that a spectral sequence with coherent terms over a Noetherian scheme will converge to a coherent sheaf, whose proof is just as easy as the quasicoherent case.

THEOREM 3.2.5. *Let Y be a locally Noetherian scheme and $f\colon X \to Y$ be a proper morphism. Then Rf_* restricts to a functor*

$$Rf_*\colon D_{\mathrm{coh}}(X) \to D_{\mathrm{coh}}(Y),$$

and in particular to $D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ if Y is Noetherian.

PROOF. A proper morphism is in particular qcqs, so Rf_* is defined. Moreover, since the claim is local on Y we may assume it is quasicompact, and then Remark 2.3.41 shows that

$$H^p(Rf_*A) = \varprojlim_n H^p(Rf_*\tau_{\geq -n}A)$$

where the RHS is eventually constant. This allows us to reduce to proving the claim for $D^+_{\mathrm{coh}}(X) \to D^+_{\mathrm{coh}}(Y)$. Also, if Y was already quasicompact then we know that Rf_* preserves D^-_{qc} by Lemma 2.3.40, so the second claim follows. Moreover, by the spectral sequence

$$R^q f_*(H^p(A)) \implies H^{p+q}(Rf_*A),$$

we further reduce to the case that $A = \mathcal{F}$ is a single coherent sheaf.

Step 1: First suppose that f is projective. Since the question is local on Y, we may assume $Y = \operatorname{Spec} R$ for R a Noetherian ring, where f factors as

$$X \xrightarrow{i} \mathbb{P}^n_R \to \operatorname{Spec} R$$

through a closed immersion i. Then $R^i f_*(\mathcal{F})$ computes cohomology of $\mathcal{F}$ or equivalently $i_*(A)$, hence we may assume $X = \mathbb{P}^n_R$ and $i = \mathrm{Id}$. So, it suffices to show $H^p(\mathbb{P}^n, \mathcal{F})$ is coherent (i.e. a finite R-module) for $\mathcal{F}$ a coherent sheaf.

We prove this by descending induction, where the $p = n + 1$ case is clear as the cohomology is 0. For $p \leq n$, take an exact sequence

$$0 \to \mathcal{K} \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{E} = \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(-a_i)$, where we may assume that $H^i(\mathbb{P}^n, \mathcal{K})$ is a finite module for $i > p$ by induction. Then we have an exact sequence

$$H^p(\mathbb{P}^n, \mathcal{E}) \to H^p(\mathbb{P}^n, \mathcal{F}) \to H^{p+1}(\mathbb{P}^n, \mathcal{K})$$

where the left side is a finite R-module since we have computed its cohomology explicitly in Thm. 3.2.4. The right side is finite by induction. By splicing, we get a short exact sequence $0 \to C \to H^p(\mathbb{P}^n, \mathcal{F}) \to C' \to 0$ where C is the image of $H^p(\mathbb{P}^n, \mathcal{E})$ hence finite, and C' is a submodule of $H^{p+1}(\mathbb{P}^n, \mathcal{K})$ which is finite as R is Noetherian.

Step 2: Now suppose f is only proper. Again we immediately reduce to the case $Y = \operatorname{Spec} R$ is affine Noetherian. We proceed by Noetherian induction on $\operatorname{Supp} \mathcal{F} \neq \varnothing$, and since $\mathcal{F}$ is coherent, we may replace X by the scheme-theoretic support $V(\operatorname{Ann} \mathcal{F})$ of $\mathcal{F}$ and assume $\operatorname{Supp} \mathcal{F} = X$. By Chow's lemma, we may find a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ & \searrow^{p} & \downarrow f \\ & & \operatorname{Spec} R \end{array}$$

where both π and $p\colon X' \to \operatorname{Spec} R$ are projective, and π is surjective and restricts to an isomorphism on a dense open set $U \subset X'$. We may assume $U \subsetneq X'$ otherwise we're done.

Consider the coherent sheaf $\mathcal{G} := \pi^* \mathcal{F}$, then by Step 1 $\pi_* \mathcal{G}$ is coherent, and moreover the unit $\phi\colon \mathcal{F} \to \pi_* \mathcal{G}$ is an isomorphism on U, as $(\pi_* \pi^* \mathcal{F})|_U = \pi|_{U,*} \pi|_U^* \mathcal{F}$ and $\pi|_U$ is an isomorphism. This in fact reduces the problem to showing that $Rf_* \pi_* \mathcal{G}$ is coherent: Indeed, apply Rf_* to the triangle

$$\operatorname{im} \phi \to \pi_* \mathcal{G} \to \operatorname{cok} \phi \to \operatorname{im} \phi[1],$$

then the observation above shows that $\operatorname{cok} \phi$ is a coherent sheaf supported on a proper closed subset, hence $Rf_* \operatorname{cok} \phi$ is coherent. So this will show $Rf_* \operatorname{im} \phi$ is coherent. Then apply Rf_* to the triangle

$$\ker \phi \to \mathcal{F} \to \operatorname{im} \phi \to \ker \phi[1],$$

and for the same reason as above this will prove $Rf_* \mathcal{F} \in D^+_{\mathrm{coh}}(Y)$. Similarly, since π is an isomorphism on U, $R^q \pi|_{U,*} = 0$ for $q > 0$, meaning $R^q \pi_* \mathcal{G}$ is supported on a proper closed subset of X and hence $R^p f_* R^q \pi_* \mathcal{G}$ is coherent for all p and $q \geq 1$.

Step 3: So let's consider the Leray spectral sequence for the composite $p = f\pi$, i.e.

$$E_2^{p,q} = R^p f_* (R^q \pi_* \mathcal{G}) \implies R^{p+q} p_* \mathcal{G}.$$

We just said $E_2^{p,q}$ is coherent for $q \geq 1$, and inductively, $E_r^{p,q}$ is coherent for $q \geq 1$ (and $r \geq 2$) because it is obtained as a quotient of a subobject of $E_{r-1}^{p,q}$. In particular $E_\infty^{p,q}$ is coherent for $q \geq 1$.

Now look at convergence: Writing $H^n = R^n p_* \mathcal{G}$, we observe that since p is projective, H^n is coherent. We have a filtration

$$H^n = F^0 H^n \supset F^1 H^n \supset \cdots \supset F^n H^n = 0$$

equipped with exact sequences

$$0 \to F^{p+1} H^n \to F^p H^n \to E_\infty^{p,n-p} \to 0.$$

If $n - p \geq 1$ i.e. $p < n$, then the right-hand term is coherent so $F^p H^n$ being coherent implies $F^{p+1} H^n$ is coherent. Since this is true for $p = 0$, inductively we conclude $F^p H^n \cong E_\infty^{p,0}$ is coherent.

To conclude, suppose $E_r^{p,0}$ is coherent for $r \geq 3$ (if $r = 2$ then we're done). By definition there's a quotient sequence

$$0 \to \operatorname{im} d \to E_{r-1}^{p,0} \to E_r^{p,0} \to 0$$

for some differential d as $E_{r-1}^{p,0}$ has no outbound differential. d has source some $E_{r-1}^{p',q'}$ for $q' \geq 1$, hence $\operatorname{im} d$ is coherent. Thus $E_{r-1}^{p,0}$ is coherent. Hence inductively $E_2^{p,0}$ is coherent. Taking $r \gg 0$ so that $E_r^{p,0} = E_\infty^{p,0}$ gives such an r, proving coherence of $E_2^{p,0}$ as desired. □

If $f\colon X \to Y$ is a qcqs morphism, we (in light of this result) say f is *cohomologically proper* if Rf_* preserves pseudocoherent objects of $D_{\mathrm{qc}}(X)$. If Y is Noetherian and f is proper (so X is Noetherian too), then we know that Rf_* preserves both D_{coh} and D_{qc}^-, hence it preserves $D_{\mathrm{coh}}^-(X)$, i.e. the pseudocoherent objects.

Example 3.2.6. Let Y be locally Noetherian and $f\colon X \to Y$ be an affine morphism. Then f is finite iff f is proper. If f is finite, then it is projective and hence proper. Conversely, the result above shows that $f_* \mathcal{O}_X$ is a finite type sheaf on Y.

Remark 3.2.7. In the theorem above, take the most trivial Y and remove all derived implications. The result implies that if X is a proper k-scheme, then $\Gamma(X, \mathcal{F})$ is a finite-dimensional vector space for any coherent sheaf $\mathcal{F}$. This is not obvious by any means from the definition, even if X was projective or even $\mathbb{P}_k^n$ itself.

Remark 3.2.8. Let us make some statements about $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}$, for which it is important to know when it lies in $D_{\mathrm{coh}}^*(X)$. This behaves relatively well only in the Noetherian (or coherent) case, so assume X is Noetherian. Then we know that if A is pseudocoherent, i.e. $A \in D_{\mathrm{coh}}^-(X)$, and $B \in D_{\mathrm{qc}}^+(X)$, then $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \in D_{\mathrm{qc}}^+(X)$. In fact, if $B \in D_{\mathrm{coh}}^+(X)$, then $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \in D_{\mathrm{coh}}^+(X)$. Indeed, the hyper-ext spectral sequence reduces us to the case where $A = \mathcal{F}$ is a coherent sheaf, and the claim is local so we may assume X is Noetherian affine. Then $\mathcal{F}$ has a resolution by a complex $\mathcal{E}^\bullet$ of finite free $\mathcal{O}_X$-modules, which is internally projective, so we have to prove the claim about $\underline{\mathcal{H}om}_{\mathcal{O}_X}(\mathcal{E}^\bullet, \mathcal{G}^\bullet)$ where $\mathcal{G}^\bullet$ has coherent cohomology. Then our usual argument using the totalization spectral

sequence implies that this has coherent cohomology, because the E_2 page consists of modules of the form $H^p(\mathcal{G}^\bullet)^m$ for some m. However we can also say the following:

1. In the case $A = E \in \mathrm{Perf}(X)$ for any (qcqs) scheme and any pseudocoherent B, $\mathbf{R}\mathcal{H}om(E, B)$ is pseudocoherent because it is isomorphic to $E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B$.
2. If X is Noetherian, $E \in \mathrm{Perf}(X)$, and $B \in D_{\mathrm{coh}}(X)$, then $\mathbf{R}\mathcal{H}om(E, B) \in D_{\mathrm{coh}}(X)$. This is for instance by the hyper-Tor spectral sequence when one of the terms is bounded, applied on a nbhd where E is a bounded complex of vector bundles.

Example 3.2.9. This is an example of the local-to-global spectral sequence we have already seen. Let X be a scheme, $A \in D^-_{\mathrm{qc}}(X)$ be pseudocoherent, and $B \in D^+_{\mathrm{qc}}(X)$. Then we know $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B) \in D^+_{\mathrm{qc}}(X)$, and we have the spectral sequence

$$H^p(X, H^q(\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(A, B))) \implies H^{p+q}(\mathrm{RHom}_{\mathcal{O}_X}(A, B))$$

of $\mathcal{O}_X(X)$-modules. In particular, suppose X is projective over a Noetherian ring R and $\mathcal{L}$ is an R-ample line bundle, with $A \in D^-_{\mathrm{coh}}(X)$ and $B \in D^+_{\mathrm{coh}}(X)$. Then for each $q \in \mathbb{Z}$, we just saw that $H^q(\mathbf{R}\mathcal{H}om(A, B)) \in \mathrm{Coh}(X)$, so by Serre vanishing there exists $n \gg 0$ such that the coherent sheaf

$$\begin{aligned} H^q(\mathbf{R}\mathcal{H}om(A, B)) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n} &= H^q(\mathbf{R}\mathcal{H}om(A, B) \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) \\ &= H^q(\mathbf{R}\mathcal{H}om(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n})) \end{aligned}$$

has no higher cohomology (we're using flatness/perfectness of $\mathcal{L}$). Then by the spectral sequence, it follows that

$$H^0(X, H^q(\mathbf{R}\mathcal{H}om(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}))) = \mathrm{Hom}_{D(X)}(A, B \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{L}^{\otimes n}[q])$$

for $n \gg 0$ (depending on q, A, and B).

Corollary 3.2.10. *Let R be a Noetherian ring and X a proper R-scheme.*

(a) If $A \in D^-_{\mathrm{coh}}(X)$ and $B \in D^+_{\mathrm{coh}}(X)$, then $\mathrm{Hom}_{D(X)}(A, B)$ is a finite R-module.

(b) If $E \in \mathrm{Perf}(X)$ and $B \in D_{\mathrm{coh}}(X)$, then $\mathrm{Hom}_{D(X)}(E, B)$ is a finite R-module.

(c) If $E \in \mathrm{Perf}(X)$ and $B \in D^b_{\mathrm{coh}}(X)$, then the whole graded object

$$\mathrm{Hom}^*_{D(X)}(E, B) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{D(X)}(E, B[n])$$

is a finite R-module.

In particular, $D^b_{\mathrm{coh}}(X)$ is enriched in the category $\mathrm{Mod}_{\mathrm{fg}}(R)$.

Proof. (a) This follows directly from the local-to-global spectral sequence for Exts, the fact that $\mathbf{R}\mathcal{H}om(A, B)$ has coherent cohomology, and the fact that the sheaf cohomology of a coherent sheaf is finite over R since X is proper. (b) The hom-module is H^0 of $E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D_{\mathrm{coh}}(X)$. (c) This is because $E^\vee \otimes^{\mathbf{L}}_{\mathcal{O}_X} B \in D^b_{\mathrm{coh}}(X)$ and (b). The final statement is an obvious consequence of (a). □

Here is one concrete application of the cohomology of $\mathbb{P}^n$.

Proposition 3.2.11. *Let k be a field. Every vector bundle on $\mathbb{P}^1_k$ splits into a sum of line bundles. For each $n \geq 2$, $\mathbb{P}^n_k$ has a rank n vector bundle which is not a sum of line bundles.*

Proof. The latter case is easy to see: take the standard Koszul resolution and cut it off after the first step to produce a sequence

$$0 \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

(which is none other than the Euler sequence, i.e. $\mathcal{E} = \Omega^1_{\mathbb{P}^n}$). Then $\mathcal{E}$ is a rank n vector bundle which has

$$k \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \xrightarrow{\sim} H^1(\mathbb{P}^n, \mathcal{E}),$$

but none of the twisting sheaves have any H^1 when $n \geq 2$.

Now consider $\mathbb{P}^1_k$, and let $\mathcal{E}$ be a vector bundle of rank $r > 1$. $\mathcal{E}(n)$ is globally generated for $n \gg 0$, and on the other hand the same is true of $\mathcal{E}^\vee$, so we may write $\mathcal{E}(-n) \subset \mathcal{O}(-1)^m$ for $n \gg 0$. Thus there is some unique middle point n_0 for which $h^1(\mathbb{P}^1, \mathcal{E}(n_0)) > 0$ but $h^1(\mathbb{P}^1, \mathcal{E}(n_0 - \ell)) = 0$ for all $\ell > 0$.

Choose a nonzero section $s\colon \mathcal{O} \to \mathcal{E}(n_0)$, which is necessarily regular, so we obtain an exact sequence

$$0 \to \mathcal{O} \xrightarrow{s} \mathcal{E}(n_0) \to \mathcal{E}'' \to 0,$$

and we claim $\mathcal{E}''$ is locally free. Indeed it suffices to prove that $V(s) = \varnothing$, as then in each local ring $R = \mathcal{O}_{\mathbb{P}^1,x}$, the map above is isomorphic to one of the form $R \to R^n$ sending $1 \mapsto e_1$, which has free quotient R^{n-1}. To show this, note that the generic point is not in $V(s)$ as $s \neq 0$, and for each closed point x of degree $d = [\kappa(x) : k]$, vanishing at x would give a section of $\mathcal{E}(n_0)(-p) = \mathcal{E}(n_0 - d)$ which we said there are none.

Therefore $\mathcal{E}''$ is free of rank $r-1$, so inductively is a sum of line bundles. Twisting the defining exact sequence by $\mathcal{O}(-1)$, we find that since $h^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, our assumption gives $h^0(\mathbb{P}^1, \mathcal{E}''(-1)) = 0$. Therefore if $\mathcal{E}'' = \bigoplus \mathcal{O}(a_i)$, then $a_i - 1 \leq -1$ for all i, i.e. $a_i \leq 0$. Finally, observe that

$$\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^1}}(\mathcal{E}'', \mathcal{O}) = H^1(\mathbb{P}^1, (\mathcal{E}'')^\vee) = \bigoplus_i H^1(\mathbb{P}^1, \mathcal{O}(-a_i)) = 0$$

since $-a_i \geq 0$. Hence the sequence splits, so $\mathcal{E}(n_0)$ is a sum of line bundles, and thus $\mathcal{E}$ is too. □

Note that this fails if k is not a field, even if it is a Noetherian UFD so that all line bundles are of the form $\mathcal{O}(d)$. Indeed, we saw in Example 2.4.35 that $\mathbb{P}^1_{k[t]}$ has a vector bundle which isn't a sum of line bundles. Here's another simple example:

Example 3.2.12. Let R be a ring and $X = \mathbb{A}^n_R \setminus 0 = \mathbb{A}^n_R \setminus V(x_1, \ldots, x_n)$. The natural map $p\colon X \to \mathbb{P}^{n-1}_R$ is affine since $p^{-1}(D_+(x_i)) = D(x_i)$. By construction $p^*\mathcal{O}(1) = \mathcal{O}_X$, and thus $p^*\mathcal{O}(d) = \mathcal{O}_X$. By adjunction,

$$\bigoplus \mathcal{O}_X \to \mathcal{O}_X \qquad \leftrightarrow \qquad \mathcal{F} \coloneqq \bigoplus_d \mathcal{O}(d) \to p_*\mathcal{O}_X,$$

the codiagonal map corresponds to a map which we claim is an isomorphism. Indeed, we know $\mathcal{F}(U_i) = R[x_1, \ldots, x_n][x_i^{-1}]$ and it is clear that the restriction of $\mathcal{F} \to p_*\mathcal{O}_X$ is the canonical identification. Since p is affine, it follows that $H^0(X, \mathcal{O}_X) = R[x_1, \ldots, x_n]$ as an R-module, and

$$H^{n-1}(X, \mathcal{O}_X) = \bigoplus_{m \geq 0} R[x_1, \ldots, x_n]_m^\vee$$

with all other cohomology modules trivial.

Next we will prove two basic results both called 'Serre's criterion', for affineness and ampleness respectively, and deduce some interesting consequences. Usually, for each affineness result there is a corresponding result for ampleness of line bundles, so we will usually state both and prove them simultaneously.

Proposition 3.2.13. *Let X be a quasicompact scheme. If $H^1(X, \mathcal{I}) = 0$ for every quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, then X is affine.*

Proposition 3.2.14. *Let X be a quasicompact scheme and $\mathcal{L}$ a line bundle on X. If for each quasicoherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$, there exists an $n \geq 1$ such that $H^1(X, \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}) = 0$, then $\mathcal{L}$ is ample.*

Proofs (of Prop. 3.2.13 and Prop. 3.2.14). Let $x \in X$ be a closed point and U an affine open nbhd of x such that $\mathcal{L}|_U$ is trivial. We can choose ideal sheaves $\mathcal{I}' \subset \mathcal{I}$ such that $V(\mathcal{I}') = (X \setminus U) \cup \{x\}$ and $V(\mathcal{I}) = X \setminus U$ by taking the reduced subscheme structures. Writing $\mathcal{F} = \mathcal{I}/\mathcal{I}'$, we see $\operatorname{Supp} \mathcal{F} = \{x\}$ and $\mathcal{F}$ is the pushforward of the structure sheaf along $i\colon \{x\} \hookrightarrow X$. Tensor the defining sequence with $\mathcal{L}^{\otimes n}$,

$$0 \to \mathcal{I}' \otimes \mathcal{L}^{\otimes n} \to \mathcal{I} \otimes \mathcal{L}^{\otimes n} \to i_*\mathcal{O}_{\{x\}} \otimes \mathcal{L}^{\otimes n} \to 0,$$

and choose n so that $H^1(X, \mathcal{I}' \otimes \mathcal{L}^{\otimes n}) = 0$. In the case of Prop. 3.2.13 we can just take $\mathcal{L} = \mathcal{O}_X$. By the projection formula, the right-most term is $i_*(\mathcal{L}^{\otimes n}(x))$ and taking global sections gives a surjection $H^0(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \to \mathcal{L}^{\otimes n}(x)$. By lifting a nonzero element of this vector space to

$$f \in H^0(X, \mathcal{I} \otimes \mathcal{L}^{\otimes n}) \subset H^0(X, \mathcal{L}^{\otimes n}),$$

it follows that $x \notin V(f)$, and $V(\mathcal{I}) \subset V(f)$, i.e. $x \in D(f) \subset U$. Therefore $D(f) = D_U(f|_U)$ for $f|_U$ a section of $\mathcal{L}^{\otimes n}|_U \cong \mathcal{O}_U$, which in particular shows that $D(f)$ is affine.

Consider the union of all affine open sets of the form X_f where $f \in H^0(X, \mathcal{L}^{\otimes n})$ for various $n > 0$, which is an open set containing all closed points by the above case. Since all points in a quasicompact scheme specialize to a closed point, this covers X, which completes the proof of Prop. 3.2.14.

For Prop. 3.2.13, we may now choose $f_1, \ldots, f_n$ sections of $\mathcal{O}_X$ such that $X = \bigcup_i X_{f_i}$ and each X_{f_i} is affine, then by the elementary affineness criterion it suffices to show $(f_1, \ldots, f_n) = (1)$. Since these cover X, we can write an exact sequence

$$0 \to \mathcal{K} \to \mathcal{O}_X^n \xrightarrow{(f_i)} \mathcal{O}_X \to 0$$

and we want to show the second map is surjective on sections, so it suffices to prove that $H^1(X, \mathcal{K}) = 0$. Indeed, letting $\mathcal{K}_i = \mathcal{K} \cap \mathcal{O}_X^i \subset \mathcal{O}_X$ for the intersection with the first i summands of $\mathcal{O}_X$, we obtain a filtration $0 = \mathcal{K}_0 \subset \mathcal{K}_n = \mathcal{K}$ such that

$$0 \to \mathcal{K}_i \to \mathcal{K}_{i+1} \to \mathcal{I}_i \to 0$$

is exact for $\mathcal{I}_i$ some ideal sheaf. Since $\mathcal{K}_0$ has trivial H^1, by induction we conclude $\mathcal{K}_n = \mathcal{K}$ does too. □

It is easy to formulate and prove relative versions of this to obtain criteria for f-ampleness and affine morphisms, but we won't need this. Future things to add here: criteria for irreducible components, the affineness of/ampleness over the reduction, Chevalley's criterion for descent of affineness/ampleness under surjective finite morphisms, etc. Future results to add here: Theorem on formal functions and Zariski's main theorem.

Next, let us give a completely elementary proof of the bounded-coherent duality for $\mathbb{P}^n_R$ over a Noetherian ring R. We will enhance this using our results about duality shortly.

3.2.2. Reprise: Standard applications. In this section we will, using some of our work from the previous sections, give new proofs of some basic parts of cohomology theory. We will provide non-standard proofs of the computation of cohomology of $\mathbb{P}^n_R$, Serre vanishing, Grothendieck coherence, and Serre duality. The emphasis is on using the extensive theory we have developed to greatly simplify some of the usual big results.

Example 3.2.15. Let R be a ring and $\mathbb{P}^n = \mathbb{P}^n_R$ projective space over R. We'll show by induction on n that $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$ for $q > 0$ and $d \geq -n$. When $n = 0$ this is clear because $\mathbb{P}^0 = \operatorname{Spec} R$ is affine. It follows from Example 2.4.29 applied to the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ and affine vanishing that

$$\varinjlim \left(H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \xrightarrow{x_0} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \xrightarrow{x_0} \cdots \right) = 0$$

for all $q > 0$. On the other hand, consider the hyperplane sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(d-1) \xrightarrow{x_0} \mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_{\mathbb{P}^{n-1}}(d) \to 0$$

twisted by $\mathcal{O}_{\mathbb{P}^n}(d)$. By induction, $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$ has no higher cohomology when $d \geq -n+1$, so

$$H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-1)) \xrightarrow{\sim} H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

is an isomorphism for $d \geq -n+1$ and $q \geq 2$, and a surjection for $q = 1$. To see that it is also an injection for $q = 1$, observe that $\mathcal{O}_{\mathbb{P}^n}(d) \to \mathcal{O}_{\mathbb{P}^{n-1}}(d)$ induces a surjection on global sections for all d. Therefore all maps in the direct system above are isomorphisms, and hence each term equals zero. The first term in the isomorphism chain is $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n)) = 0$, completing the proof.

THEOREM 3.2.16. *Let R be a ring, $n > 0$, and denote by $p\colon \mathbb{P}^n_R \to \operatorname{Spec} R$ the structure map.*

(a) p is quasi-perfect and hence quasi-proper.

(b) There is a canonical isomorphism $\operatorname{tr}\colon H^n(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-n-1)) \xrightarrow{\sim} R$ *inducing an isomorphism* $\mathcal{O}_{\mathbb{P}^n_R}(-n-1)[n] \xrightarrow{\sim} p^\times \mathcal{O}_R$.

(c) The isomorphism of (b) induces isomorphisms

$$R\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(-n-1-d)) \cong R\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d))^\vee[-n].$$

for all d. In particular,

$$R\Gamma(\mathbb{P}^n_R, \mathcal{O}_{\mathbb{P}^n_R}(d)) \cong \begin{cases} (R[x_0, \ldots, x_n]_d)[0] & d \geq 0 \\ 0 & -n \leq d < 0 \\ (R[x_0, \ldots, x_n]^\vee_{n+1-d})[-n] & d \leq -n-1 \end{cases}$$

canonically induced by the isomorphisms of (b).

PROOF. (a) In Example 2.5.34 we showed that $\mathcal{O}_{\mathbb{P}^n}(-n), \ldots, \mathcal{O}_{\mathbb{P}^n}$ form a generating set for $D_{\mathrm{qc}}(\mathbb{P}^n)$. Example 3.2.15 shows that Rp_* of the first $n-1$ of these are 0, and $Rp_*\mathcal{O}_{\mathbb{P}^n}$ is $\mathcal{O}_R[0]$. These are all perfect, so p is quasiperfect by Lemma 2.4.56.

(b) Consider the standard Koszul exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \mathcal{O}_{\mathbb{P}^n}(-n)^{n+1} \to \cdots \to \mathcal{O}_{\mathbb{P}^n}(-1)^{n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

on $\mathbb{P}^n$, which we view as a resolution of $\mathcal{O}_{\mathbb{P}^n}(-n-1)[0]$ concentrated in $[0,n]$. Now apply the hypercohomology spectral sequence; by Example 3.2.15, the only nonzero term is $E_{n,0} = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = R$, so it degenerates, giving $H^q(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) = 0$ for $q \neq n$, and the edge map provides an isomorphism

$$\mathrm{tr}\colon H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \xrightarrow{\sim} R.$$

In particular, $Rp_*\mathcal{O}_{\mathbb{P}^n}(-n-1) \cong \mathcal{O}_R[-n]$, so by adjunction there is a canonical map $\mathcal{O}_{\mathbb{P}^n}(-n-1)[n] \to p^\times \mathcal{O}_R$. We can check that this is an isomorphism after applying $\mathrm{RHom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(d), -)$ for $d = -n, \ldots, 0$ by the generating set listed in (a). This gives

$$\begin{array}{ccc} \mathrm{RHom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(d), \mathcal{O}_{\mathbb{P}^n}(-n-1)[n]) & \longrightarrow & \mathrm{RHom}_{\mathcal{O}_{\mathbb{P}^n}}(\mathcal{O}_{\mathbb{P}^n}(d), p^\times \mathcal{O}_R) \\ \| & & \| \\ R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1-d))[n] & \longrightarrow & \mathrm{RHom}_{\mathcal{O}_R}(Rp_*\mathcal{O}_{\mathbb{P}^n}(d), \mathcal{O}_R) \end{array}$$

The only interesting case is $d = 0$ as both sides are zero otherwise. In the $d = 0$ case, we obtain the shift of the isomorphism

$$R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-n-1)) \to \mathrm{RHom}_{\mathcal{O}_R}(\mathcal{O}_R, \mathcal{O}_R)[-n] \cong R[-n]$$

that we started with. This proves (b).

(c) Now that we know $\mathcal{O}_{\mathbb{P}^n}(-n-1)[n] \cong p^\times \mathcal{O}_R$, the same horizontal maps in the square above are isomorphisms *for all* d. By (a), $Rp_*\mathcal{O}_{\mathbb{P}^n}(d)$ is perfect, so the RHS above equals $R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^\vee$, giving the first part of (c). The second statement is just the fact that this is represented as the strictly perfect complex

$$R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = \big(R[x_0, \ldots, x_n]_d\big)[0]$$

when $d \geq 0$. □

Corollary 3.2.17. *Let $f\colon X \to Y$ be a proper morphism of Noetherian schemes. Then f is quasi-proper, so in particular Rf_* preserves D^*_{coh} for $* \in \{(\ \ ), +, -, b\}$.*

Proof. We showed that the second conclusion is equivalent to quasi-properness in Lemma 2.5.64. First assume f is projective; in this case, since the claim is target-local, we may assume $Y = \operatorname{Spec} R$ for R a Noetherian ring, so f factors through a closed immersion as $X \hookrightarrow \mathbb{P}^n_R \to \operatorname{Spec} R$ for some n. By Noetherianity, the first map is a pseudocoherent closed immersion, hence quasi-proper. Since the second map is quasi-proper by Thm. 3.2.16, we conclude that f is quasi-proper.

Now consider the general case. We proceed by Noetherian induction, assuming that the composite $Z \hookrightarrow X \to Y$ is quasi-proper for all proper closed subschemes Z. Again we may assume Y is affine Noetherian. By Chow's lemma, choose a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ p\ } & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\ & Y & \end{array}$$

where p and g are projective, and p is surjective and restricts to an isomorphism on a dense open set $U \subset X'$. Let Z be the reduced subscheme structure on $X' \setminus U$, and choose a triangle

$$\mathcal{O}_X \to Rp_*\mathcal{O}_{X'} \to K \to \mathcal{O}_X[1].$$

It follows that $K|_U = 0$, and since p is projective, $Rp_*\mathcal{O}_{X'}$ is pseudocoherent, so $K \in D^b_{\mathrm{coh},|Z|}$. Finally, let

$$\mathcal{E} = \{A \in D_{\mathrm{qc}}(X) : Rf_*A \text{ is pseudocoherent}\},$$

which we know is a thick subcategory. By the induction hypothesis, the restriction fi is quasi-proper, so it follows that $i_*D^b_{\mathrm{coh}}(Z) \subset \mathcal{E}$, and hence $D^b_{\mathrm{coh},|Z|}(X) \subset \mathcal{E}$ by Lemma 2.5.66.

It suffices to show that if $E \in \mathrm{Perf}(X)$ then $E \in \mathcal{E}$. Indeed, by the projection formula we obtain an exact triangle

$$E \to Rp_*Lp^*E \to E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \to E[1],$$

and then applying Rf_* we get

$$Rf_*E \to Rg_*Lp^*E \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K) \to Rf_*E[1].$$

Since Lp^*E is perfect and g is projective, the second term is pseudocoherent. Since E is perfect and $K \in D^b_{\mathrm{coh},|Z|}$, $E \otimes^{\mathbf{L}}_{\mathcal{O}_X} K \in D^b_{\mathrm{coh},|Z|} \subset \mathcal{E}$, hence the third term is pseudocoherent as well. Therefore Rf_*E is pseudocoherent. □

3.2.3. Cohomology and base change. The central focus of this section is proving some standard results about tor-independent base change, and then relating this to the twisted inverse image functor. We will see that, while the proof of the former will be somewhat direct, the latter will have some interesting geometric steps.

Let R be a ring and A and B be R-algebras. We say A and B are *tor-independent* if $\operatorname{Tor}_i^R(A,B) = 0$ for $i > 0$, or in other words if the natural map $A \otimes_R^{\mathbf{L}} B \to A \otimes_R B$ is an isomorphism. Let us first see that this behaves well as a local notion. We will use the following fact: $A \otimes_R^{\mathbf{L}} B$ is the underlying R-module of the derived base change of A to a B-module and B to an A-module, compatible with the R-action. Thus its cohomology $\operatorname{Tor}_*^R(A,B)$ is naturally an $A \otimes_R B$-module.

Lemma 3.2.18. *Let R be a ring, A and B be R-algebras and R' a flat R-algebra. Set*

$$A_0 \coloneqq A \otimes_R R' \qquad B_0 \coloneqq B \otimes_R R' \qquad C = A \otimes_R B,$$

assume that A' and B' are flat A_0- and B_0-algebras respectively, and write $C' = A' \otimes_{R'} B'$. Then canonically $\operatorname{Tor}_*^{R'}(A',B') = \operatorname{Tor}_*^R(A,B) \otimes_C C'$.

Proof. We use the standard consequence of the fact that colimits commute with colimits that

$$\begin{aligned}
\operatorname{Tor}_*^R(A,B) \otimes_C C' &= \operatorname{Tor}_*^R(A,B) \underset{\substack{A \otimes B \\ R}}{\otimes} (A' \underset{R'}{\otimes} B') \\
&= \operatorname{Tor}_*^R(A,B) \underset{R}{\otimes} R' \underset{\substack{A \otimes R' \\ R}}{\otimes} A' \underset{\substack{B \otimes R' \\ R}}{\otimes} B' \\
&= \operatorname{Tor}_*^R(A,B) \otimes_R R' \otimes_{A_0} A' \otimes_{B_0} B' \\
&= \operatorname{Tor}_*^{R'}(A_0,B_0) \otimes_{A_0} A' \otimes_{B_0} B',
\end{aligned}$$

where the last step holds by flat base change for Tor. Now observe that

$$\begin{aligned}
((A_0 \otimes_{R'}^{\mathbf{L}} B_0) \otimes_{A_0}^{\mathbf{L}} A') \otimes_{B_0}^{\mathbf{L}} B' &= (A' \otimes_{R'}^{\mathbf{L}} B_0) \otimes_{B_0}^{\mathbf{L}} B' \\
&= A' \otimes_{R'}^{\mathbf{L}} B',
\end{aligned}$$

and if we take cohomology the LHS becomes

$$\begin{aligned}
H^*((A_0 \otimes_{R'}^{\mathbf{L}} B_0) \otimes_{A_0}^{\mathbf{L}} A') \otimes_{B_0} B' &= H^*(A_0 \otimes_{R'}^{\mathbf{L}} B_0) \otimes_{A_0} A' \otimes_{B_0} B' \\
&= \operatorname{Tor}_*^{R'}(A_0,B_0) \otimes_{A_0} A' \otimes_{B_0} B' \\
&= \operatorname{Tor}_*^R(A,B) \otimes_C C'
\end{aligned}$$

by the flatness assumptions, which produces the desired result. □

Corollary 3.2.19. *Let R be a ring and A and B be R-algebras. The following are equivalent:*

(a) A and B are tor-independent over R.
(b) For each prime $\mathfrak{r}$ of R and primes $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$ lying over $\mathfrak{r}$, $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are tor-independent over $R_\mathfrak{r}$.

PROOF. Let $C = A \otimes_R B$, then as mentioned $\mathrm{Tor}^R_*(A,B)$ is a C-module, so (a) is equivalent to $\mathrm{Tor}^R_i(A,B)_{\mathfrak{s}} = 0$ for primes $\mathfrak{s} \subset C$ and $i > 0$. Using the notation of (b), Lemma 3.2.18 applied to $R' = R_{\mathfrak{r}}$ and $A' = A_{\mathfrak{p}}$, $B' = B_{\mathfrak{q}}$ shows that

$$\mathrm{Tor}^{R_{\mathfrak{r}}}_*(A_{\mathfrak{p}}, B_{\mathfrak{q}}) = \mathrm{Tor}^R_*(A,B) \otimes_C (A_{\mathfrak{p}} \otimes_{R_{\mathfrak{r}}} B_{\mathfrak{q}}).$$

Therefore it is clear that $(a) \Rightarrow (b)$. On the other hand, if $\mathfrak{s} \subset C$ lies over $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$ as above, then $C_{\mathfrak{s}}$ is canonically a $(A_{\mathfrak{p}} \otimes_{R_{\mathfrak{r}}} B_{\mathfrak{q}})$-module by the contraction assumption. Therefore if the LHS above is zero, then tensoring over this base with $C_{\mathfrak{s}}$ shows that

$$\mathrm{Tor}^R_i(A,B) \otimes_C C_{\mathfrak{s}} = \mathrm{Tor}^R_i(A,B)_{\mathfrak{s}} = 0,$$

and since this holds for all $\mathfrak{s}$ this proves (a). □

In the following result, we use the convention that if $f\colon W \to X$ is a morphism of schemes, then we write $\mathcal{O}_X$ also for the $\mathcal{O}_W$-module $f^{-1}\mathcal{O}_X$ via restriction of scalars along $f^{-1}\mathcal{O}_X \to \mathcal{O}_W$.

Proposition 3.2.20. *Let S be a scheme and X and Y be S-schemes. The following conditions are equivalent:*

(a) For each $x \in X$ and $y \in Y$ lying over a point $s \in S$, the $\mathcal{O}_{S,s}$-algebras $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ are tor-independent.

(b) For each $x \in X$ and $y \in Y$ lying over a point $s \in S$, there are affine open nbhds U, V, W of x, y, s respectively, for which U and V have image in W, such that the $\mathcal{O}_S(W)$-algebras $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ are tor-independent.

(c) For all affine opens $W \subset S$ and affine opens $U \subset X$ and $V \subset Y$ whose image lies in W, the $\mathcal{O}_S(W)$-algebras $\mathcal{O}_X(U)$ and $\mathcal{O}_Y(V)$ are tor-independent.

PROOF. $(c) \Rightarrow (b)$ is trivial, and it follows immediately from the forward implication of Cor. 3.2.19 that $(b) \Rightarrow (a)$. It then follows from the reverse direction of Cor. 3.2.19 that $(a) \Rightarrow (c)$. □

Definition 3.2.21. Let X and Y be S-schemes. X and Y are *tor-independent* over S if they satisfy the equivalent conditions of Prop. 3.2.20. We also sometimes say that the cospan $X \to S \leftarrow Y$ is tor-independent, or that a commutative square containing the cospan is tor-independent.

Example 3.2.22. If X and Y are S-schemes and either X or Y is *flat* over S, then X and Y are tor-independent. Indeed, in that case $\mathcal{O}_{X,x}$ or $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{S,s}$, so the pair is clearly tor-independent over $\mathcal{O}_{S,s}$. In particular, any pair of schemes over a field are tor-independent.

Example 3.2.23. Let $i_k\colon Z_k \hookrightarrow X$ be closed immersions for $k = 1, 2$. Because of the description of the stalks of $i_{k,*}\mathcal{O}_{Z_k}$, it is clear that Z_1 and Z_2 are tor-independent iff $\mathcal{T}or_i^{\mathcal{O}_X}(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = 0$ for $i > 0$, iff

$$\mathcal{O}_{Z_1} \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{O}_{Z_2} \to \mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_2} = \mathcal{O}_{Z_1 \cap Z_2}$$

is an isomorphism. In other words, closed subschemes are tor-independent iff their scheme-theoretic intersection coincides with their derived intersection. More

generally, we think of tor-independence of S-schemes as a condition that they 'intersect well' over S.

Example 3.2.24. Tor-independence of a cospan is preserved under flat base change. More precisely, consider a diagram of the form

$$\begin{array}{ccccc} X_T & \longrightarrow & T & \longleftarrow & Y_T \\ \downarrow & \lrcorner & \downarrow & \llcorner & \downarrow \\ X & \longrightarrow & S & \longleftarrow & Y \end{array}$$

in which $T \to S$ is flat, and X and Y are tor-independent over S. We claim that X_T and Y_T are tor-independent over T. Indeed, we can choose an affine open $W = \operatorname{Spec} R$ of S and affine opens $U = \operatorname{Spec} A, V = \operatorname{Spec} B, W' = \operatorname{Spec} R'$ of X, Y, and T respectively, all of which have image in W. Such W' of this form cover T, and the corresponding base changes U_T and V_T over W' allow us to reduce to the case where everything is affine. By flat base change for tor,

$$0 = \operatorname{Tor}_i^R(A, B) \otimes_R R' = \operatorname{Tor}_i^{R'}(A \otimes_R R', B \otimes_R R')$$

for $i > 0$, which is what we wanted to show.

Now let us prove tor-independent base change. Suppose we are given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

of schemes. Assume for convenience that f and $\overline{f}$ are qcqs, so that we have the unbounded adjunctions $Lf^* \dashv Rf_*$ and $L\overline{f}^* \dashv R\overline{f}_*$ on D_{qc}. Take the counit $Lf^*Rf_* \to \mathrm{Id}_{D_{\mathrm{qc}}(X)}$ and postcompose to obtain a natural map

$$L\overline{f}^* Lg^* Rf_* = L\overline{g}^* Lf^* Rf_* \to L\overline{g}^*,$$

which is then adjoint to a map

$$\beta \colon Lg^* Rf_* \to R\overline{f}_* L\overline{g}^*$$

of functors $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y')$. We call this the *base change map* (even though we haven't assumed the square is cartesian). We can also obtain the same map by instead starting with the unit $\mathrm{Id} \to R\overline{f}_* L\overline{f}^*$, and it is clear we get the same answer.

Example 3.2.25. One case where we already know that the base change map is an isomorphism is in the case of open base change, i.e.:

$$\begin{array}{ccc} f^{-1}(U) & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Indeed, this is simply because of the behavior of Rf_* under restriction to an open and that $Lj^* = (-)|_U$ for j the inclusion.

This base change map has the following compatibility property. Namely, suppose we have a commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{f''} & Y'' \\ {\scriptstyle \overline{h}}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{f'} & Y' \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with two squares, then we label the base change maps for the bottom square, top square, and outer square as $\beta_1, \beta_2, \beta_{12}$ respectively. We can write β_{12} canonically as

$$\beta_{12} \colon Lh^* Lg^* Rf_* \to Rf'_* L\overline{h}^* L\overline{g}^*,$$

and it is easy to see that the diagram

$$\begin{array}{ccc} Lh^* Lg^* Rf_* & \xrightarrow{Lh^*(\beta_1)} & Lh^*\, Rf'_*\, L\overline{g}^* \\ & \searrow^{\beta_{12}} & \downarrow{\scriptstyle \beta_{2,L\overline{g}^*}} \\ & & Rf''_*\, L\overline{h}^* L\overline{g}^* \end{array}$$

of functors commutes by adjunction, i.e. $\beta_{12} = \beta_{2,L\overline{g}^*} \circ Lh^*(\beta_1)$.

Here is a simple consequence of this compatibility and open base change (Example 3.2.25). Let β be the base change map for a square as in the definition, and place the open base change square above this. β is an isomorphism iff $\beta|_U$ is an isomorphism for all U in an open cover of Y', and thus by the above this is equivalent to the composite square with upper map $\overline{f}^{-1}(U) \to U$ having a base change isomorphism for all U in an open cover of Y'.

We can apply this again to the commutative tower:

$$\begin{array}{ccc} \overline{f}^{-1}(U) & \longrightarrow & U \\ \downarrow & & \downarrow \\ f^{-1}(V) & \longrightarrow & V \\ {\scriptstyle j_{f^{-1}(V)}}\downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Once again, by open base change and the compatibility result, we obtain two immediate consequences:

(a) If the base change map for the restricted upper square is an isomorphism, then the one for the outer square is too.

(b) If $j^*_{f^{-1}(V)}$ is essentially surjective, then the base change map for the outer square will imply that the one for the upper square is an isomorphism. Note that we already know $j^*_{f^{-1}(V)}$ is essentially surjective whenever

$Rj_{f^{-1}(V),*}$ lands in D_{qc}, i.e. $f^{-1}(V) \hookrightarrow X$ is qcqs. Since the source is already qcqs, this happens when X is quasiseparated.

We obtain a similar comparison map if we start with a horizontal diagram:

$$\begin{array}{ccccc} X' & \xrightarrow{\overline{f}} & Y' & \xrightarrow{\overline{h}} & Z' \\ \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y & \xrightarrow{h} & Z \end{array}$$

Namely, labeling the maps of the left, right, and outer squares as $\beta_1, \beta_2, \beta_{12}$, we again immediately see that

$$\beta_{12} = R\overline{f}_*(\beta_1) \circ \beta_{2,Rf_*}$$

via the same proof as before. At the very least, this shows that if the two squares have the base change isomorphism, then the outer one does as well.

THEOREM 3.2.26. *Let $f\colon X \to Y$ be a qcqs morphism of schemes and*

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ \overline{g}\downarrow & \lrcorner & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

be a cartesian square. The base change map

$$Lg^* Rf_* \to R\overline{f}_* L\overline{g}^*$$

is an isomorphism if X and Y' are tor-independent over Y. If in addition X is quasiseparated, then the converse holds.

PROOF. It follows from the remark above the proof, for both the forward and converse direction, that we may reduce to the case where Y and Y' are affine, where for the converse we are also using Prop. 3.2.20. Then g is affine, and by conservativity of affine pushforward (Example 2.3.46), the base change map is an isomorphism iff

$$Rg_* Lf^* Rf_* \to Rg_* R\overline{f}_* L\overline{g}^* = Rf_* R\overline{g}_* L\overline{g}^*$$

is an isomorphism. By the projection formula, this is an isomorphism iff

$$Rf_* E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} g_*\mathcal{O}_{Y'} \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \overline{g}_*\mathcal{O}_{X'})$$

is an isomorphism for all $E \in D_{\mathrm{qc}}(X)$. We know from classical affine base change that $\overline{g}_*\mathcal{O}_{X'} = \overline{g}_*\overline{f}^*\mathcal{O}_{Y'} = f^* g_*\mathcal{O}_{Y'}$, and projection again allows us to rewrite the LHS to give the map

$$(*) \qquad Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^* g_*\mathcal{O}_{Y'}) \to Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} f^* g_*\mathcal{O}_{Y'})$$

which is $Rf_*(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} -)$ applied to the natural map $Lf^*(g_*\mathcal{O}_Y) \to f^* g_*\mathcal{O}_Y$. Upon restriction to an affine open $U = \operatorname{Spec} A \subset X$, this map corresponds to

$$A \otimes^{\mathbf{L}}_R B \to A \otimes_R B \qquad \text{in } D_{\mathrm{qc}}(X)$$

where $B = \mathcal{O}_{Y'}(Y')$ and $R = \mathcal{O}_Y(Y)$. Therefore, $Lf^*(g_*\mathcal{O}_Y) \to f^*g_*\mathcal{O}_Y$ is an isomorphism iff X and Y' are tor-independent.

This proves the forward implication as it shows that $(*)$ is an isomorphism. It also proves the converse in the case that X were affine, because then f is affine, so Rf_* is conservative, and we can take $E = \mathcal{O}_X$ to obtain $Lf^*(g_*\mathcal{O}_Y) \xrightarrow{\sim} f^*g_*\mathcal{O}_Y$. To finish the converse, use the diagram

$$\begin{array}{ccccc} \overline{g}^{-1}(U) & \hookrightarrow & X' & \xrightarrow{\overline{f}} & Y' \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \overline{g}} & & \downarrow{\scriptstyle g} \\ U & \hookrightarrow & X & \xrightarrow{f} & Y \end{array}$$

for some affine open $U \subset X$. Since the isomorphism is known for both squares, the remark above the proof shows that the outer square has the base change isomorphism. Thus U and Y' are tor-independent over Y, and since U was arbitrary we conclude X and Y' are tor-independent. □

Example 3.2.27. Let $f\colon X \to S$ be a flat qcqs morphism (i.e. a 'family') and $s \in S$ a point. Write $i_s\colon \operatorname{Spec}\kappa(s) \to S$ and $i\colon X_s \hookrightarrow X$. Then tor-independent base change gives

$$Li_s^*Rf_*A = (Rf_*A)_s \otimes^{\mathbf{L}}_{\mathcal{O}_{S,s}} \kappa(s) = R\Gamma(X_s, Li^*A)$$

for $A \in D_{\mathrm{qc}}(X)$. For fun, consider the case where $A = \mathcal{E}$ is a vector bundle, and apply the hyper-tor spectral sequence. Then we get

$$\operatorname{Tor}_{-p}^{\mathcal{O}_{S,s}}((R^qf_*\mathcal{E})_s, \kappa(s)) \implies H^{p+q}(X_s, \mathcal{E}|_{X_s}),$$

relating the cohomology of the fibers to the higher direct images. Since this is happening over a field, we often want to consider these as numerical spectral sequences. If S is locally Noetherian, then we have called the LHS the Betti numbers:

$$\beta_p^{\mathcal{O}_{S,s}}((R^qf_*\mathcal{E})_s) \implies h^{q-p}(X_s, \mathcal{E}|_{X_s})$$

In particular, we see that if the higher direct images are all vector bundles, then the cohomology of the fiber coincides with the stalk of the higher direct images.

Proposition 3.2.28. *Suppose that*

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is a tor-independent cartesian square of schemes.

- *(a) If g is locally of finite tor dimension, then $\overline{g}$ is locally of finite tor dimension. If g has tor dimension $\leq d$ then $\overline{g}$ has tor dimension $\leq d$.*
- *(b) If f is quasi-proper, then $\overline{f}$ is quasi-proper.*
- *(c) If f is quasi-perfect, then $\overline{f}$ is quasi-perfect.*

PROOF. (a) Let $U \subset X$ and $V \subset Y$ be affine opens so that $f(U) \subset V$, then we can replace the horizontal maps by $U \to V$ and $\overline{g}^{-1}(U) \to g^{-1}(V)$, then the square remains tor-independent and cartesian. Thus we may assume f and hence $\overline{f}$ are affine, and prove the second statement. By tor-independent base change, $Lg^* Rf_* \cong R\overline{f}_* L\overline{g}^*$, so this composite takes a sheaf in degree zero to $[-d, 0]$ by Example 2.3.47. The same example applied to $R\overline{f}_*$ shows that $L\overline{f}^*$ takes a sheaf to $[-d, 0]$, proving the claim.

(b,c) We have seen in Prop. 2.5.45 and Prop. 2.5.46 that the claim is local on quasicompact opens of the target; thus by replacing f with $f^{-1}(V) \to V$ and $\overline{f}$ with $\overline{f}^{-1}(U) \to U$, we may assume g and thus $\overline{g}$ are affine.

Let $E \in \mathrm{Perf}(X)$ be a generator of $D_{\mathrm{qc}}(X)$, and observe that

$$\mathrm{RHom}_{\mathcal{O}_{X'}}(L\overline{g}^* E, -) = \mathrm{RHom}_{\mathcal{O}_X}(E, R\overline{g}_*(-)),$$

where the RHS is conservative as a composite of conservative functors since $\overline{g}$ is affine (Example 2.3.46). Thus $L\overline{g}^* E$ generates $D_{\mathrm{qc}}(X')$, so it suffices to show that $R\overline{f}_* L\overline{g}^* E$ is pseudocoherent (resp. perfect) by the two cited criteria. This is clear by tor independent base change since $Lg^* Rf_* E$ is pseudocoherent (resp. perfect) if f is quasi-proper (resp. quasi-perfect). □

Remark 3.2.29. (a) and (c) above are the more interesting statements, and especially for (c) I cannot see a simple 'classical' way to prove this even under good finiteness hypotheses. On the other hand, (b) is only useful in the non-Noetherian case, because if f is proper then $\overline{f}$ is already proper without any tor-independence.

Example 3.2.30. Suppose we have a tor-independent cartesian square of qcqs schemes as above. Since $Lg^* Rf_* \xrightarrow{\sim} R\overline{f}_* L\overline{g}^*$ by tor-independent base change, by taking right adjoints we obtain $f^\times Rg_* \xrightarrow{\sim} R\overline{g}_* \overline{f}^\times$. We will prove a much cooler tor-independent base change theorem for $f^\times$ below.

Let X and Y be qcqs S-schemes, and consider the natural projections

$$X \xleftarrow{p_X} X \times_S Y \xrightarrow{p_Y} Y,$$

which are qcqs by qcqs base change. Recall (Example 2.4.36) that we defined a Fourier-Mukai functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$ to be a functor isomorphic to one of the form

$$A \mapsto Rp_{Y,*}(K \otimes^{\mathbf{L}}_{\mathcal{O}_{X\times Y}} Lp_Y^* A)$$

for some kernel $K \in D_{\mathrm{qc}}(X \times Y)$. We showed that this is equivalent to having a qcqs S-scheme W and S-maps $f\colon W \to X$ and $g\colon W \to Y$ such that

$$A \mapsto Rg_*(K \otimes^{\mathbf{L}}_{\mathcal{O}_W} Lf^* A)$$

for some $K \in D_{\mathrm{qc}}(W)$, in which case the corresponding kernel is the pushforward of K along $W \to X \times_S Y$.

Proposition 3.2.31. *Let Y be a qcqs flat S-scheme and X, Y be qcqs tor-independent S-schemes. If*

$$\Phi_{K_1}\colon D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y) \qquad \Phi_{K_2}\colon D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(Z)$$

are the Fourier-Mukai functors with kernels K_1 and K_2 respectively, then $\Phi_{K_2} \circ \Phi_{K_1}$ is Fourier-Mukai with kernel

$$R\pi_{XZ,*}(L\pi_{XY}^* K_1 \otimes^{\mathbf{L}} L\pi_{YZ}^* K_2),$$

where $\pi_\bullet$ denote the corresponding projections from $X \times_S Y \times_S Z$.

Proof. We consider the cartesian square:

$$\begin{array}{ccc} X \times_S Y \times_S Z & \xrightarrow{\pi_{YZ}} & Y \times_S Z \\ {\scriptstyle \pi_{XY}}\downarrow & \lrcorner & \downarrow{\scriptstyle q_Y} \\ X \times_S Y & \xrightarrow{p_Y} & Y \end{array}$$

This is tor-independent by the assumptions and Example 3.2.24 since $Y \to S$ is flat. Thus tor-independent base change applies, so we compute:

$$\begin{aligned} \Phi_{K_2}(\Phi_{K_1}(A)) &= Rq_{Z,*}\Big(K_2 \otimes^{\mathbf{L}} Lq_Y^* Rp_{Y,*}(K_1 \otimes^{\mathbf{L}} Lp_X^* A)\Big) \\ &= Rq_{Z,*}\Big(K_2 \otimes^{\mathbf{L}} R\pi_{YZ,*} L\pi_{XY}^*(K_1 \otimes^{\mathbf{L}} Lp_X^* A)\Big) \\ &= Rq_{Z,*}\Big(K_2 \otimes^{\mathbf{L}} R\pi_{YZ,*}(L\pi_{XY}^* K_1 \otimes^{\mathbf{L}} L\pi_X^* A)\Big) \\ &= R(q_Z\pi_{YZ})_*\Big(L\pi_{YZ}^* K_2 \otimes^{\mathbf{L}} L\pi_{XY}^* K_1 \otimes^{\mathbf{L}} L\pi_X^* A\Big) \\ &= R(r_Z\pi_{XZ})_*\Big(L\pi_{YZ}^* K_2 \otimes^{\mathbf{L}} L\pi_{XY}^* K_1 \otimes^{\mathbf{L}} L\pi_{XZ}^* Lr_Z^* A\Big) \\ &= Rr_{Z,*}\Big((L\pi_{YZ}^* K_2 \otimes^{\mathbf{L}} L\pi_{XY}^* K_1) \otimes^{\mathbf{L}} Lr_Z^* A\Big) \end{aligned}$$

by a few applications of the projection formula, where $r_\bullet$ are the projections from $X \times_S Z$. This is what we wanted. □

As a corollary, the collection of qcqs flat S-schemes (i.e. families over S) behaves very well with respect to Fourier-Mukai functors, because then Fourier-Mukai functors are closed under composition. We will explain a categorical perspective on this momentarily. First, if X, Y, Z are qcqs S-schemes and $K_1 \in D_{\mathrm{qc}}(X \times_S Y), K_2 \in D_{\mathrm{qc}}(Y \times_S Z)$, we define

$$K_1 \star_S K_2 \coloneqq R\pi_{XZ,*}(L\pi_{XY}^* K_1 \otimes^{\mathbf{L}} L\pi_{YZ}^* K_2) \in D_{\mathrm{qc}}(X \times_S Y)$$

to be the *convolution* of the two kernels. This behaves well under flatness assumptions. Indeed, assume W, X, Y, Z are qcqs and flat over S (or more precisely X and Y are flat and W and Z are tor-independent), and

$$K_1 \in D_{\mathrm{qc}}(W \times_S X), K_2 \in D_{\mathrm{qc}}(X \times_S Y), K_3 \in D_{\mathrm{qc}}(Y \times_S Z).$$

Then both $(K_1 \star_S K_2) \star_S K_3$ and $K_1 \star_S (K_2 \star_S K_3)$ are canonically isomorphic to

$$R\sigma_{WZ,*}(L\sigma_{WX}^* K_1 \otimes^{\mathbf{L}} L\sigma_{XY}^* K_2 \otimes^{\mathbf{L}} L\sigma_{YZ}^* K_3)$$

where σ denotes projections from $W \times_S X \times_S Y \times_S Z$, via the natural map from either association to the above pushforward (by adjunction). Indeed, write π for

projections from $W \times_S X \times_S Y$ and τ for projections from $W \times_S Y \times_S Z$. Consider the tor-independent cartesian square:

$$\begin{array}{ccc} W \times_S X \times_S Y \times_S Z & \xrightarrow{\sigma_{WYZ}} & W \times_S Y \times_S Z \\ {\scriptstyle \sigma_{WXY}}\downarrow & \lrcorner & \downarrow{\scriptstyle \tau_{WY}} \\ W \times_S X \times_S Y & \xrightarrow{\pi_{WY}} & W \times_S Y \end{array}$$

Then it follows from tor-independent base change and the projection formula that we have natural isomorphisms

$$\begin{aligned} (K_1 \star K_2) \star K_3 &= R\tau_{WZ}\Big(L\tau_{WY}R\pi_{WY}(L\pi_{WX}K_1 \otimes^{\mathbf{L}} L\pi_{XY}K_2) \otimes^{\mathbf{L}} L\tau_{YZ}K_3\Big) \\ &= R\tau_{WZ}\Big(R\sigma_{WXY}(L\sigma_{WX}K_1 \otimes^{\mathbf{L}} L\sigma_{XY}K_2) \otimes^{\mathbf{L}} L\tau_{YZ}K_3\Big) \\ &= R\sigma_{WZ}(L\sigma_{WX}K_1 \otimes^{\mathbf{L}} L\sigma_{XY}K_2 \otimes^{\mathbf{L}} L\sigma_{YZ}K_3) \end{aligned}$$

(To make everything fit on the page, some stars and subscripts are omitted.) Symmetrically, the other associativity equation also equals the same expression. This provides the recipe for computing the convolution of a finite collection of convolvable kernels: pull them all back to the relevant product, tensor, and push down.

Based on this verification, we naturally obtain a bicategory (i.e. weak 2-category) $\mathbf{Kern}_S$ defined as follows: its objects are flat qcqs S-schemes, and hom-spaces are given by

$$\mathbf{Kern}_S(X, Y) \coloneqq D_{\mathrm{qc}}(X \times Y).$$

The composition law is given by convolution, and associators are given by the natural isomorphisms deduced above. It follows from the projection formula that if $\Delta_X \colon X \to X \times_S X$ denotes the diagonal, then

$$(R\Delta_{X,*}\mathcal{O}_X) \star_S K = K \qquad K' \star_S (R\Delta_{X,*}\mathcal{O}_X) = K',$$

so $R\Delta_{X,*}\mathcal{O}_X \in D_{\mathrm{qc}}(X \times_S X)$ is the identity. The point of doing this is that there is a canonical pseudofunctor

$$\mathrm{FM} \colon \mathbf{Kern}_S \to \mathbf{Cat}_1$$

sending $X \mapsto D_{\mathrm{qc}}(X)$, where

$$\mathbf{Kern}_S(X, Y) \to \mathrm{Fun}(D_{\mathrm{qc}}(X), D_{\mathrm{qc}}(Y)) \qquad K \mapsto \Phi_K$$

is the functor sending a kernel to its associated Fourier-Mukai functor. Prop. 3.2.31 shows that this indeed respects composition. An important question is to determine what the essential image of these functors is, but if we address this it will be done once we make our derived categories into ∞-categories.

Remark 3.2.32. Suppose X is in addition quasiproper (resp. quasiperfect) over S. Then for Y flat over S, the projection map $X \times_S Y \to Y$ is also quasiproper (resp. quasi-perfect) by tor-independent base change. Therefore, a Fourier-Mukai functor restricts to $D_{\mathrm{pc}}(X) \to D_{\mathrm{pc}}(Y)$ assuming that K is a pseudocoherent kernel. If K is a perfect kernel and X is quasi-perfect, then it restricts to $\mathrm{Perf}(X) \to \mathrm{Perf}(Y)$.

Now if X and Y are both flat, then in this case X is quasi-perfect over S so we obtain the above restriction again. Note also that by Prop. 2.5.29, if ever such a Fourier-Mukai functor $\mathrm{Perf}(X) \to \mathrm{Perf}(Y)$ is an equivalence, then $D_{\mathrm{qc}}(X) \xrightarrow{\sim} D_{\mathrm{qc}}(Y)$. Thus for quasi-proper schemes flat over a base, Perf 'detects' the whole unbounded derived category for Fourier-Mukai functors with perfect kernel.

Example 3.2.33. As a continuation of the previous remark, suppose S is a Noetherian base and X, Y are proper flat S-schemes. If $E \in \mathrm{Perf}(X \times_S Y)$ is a perfect kernel, then $\Phi_E \colon \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$, and also $\Phi_E \colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$. If we only have $K \in D^b_{\mathrm{coh}}(X \times Y)$, then Φ_K preserves $D_{\mathrm{pc}} = D^-_{\mathrm{coh}}(X)$, but possibly doesn't stay bounded. For this reason, in geometric settings, we often want a perfect kernel to get a Fourier-Mukai functor on D^b_{coh}. This situation is nice for other reasons: if $K = E$ is perfect, then by Example 2.4.50 we have

$$\Phi_E \dashv (A \mapsto Rp_{Y,*}\, \mathbf{R}\mathcal{H}om(E, p_X^\times A)).$$

Since $Y \to S$ is quasi-perfect, p_X is quasi-perfect, so $p_X^\times(A) = p_X^\times(\mathcal{O}_X) \otimes^{\mathbf{L}} Lp_X^*$, and thus the right adjoint to Φ_E is

$$A \mapsto Rp_{Y,*}(E^\vee \otimes^{\mathbf{L}} p_X^\times(\mathcal{O}_X) \otimes^{\mathbf{L}} Lp_X^* A),$$

hence the right adjoint is Fourier-Mukai with kernel $\mathbf{R}\mathcal{H}om(E, p_X^\times \mathcal{O}_X)$. We will be able to give a better way to compute $p_X^\times$ shortly.

Let X and Y be qcqs over a base scheme S, and consider the cartesian square:

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\pi_X} & Y \\ {\scriptstyle \pi_Y}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle p_X} \\ X & \xrightarrow{p_Y} & S \end{array}$$

Given the counits $Lp_X^* Rp_{X,*} \to \mathrm{Id}$ and $Lp_Y^* Rp_{Y,*} \to \mathrm{Id}$, we can postcompose with the pullbacks to $X \times_S Y$ and tensor to obtain a map

$$L\pi_X^* Lp_X^* Rp_{X,*} A \underset{\mathcal{O}_S}{\otimes^{\mathbf{L}}} L\pi_Y^* Lp_Y^* Rp_{Y,*} B \to L\pi_X^* A \underset{\mathcal{O}_{X\times_S Y}}{\otimes^{\mathbf{L}}} L\pi_Y^* B$$

natural in $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(Y)$. In this setting we usually denote the RHS by $A \boxtimes^{\mathbf{L}}_{\mathcal{O}_S} B$, called the *external product*, which is a multiplication functor

$$D_{\mathrm{qc}}(X) \times D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X \times_S Y).$$

The LHS is equal to $L\pi^* Rp_{X,*} A \otimes^{\mathbf{L}} L\pi^* Rp_{Y,*} B$, so the map is adjoint to a natural map

$$Rp_{X,*} A \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rp_{Y,*} B \to R\pi_*(A \boxtimes^{\mathbf{L}}_{\mathcal{O}_S} B),$$

which is called the *Kunneth map*. A simple consequence of our base change result is that this is an isomorphism in the tor-independent case.

Proposition 3.2.34. *Let X and Y be tor-independent, qcqs S-schemes. Using the notation above, the Kunneth map*

$$Rp_{X,*} A \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rp_{Y,*} B \to R\pi_*(A \boxtimes^{\mathbf{L}}_{\mathcal{O}_S} B)$$

is a natural isomorphism for all $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(Y)$.

PROOF. By the qcqs projection formula and tor-independent base change, we deduce that

$$\begin{aligned} Rp_{X,*}A \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rp_{Y,*}B &= Rp_{X,*}(A \otimes^{\mathbf{L}} Lp_X^* Rp_{Y,*}B) \\ &= Rp_{X,*}(A \otimes^{\mathbf{L}} R\pi_{X,*}L\pi_Y^* B) \\ &= Rp_{X,*}R\pi_{X,*}(L\pi_X^* A \otimes^{\mathbf{L}} L\pi_Y^* B) \\ &= R\pi_*(A \boxtimes^{\mathbf{L}}_{\mathcal{O}_S} B), \end{aligned}$$

completing the proof. □

Example 3.2.35. Let S be any scheme and $X \coloneqq \mathbb{P}^n_S \to S$ the corresponding projective space. This is flat because $\mathbb{P}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$ is flat, as it is locally $\mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec}\mathbb{Z}$. Moreover, the line bundles $\mathcal{O}_X(d)$ for $d \in \mathbb{Z}$ are all pullbacks of $\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(d)$ from $\mathbb{Z}$, so the qcqs and tor-independent base change square

$$\begin{array}{ccc} X & \xrightarrow{p} & S \\ {\scriptstyle \overline{h}}\downarrow & \lrcorner & \downarrow{\scriptstyle h} \\ \mathbb{P}^n_{\mathbb{Z}} & \xrightarrow{q} & \operatorname{Spec}\mathbb{Z} \end{array}$$

allows us to compute their cohomology: indeed,

$$Lh^* Rq_* = Rp_* L\overline{h}^*,$$

so $Rp_*\mathcal{O}_X(d) = Lh^* Rq_*\mathcal{O}_{\mathbb{P}^n}(d) = R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \otimes^{\mathbf{L}} \mathcal{O}_S$. Thus

$$Rp_*\mathcal{O}_X(d) = (\mathcal{O}_S[x_0, \dots, x_n]_d)[0] \cong \mathcal{O}_S^{\binom{n+d}{d}}[0]$$

for $d \geq 0$, $Rp_*\mathcal{O}_X(d) = 0$ for $d \in [-n, -1]$, and

$$Rp_*\mathcal{O}_X(d) = (\mathcal{O}_S[x_0, \dots, x_n]^\vee_{n+1-d})[0] \cong \mathcal{O}_S^{\binom{-d-1}{-n-d-1}}[-n].$$

The Künneth formula above then allows us to compute the cohomology of external product bundles on $\mathbb{P}^{n_1}_S \times_S \cdots \times_S \mathbb{P}^{n_k}_S$. For example, on $X = \mathbb{P}^n_S \times_S \mathbb{P}^m_S$, we write $\mathcal{O}_X(i,j)$ for the obvious external product. Then the result above shows that

$$R\pi_*\mathcal{O}_X(i,j) = Rp_{\mathbb{P}^n_S,*}\mathcal{O}_{\mathbb{P}^n_S}(i) \otimes^{\mathbf{L}}_{\mathcal{O}_S} Rp_{\mathbb{P}^m_S,*}\mathcal{O}_{\mathbb{P}^m_S}(j)$$

which is computed by the formula above. For example,

$$R\pi_*\mathcal{O}_X(-n-1, -m-1) = \mathcal{O}_S[-n] \otimes^{\mathbf{L}}_{\mathcal{O}_S} \mathcal{O}_s[-m] = \mathcal{O}_S[-n-m],$$

so we should expect $\mathcal{O}_X(-n-1, -m-1)$ to be the dualizing object for X over S. We will formalize this below.

Example 3.2.36. Let X and Y be qcqs k-schemes for a field k. Identify $D(k)$ with the category of $\mathbb{Z}$-graded k-vector spaces, and recall that all k-complexes are K-injective because an acyclic k-complex is contractible. Thus for any $A \in D_{\mathrm{qc}}(X)$ and $B \in D_{\mathrm{qc}}(Y)$, the Kunneth formula gives

$$\mathbb{H}^n(X \times_k Y, A \boxtimes^{\mathbf{L}} B) \cong \bigoplus_{p+q=n} \mathbb{H}^p(X, A) \otimes \mathbb{H}^q(Y, B),$$

just as in the classical formula. If $A \simeq \mathcal{E}^\bullet$ is a bounded complex of vector bundles and $B \simeq \mathcal{F}^\bullet$, then $A \boxtimes^{\mathbf{L}} B$ is computed completely classically, namely

$$A \boxtimes^{\mathbf{L}} B = p_X^* \mathcal{E}^\bullet \otimes_{X \times_k Y} p_Y^* \mathcal{F}^\bullet = \mathcal{E}^\bullet \boxtimes \mathcal{F}^\bullet,$$

as we expect.

Example 3.2.37. The purpose of this example is to explain why Example 3.2.36 fails when we're not over a field. In fact, we will use our formalism to give an example of a (polynomial) ring R, a smooth projective R-scheme X, and a locally free sheaf $\mathcal{E}$ on X such that

$$H^0\Big(R\Gamma(X, \mathcal{E}) \otimes^{\mathbf{L}}_R R\Gamma(X, \mathcal{E})\Big) \neq H^0(X, \mathcal{E}) \otimes_R H^0(X, \mathcal{E}).$$

By the Kunneth formula, this will show that

$$H^0(X \times_R X, \mathcal{E} \boxtimes \mathcal{E}) \neq H^0(X, \mathcal{E}) \otimes H^0(X, \mathcal{E}),$$

which is a question one could ask in entirely classical terms.

Let k be a field, $R = k[u, v, w]$, and $X = \mathbb{P}^1_R$. Note that

$$\operatorname{Ext}^1(\mathcal{O}_X^3, \mathcal{O}_X(-2)) = H^1(X, \mathcal{O}_X(-2))^3 = R^3,$$

so choose an exact sequence of the form

$$0 \to \mathcal{O}_X(-2) \to \mathcal{E} \to \mathcal{O}_X^{\oplus 3} \to 0$$

with extension class $(u, v, w) \in R^3$. Then $\mathcal{E}$ is locally free of rank 2, which produces a triangle $\mathcal{E} \to \mathcal{O}_X \to \mathcal{O}_X(-2)[1] \to \mathcal{E}[1]$. Thus

$$R\Gamma(X, \mathcal{E}) \to R\Gamma(X, \mathcal{O}_X^{\oplus 3}) = R^3 \to R\Gamma(X, \mathcal{O}_X(-2)[1]) = R \to$$

is exact, which means that

$$R\Gamma(X, \mathcal{E}) \simeq \operatorname{Cone}(R^3 \xrightarrow{(u,v,w)} R) = \big[R^3 \xrightarrow{(u,v,w)} R\big]$$

in $D(R)$. Letting $M = H^0(X, \mathcal{E})$, we have $M = \ker(R^3 \to R)$; this map is also contained in the Koszul resolution for $k = k[u, v, w]/(u, v, w)$, which is

$$0 \to R \xrightarrow{(w,-v,u)} R^3 \to R^3 \xrightarrow{(u,v,w)} R \to 0.$$

Therefore we have an exact sequence

$$0 \to R \xrightarrow{(w,-v,u)} R^3 \to M \to 0$$

by splicing, which is a minimal free resolution. Tensoring this with M, we observe that $[M \to M^3] \simeq M \otimes^{\mathbf{L}}_R M$. But $x \mapsto (wx, -vx, ux)$ is still injective because M is a submodule of R^3, so $M \otimes^{\mathbf{L}}_R M \simeq M \otimes_R M$. Therefore

$$M \otimes_R M \simeq [R \to R^3 \to R^6][2]$$

(in degrees $-2, -1, 0$). Because this was true before totalization, the matrix entries above all lie in $\mathfrak{m} = (u, v, w) \subset R$, so this becomes a minimal free resolution after localizing at $\mathfrak{m}$. In particular $\operatorname{pd}_{R_\mathfrak{m}}(M_\mathfrak{m} \otimes_{R_\mathfrak{m}} M_\mathfrak{m}) = 2$.

On the other hand, we also have in degrees $0, 1, 2$

$$R\Gamma(X, \mathcal{E}) \otimes^{\mathbf{L}}_R R\Gamma(X, \mathcal{E}) = \big[R^9 \xrightarrow{d} R^6 \to R\big],$$

so let N be H^0 of the equation above, which fits into an exact sequence $0 \to N \to R^9 \to \operatorname{im}(d) \to 0$. Since $\operatorname{im}(d)_{\mathfrak{m}}$ is a nonzero submodule of $R^6_{\mathfrak{m}}$, it has depth ≥ 1 over $R_{\mathfrak{m}}$. Therefore the exact sequence implies $\operatorname{depth}_{R_{\mathfrak{m}}}(N_{\mathfrak{m}}) \geq \operatorname{depth}(\operatorname{im}(d)_{\mathfrak{m}}) + 1 \geq 2$. By the Auslander-Buchsbaum formula, $\operatorname{pd}_{R_{\mathfrak{m}}}(N_{\mathfrak{m}}) \leq 3 - 2 = 1$, so we conclude $N_{\mathfrak{m}} \neq (M \otimes_R M)_{\mathfrak{m}}$, hence $N \neq M \otimes_R N$ as desired.

Our next subject is tor-independent base change for the twisted inverse image. The setup and proof will be somewhat similar to the proof of regular tor-independent base change: we use general pullback facts to reduce to the case of an affine morphism or open immersion, and prove these cases separately. One annoyance of the theory is that the base change map will only exist in the tor-independent case.

Suppose we are given a tor-independent cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

where X, Y, X', and Y' are all qcqs schemes, so in particular all maps are qcqs. The *twisted base change map* is a natural map $\gamma\colon L\overline{g}^* f^\times \to \overline{f}^\times Lg^*$ of functors $D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X')$. It is defined as the adjunct map corresponding to

$$R\overline{f}_* L\overline{g}^* f^\times \xrightarrow{\beta^{-1} f^\times} Lg^* Rf_* f^\times \to Lg^*,$$

where β is the tor-independent base change isomorphism, and the second map is induced by the counit of $Rf_* \dashv f^\times$. γ satisfies the same compatibilities as the standard derived base change map; namely, suppose we stack tor-independent base change diagrams like:

$$\begin{array}{ccccc} X'' & \xrightarrow{f_1} & Y'' & & \\ {\scriptstyle \overline{h}}\downarrow & \lrcorner & \downarrow{\scriptstyle h} & & \\ X' & \xrightarrow{\overline{f}} & Y' & \xrightarrow{\overline{h}_1} & Z' \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} & \lrcorner & \downarrow{\scriptstyle g_1} \\ X & \xrightarrow{f} & Y & \xrightarrow{h_1} & Z \end{array}$$

Denote the base change maps for the left-bottom, left-top, and right squares as γ_1, γ_2, and γ_3 respectively. Let γ_{12} and γ_{13} be the base change maps for the pasted squares.

Remark 3.2.38. For this to make sense, we need to know that the composite square of adjacent tor-independent base change squares is tor-independent. This is easy to check from the definition, but also we know that the base change map of the composite square is an isomorphism by pasting compatibility, and since we are assuming that the schemes are qcqs the converse of Thm. 3.2.26 holds.

Now, view γ_{12} and γ_{13} as maps:

$$\gamma_{12}\colon L\overline{h}^* L\overline{g}^* f^\times \to f_1^\times Lh^* Lg^* \qquad \gamma_{13}\colon L\overline{g}^* f^\times h_1^\times \to \overline{f}^\times \overline{h}_1^\times Lg_1^*$$

Then we have the commutative diagrams

$$\begin{array}{ccc} L\overline{h}^* L\overline{g}^* f^\times & \xrightarrow{Lh^*\gamma_1} & L\overline{h}^*\overline{f}^\times Lg^* \\ & \searrow^{\gamma_{12}} & \downarrow{\scriptstyle \gamma_{2,Lg^*}} \\ & & f_1^\times Lh^* Lg^* \end{array} \qquad \begin{array}{ccc} L\overline{g}^* f^\times h_1^\times & \xrightarrow{\gamma_{1,h_1^\times}} & \overline{f}^\times Lg^* h_1^\times \\ & \searrow^{\gamma_{13}} & \downarrow{\scriptstyle \overline{f}^\times\gamma_3} \\ & & \overline{f}^\times \overline{h}_1^\times Lg_1^* \end{array}$$

expressing the obvious compatibilities. These follow purely formally from the corresponding identities for the standard derived base change map, as well as the unit-counit identities for an adjunction.

3.2.4. Grothendieck duality I.

3.3. Derived intersection theory

3.3.1. Grothendieck groups of derived categories. Let $\mathcal{D}$ be a triangulated category. In this section we will define a corresponding group $K_0(\mathcal{D})$, and then specialize to the algebro-geometric case. We will as usual ignore any set-theoretic issues unless they actually become relevant to the considerations at hand.

Let $K_0(\mathcal{D})$ be the free abelian group on the collection of isomorphism classes of objects of $\mathcal{D}$ modulo the subgroup generated by elements of the form $[A]-[B]+[C]$ whenever there is an exact triangle

$$A \to B \to C \to A[1].$$

We call $K_0(\mathcal{D})$ the *Grothendieck group* of $\mathcal{D}$. We also recall the classical definition that if $\mathcal{A}$ is an exact category, then $K_0(\mathcal{A})$ has the corresponding definition except using exact sequences $0 \to A \to B \to C \to 0$ instead of exact triangles.

This is functorial: if $F\colon \mathcal{D} \to \mathcal{E}$ is an exact functor of triangulated categories, then we obtain an induced functor $F_*\colon K_0(\mathcal{D}) \to K_0(\mathcal{E})$ sending $[A] \mapsto [F(A)]$. This is well-defined because F preserves isomorphisms and exact triangles. In particular, triangulated categories which are triangulated-equivalent have isomorphic Grothendieck groups. The corresponding remarks hold for exact categories.

This notion is well-behaved for arbitrary triangulated categories. First, given any $A, B \in \mathcal{D}$, we always have an explicit model for the difference $[A]-[B]$, namely it is the cone C in $A \to B \to C \to A[1]$. The triangle $A \to A \to 0 \to A[1]$ shows that $[0] = 0$ in $K_0(\mathcal{D})$, and rotating the triangle to $A \to 0 \to A[1] \to$ implies that $[A] = -[A[1]]$. In particular we have

$$[A[n]] = (-1)^n[A]$$

for all $n \in \mathbb{Z}$. We remark that the octahedral axiom doesn't give us anything new, namely it just says

$$([A] - [B]) - ([B] - [C]) = [A] - [C]$$

as was true already. Since in any triangulated category we always have $A \to A \oplus B \to B \to A[1]$ exact, it follows that $[A \oplus B] = [A] + [B]$, giving an easy model for the sum of two objects.

Example 3.3.1. Let $\mathcal{A}$ be an abelian category and $C^\bullet \in \mathrm{Ch}^b(\mathcal{A})$ a bounded chain complex. Then

$$\sum_{i\in\mathbb{Z}} (-1)^i [H^i(C)] = \sum_{i\in\mathbb{Z}} (-1)^i [C^i]$$

in $K_0(\mathcal{A})$. Indeed, consider the coboundary/cocycle sequences

$$0 \to Z^i \to C^i \to B^{i+1} \to 0 \qquad 0 \to B^i \to Z^i \to H^i(C) \to 0.$$

Therefore in K-theory we have

$$[C^i] = [Z^i] + [B^{i+1}] = [H^i(C)] + [B^i] + [B^{i+1}]$$

for all $i \in \mathbb{Z}$. Taking the alternating sum over both sides, the $[B^i] + [B^{i+1}]$ term telescopes to zero, because $B^i = 0$ for $i \ll 0$ and $i \gg 0$. In particular, if $C^\bullet$ is acyclic then its Euler characteristic in $K_0(\mathcal{A})$ is zero.

Now let $\mathcal{A}$ be an abelian category and $\mathcal{D}$ be a triangulated category. Let $H\colon \mathcal{D} \to \mathcal{A}$ be a cohomological functor, and assume that it is *bounded*: this means that for each $A \in \mathcal{D}$, the object $H(A[i])$ is nonzero for only finitely many i. Given a bounded cohomological functor H, there is an associated map

$$\chi_H\colon K_0(\mathcal{D}) \to K_0(\mathcal{A})$$

defined by $\chi_H([A]) = \sum_{i\in\mathbb{Z}} H(A[i])$. First, this is well-defined on isomorphism classes because of boundedness; to check the relation, let $A \to B \to C \to A[1]$ be exact, and consider the long exact sequence

$$\cdots \to H(A[i]) \to H(B[i]) \to H(C[i]) \to H(A[i+1]) \to \cdots$$

which is zero far enough to the left and right. By Example 3.3.1 applied to the bounded acyclic complex shown above, we see that

$$\begin{aligned} 0 &= \sum_{i\in\mathbb{Z}} (-1)^i ([H(A[i])] - [H(B[i])] + [H(C[i])]) \\ &= \chi_H(A) - \chi_H(B) + \chi_H(C), \end{aligned}$$

which proves that χ_H is well-defined.

The most important case of this construction is when $\mathcal{D} = D^b(\mathcal{A})$ is the bounded-derived category of $\mathcal{A}$, and $H = H^0$. Note firstly that there is in fact an induced map

$$K_0(\mathrm{Ch}^b(\mathcal{A})) \to H_0(D^b(\mathcal{A}))$$

sending $[C^\bullet] \mapsto [C^\bullet]$. It is clear that this is well-defined on isomorphism classes, and it satisfies the defining relation since a short exact sequence of complexes corresponds to a distinguished triangle in the derived category.

Example 3.3.2. This is the derived example corresponding to Example 3.3.1. If $A \in D^b(\mathcal{A})$, then we claim that

$$[A] = \sum_{i\in\mathbb{Z}} (-1)^i [H^i(A)[0]]$$

inside $K_0(D^b(\mathcal{A}))$. We show this by induction on the length of cohomological amplitude; if $A = 0$ then this is clear. Otherwise, let the bottommost cohomology degree of A be n, and consider the triangle

$$H^n(A)[-n] \to A \to \tau_{\geq n+1} A \to H^n(C)[-n+1].$$

Then by induction we have

$$\begin{aligned} [A] &= [H^n(A)[-n]] + [\tau_{\geq n+1} A] \\ &= (-1)^n [H^n(A)[0]] + \sum_{i\geq n+1} (-1)^i [H^i(A)[0]] \\ &= \sum_{i\in\mathbb{Z}} (-1)^i [H^i(A)[0]] \end{aligned}$$

as desired.

Proposition 3.3.3. *Let $\mathcal{A}$ be an abelian category. Then the map*

$$\chi \coloneqq \chi_{H^0} \colon K_0(D^b(\mathcal{A})) \to K_0(\mathcal{A})$$

is an isomorphism with inverse $[A] \mapsto [A[0]]$.

Proof. The proposed inverse is well-defined because it is a composite of the functor $K_0(\mathcal{A}) \to K_0(\mathrm{Ch}^b(\mathcal{A}))$ induced from the 0-embedding $\mathcal{A} \to \mathrm{Ch}^b(\mathcal{A})$, and the functor $K_0(\mathrm{Ch}^b(\mathcal{A})) \to K_0(D^b(\mathcal{A}))$ that we just defined. The composite

$$[A] \mapsto [A[0]] \mapsto H^0(A[0]) = [A]$$

is the identity. That the other direction of composing is the identity is exactly the statement of Example 3.3.2. □

Remark 3.3.4. Let X be a scheme. We have considered subcategories like

$$D^b_{\mathrm{qc}}(X) \qquad D^b_{\mathrm{coh}}(X)$$

and have not yet discussed their relation to $D^b\,\mathrm{QCoh}(X)$ and $D^b\,\mathrm{Coh}(X)$. To resolve any ambiguity, consider the following situation: let $\mathcal{A}$ be an abelian category and $\mathcal{C} \subset \mathcal{A}$ a *wide subcategory*: this means that $\mathcal{C}$ is closed under extensions, kernels, and cokernels, and is in particular an abelian subcategory. Let $D^b_{\mathcal{C}}(\mathcal{A}) \subset D^b(\mathcal{A})$ denote the full subcategory of objects of $D^b(\mathcal{A})$ whose cohomology objects lie in $\mathcal{C}$. Then we have a commutative diagram

$$\begin{array}{ccc} K_0(D^b_{\mathcal{C}}(\mathcal{A})) & \longrightarrow & K_0(\mathcal{C}) \\ \downarrow & & \downarrow \\ K_0(D^b(\mathcal{A})) & \xrightarrow{\cong} & K_0(\mathcal{A}) \end{array}$$

because the functor $H^0 \colon D^b_{\mathcal{C}}(\mathcal{A}) \to \mathcal{C}$ is a bounded cohomological functor. Now the map $[C] \mapsto [C[0]]$ is still well-defined from $K_0(\mathcal{C}) \to K_0(D^b_{\mathcal{C}}(\mathcal{A}))$, and the decomposition of Example 3.3.2 still takes place in $K_0(D^b_{\mathcal{C}}(\mathcal{A}))$. (Note that we are using the fact that $\mathcal{C}$ is Serre to say that the truncation functors preserve $D_{\mathcal{C}}(\mathcal{A})$.) Thus the same proof as for Prop. 3.3.3 shows that the top map is an isomorphism. In particular, we can still assert isomorphisms

$$K_0(D^b_{\mathrm{qc}}(X)) \cong K_0(\,\mathrm{QCoh}(X)) \qquad K_0(D^b_{\mathrm{coh}}(X)) = K_0(\,\mathrm{Coh}(X))$$

without ever having to pass through the derived categories of $\mathrm{Coh}(X)$ and $\mathrm{QCoh}(X)$.

Example 3.3.5. Let X be a scheme and $\mathcal{F}$ any quasicoherent sheaf; write $\mathcal{G} = \bigoplus_{i=1}^{\infty} \mathcal{F}$. Then $\mathcal{G} \oplus \mathcal{F} \cong \mathcal{G}$, which implies that $[\mathcal{G}] + [\mathcal{F}] = [\mathcal{G}]$, i.e. $[\mathcal{F}] = 0$. Therefore, $K_0(\mathrm{QCoh}(X)) = K_0(D^b_{\mathrm{qc}}(X)) = 0$. So, we will mostly only be interested in the coherent case. More generally, the same argument applies to any abelian category which admits countable direct sums.

Definition 3.3.6. Let X be a Noetherian scheme. We write $G_0(X) := K_0(D^b_{\mathrm{coh}}(X))$ and often identify this with $K_0(\mathrm{Coh}(X))$. We write $K_0(X) := K_0(\mathrm{Perf}(X))$.

We note that our assumption of Noetherianity here is a bit overkill, and qcqs is usually fine, but then the theory needs a bit more care. Part of the power of these definitions is that K_0 and G_0 have obvious ring and module structures respectively. First, for each $E \in \mathrm{Perf}(X)$ we have a triangulated functor $\mathrm{Perf}(X) \to \mathrm{Perf}(X)$ given by $F \mapsto E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F$, which in particular induces a group homomorphism $K_0(X) \to K_0(X)$. Since this is natural with respect to E, it gives a map $K_0(X) \to \mathrm{Hom}_{\mathbb{Z}}(K_0(X), K_0(X))$, and thus a bilinear product

$$K_0(X) \times K_0(X) \to K_0(X),$$

which we write by $(\alpha, \beta) \mapsto \alpha \cdot \beta$. On the basis elements, we have $[E] \cdot [F] = [E \otimes^{\mathbf{L}} F]$. Commutativity and associativity follow from the corresponding properties of the derived tensor product. Thus $K_0(X)$ is naturally a ring, with unit $1 = \mathcal{O}_X$.

Since X is Noetherian, being in $D^b_{\mathrm{coh}}(X)$ is equivalent to being bounded and pseudocoherent. Derived tensor product preserves pseudocoherent objects, and hence the same argument above gives a bilinear map

$$K_0(X) \times G_0(X) \to G_0(X)$$

which on basis elements is also given by $[E] \cdot [A] = [E \otimes^{\mathbf{L}} A]$. This makes $G_0(X)$ a $K_0(X)$-module. The inclusion $\mathrm{Perf}(X) \to D^b_{\mathrm{coh}}(X)$ (again by Noetherianity) induces a map $K_0(X) \to G_0(X)$ of abelian groups. It is also a map of $K_0(X)$-modules since the ring and module structure coincide on perfect objects. Our machinery makes it relatively easy to say things about this:

Proposition 3.3.7. *Let X be a Noetherian regular scheme. The natural map $K_0(X) \to G_0(X)$ is an isomorphism of $K_0(X)$-modules.*

Proof. In this case we have shown that $\mathrm{Perf}(X) \to D^b_{\mathrm{coh}}(X)$ is an exact equivalence, hence the induced map on K_0 is an isomorphism. □

Note that equivalently we have given $K_0(\mathrm{Coh}(X))$ a module structure. Given a coherent sheaf $\mathcal{G}$, we have by definition of χ that

$$[E] \cdot [\mathcal{F}] = [E \otimes^{\mathbf{L}}_{\mathcal{O}_X} F] = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} \mathcal{G})] \in K_0(\mathrm{Coh}(X)).$$

In particular if $\mathcal{F}$ is a perfect sheaf, then

$$[\mathcal{F}] \cdot [\mathcal{G}] = \sum_{i \leq 0} (-1)^i [\mathrm{Tor}^i_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})].$$

So, if $\mathcal{F} = \mathcal{E}$ is a vector bundle, then $[\mathcal{E}] \cdot [\mathcal{G}] = [E \otimes_{\mathcal{O}_X} \mathcal{G}]$.

Suppose $f \colon X \to Y$ is a morphism of Noetherian schemes. Then we have the canonical pullback map $Lf^* \colon \mathrm{Perf}(Y) \to \mathrm{Perf}(X)$ which induces a pullback map $f^* \colon K_0(Y) \to K_0(X)$. This is a ring homomorphism because (derived) pullback commutes with (derived) tensor products.

Example 3.3.8. Let k be a field. Then

$$K_0(\operatorname{Spec} k) = G_0(\operatorname{Spec} k) = K_0(\operatorname{Vect}_{\mathrm{fd}}(k)),$$

and there is a natural map $K_0(\operatorname{Vect}_{\mathrm{fd}}(k)) \to \mathbb{Z}$ given by taking the dimension: indeed it depends only on isomorphism class, and the dimension is additive in exact sequences of vector spaces. This map sends $k \mapsto 1$, and hence is surjective. An arbitrary element of K_0 is a finite sum $\sum_i k_i[V_i]$ where V_i are vector spaces, and we can group all terms with a positive and negative coefficient with one another. Taking direct sums shows that it can be represented as $[V] - [W]$, which maps to $\dim(V) - \dim(W)$. If this is zero, then $V \cong W$, so $[V] - [W] = 0$. Thus

$$K_0(\operatorname{Spec} k) = G_0(\operatorname{Spec} k) \cong \mathbb{Z}$$

by the dimension map. Moreover, the composite $\dim_k \colon K_0(\operatorname{Spec} k) \to \mathbb{Z}$ is a ring homomorphism. Indeed, the ring structure on $K_0(\operatorname{Spec} k)$ corresponds to a ring structure on $K_0(\operatorname{Vect}_{\mathrm{fd}}(k))$ given by a tensor product of vector spaces, as all objects are their own free resolution. But $\dim(V \otimes W) = \dim(V)\dim(W)$, hence $K_0(\operatorname{Spec} k) \cong \mathbb{Z}$ as rings. From what we have said previously, to compute dim of an object $E \in \operatorname{Perf}(k)$, we just get $\sum_i (-1)^i \dim_k H^i(E)$. If $E = V^\bullet$ is given as a complex, then we know by the corresponding G_0 result that $\dim(E) = \sum_i (-1)^i \dim_k V^i$.

Note that decomposing elements of $K_0(X)$ via cohomology is not available to us anymore because the cohomology of a perfect object may not be perfect. But we can still say the following: Let $\mathcal{F}^\bullet$ be a bounded complex of perfect sheaves. Since X is Noetherian, this means that each $\mathcal{F}^i$ is coherent and has finite tor dimension, which implies that $\mathcal{F}^\bullet$ is pseudocoherent and has bounded tor amplitude, hence is perfect. We claim that

$$[\mathcal{F}^\bullet] = \sum_{i \in \mathbb{Z}} (-1)^i [\mathcal{F}^i[0]] \qquad \text{in } K_0(X).$$

Indeed, proceed by induction on the length, except this time use the stupid filtration triangle

$$\sigma_{\geq n+1} F^\bullet \to \mathcal{F}^\bullet \to \mathcal{F}^n[-n] \to \sigma_{\geq n+1}\mathcal{F}^\bullet[1]$$

where n is the bottom nonzero term, which remains in $\operatorname{Perf}(X)$. By the inductive hypothesis applied to $\sigma_{\geq n+1}\mathcal{F}^\bullet$ and the addition in K_0, we conclude the result.

In light of the above example, for arbitrary Noetherian X and $x \in X$, the corresponding map $i_x \colon \operatorname{Spec} \kappa(x) \to X$ induces a pullback map

$$i_x^* \colon K_0(X) \to K_0(\operatorname{Spec} \kappa(x)) \xrightarrow{\sim} \mathbb{Z},$$

which we denote by $\operatorname{rk}_x \colon K_0(X) \to \mathbb{Z}$. For instance, if $\mathcal{E}^\bullet$ is a complex of vector bundles, then

$$\operatorname{rk}_x(\mathcal{E}^\bullet) = \sum_i (-1)^i \operatorname{rk}_x \mathcal{E}^i = \sum_i (-1)^i \dim_{\kappa(x)} \mathcal{E}^i(x),$$

(where $\mathcal{E}^i(x) = \mathcal{E}^i_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$) since the derived pullback is just pullback on a locally free sheaf.

Next, suppose the morphism $f\colon X \to Y$ has finite tor-dimension, then in this case we obtain an exact functor $Lf^*\colon D^b_{\text{coh}}(Y) \to D^b_{\text{coh}}(X)$, which now induces a pullback $f^*\colon G_0(Y) \to G_0(X)$. By definition, we get a commutative diagram

$$\begin{array}{ccc} K_0(Y) & \longrightarrow & G_0(X) \\ \downarrow & & \downarrow \\ K_0(X) & \longrightarrow & G_0(Y) \end{array}$$

e.g. of abelian groups, or of $K_0(Y)$-modules via restriction of scalars. We have seen that regular immersions $i\colon Z \to X$ have finite tor dimension. In this case one writes the pullback as $i^!$ and calls it the *Gysin pullback* (not to be confused with the functor from Grothendieck duality). We will be able to say more about Gysin pullbacks shortly.

Example 3.3.9. Let X be Noetherian and $x \in X$ a point. We know that $\operatorname{Spec}\kappa(x) \to X$ has finite tor dimension iff $\kappa(x)$ has finite tor dimension over $\mathcal{O}_{X,x}$, i.e. $x \in X_{\text{reg}}$. Therefore for a regular point we may define $\text{rk}_x\colon G_0(X) \to G_0(\operatorname{Spec}\kappa(x)) \cong \mathbb{Z}$ by the dimension function. Viewing $\mathbb{Z}$ as a $K_0(X)$-module via the rk_x function which always exists, the homomorphism $G_0(X) \to \mathbb{Z}$ is a $K_0(X)$-module map.

The rank at a smooth point x is computable in a much nicer way. We know that for $A \in D^b_{\text{coh}}(X)$, $[A] = \sum_i (-1)^i [H^i(A)]$, and applying Li_x^* we obtain

$$\text{rk}_x(A) = \sum_i (-1)^i \dim_{\kappa(x)} H^i(A_x \otimes^{\mathbf{L}}_{\mathcal{O}_{X,x}} \kappa(x)),$$

which is some kind of measurement of how A meets the point x. This is very different from a local length invariant which we will define momentarily.

Next, let $f\colon X \to Y$ be a proper morphism of Noetherian schemes (or slightly more generally, it can be cohomologically proper). Then we have an exact functor $Rf_*\colon D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$, which induces a pushforward map

$$f_*\colon G_0(X) \to G_0(Y)$$

given on the basis by $[A] \mapsto [Rf_*(A)]$. For example, if X is a proper k-scheme, then we get a map $G_0(X) \to G_0(k) \cong \mathbb{Z}$ given by

$$[A] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \dim_k \mathbb{H}^i(X, A),$$

which shows that $G_0(X)$ does contain quite a bit of cohomological information. In this situation, one usually writes $\chi(A)$ for the RHS above. A bit more generally, if X is proper over a ring A, then we can still obtain its cohomological Euler characteristic but as an element of $G_0(A)$.

Of course, based on our definitions, it is obvious that these induced maps on G_0 and K_0 are all functorial with respect to compositions. Now, if f is proper and of finite tor dimension (which was the right setting for Grothendieck duality as well), then we obtain both a pullback and pushforward on G_0. One basic use case is when $i\colon Z \to X$ is a regular immersion.

Example 3.3.10. The best setting is a morphism of smooth projective varieties $f\colon X \to Y$ over a regular Noetherian ring R (e.g. a field). In this case, $K_0(X) = G_0(X)$ and $K_0(Y) = G_0(Y)$, so everything is a ring, and we have a ring homomorphism $f^*\colon K_0(Y) \to K_0(X)$ and a map of abelian groups $f_*\colon K_0(X) \to K_0(Y)$. In practice, this is the setting we usually use, alongside the case of a regular immersion.

So far, we have seen that using different descriptions of the groups K_0 and G_0 allows us to easily define new invariants without much work. We want to do this for the determinant, which will be a map $\det\colon K_0(X) \to \operatorname{Pic}(X)$. This is possible in general, but it does take some extra work, so instead we will do this in the presence of the resolution property.

Let X be a scheme, and recall that we define $\operatorname{Vect}(X)$ to be the exact category of vector bundles, meaning locally free sheaves of finite (possibly non-constant) rank. Let $K_0^{vb}(X) \coloneqq K_0(\operatorname{Vect}(X))$. For each $x \in X$ there is a map $\operatorname{rk}_x\colon K_0^{vb}(X) \to \mathbb{Z}$ given by $[\mathcal{E}] \mapsto \operatorname{rk}_x \mathcal{E}$, since the rank of this finite free module is additive in exact sequences. For a vector bundle $\mathcal{E}$, the assignment $X \to \mathbb{Z}$ given by $x \mapsto \operatorname{rk}_x \mathcal{E}$ is locally constant: for each $x \in X$, we can by definition find a nbhd where $\mathcal{E}|_U \cong \mathcal{O}_U^n$, and $n = \operatorname{rk}_x \mathcal{E}$. Therefore $\operatorname{rk}(\mathcal{E})$ gives a section of $H^0(X, \mathbb{Z})$ (where $\mathbb{Z}$ means the $\underline{\mathbb{Z}}$) of the constant sheaf whose stalk at x is $\operatorname{rk}_x(\mathcal{E})$.

Moreover, the assignment $[\mathcal{E}] \mapsto \operatorname{rk}(\mathcal{E}) \in H^0(X, \mathbb{Z})$ gives a well-defined map $\operatorname{rk}\colon K_0^{vb}(X) \to H^0(X, \mathbb{Z})$, since to check additivity in exact sequences we can just check on every stalk and reduce to the previous case. Now, each vector bundle $\mathcal{E}$ gives a decomposition of $X = \bigsqcup_n X_n$ where $X_n = \{x \in X : \operatorname{rk}_x(\mathcal{E}) = n\}$, where by continuity X_n is clopen in X. Since these are clopen, there is a unique sheaf $\det(\mathcal{E})$ up to isomorphism such that

$$\det(\mathcal{E})|_{X_n} \cong \Lambda^n_{\mathcal{O}_X} \mathcal{E},$$

which we know is a line bundle (by exterior power notes). Now consider a short exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

in $\operatorname{Vect}(X)$. We claim that there is a natural isomorphism $\det(\mathcal{E}_1) \otimes \det(\mathcal{E}_3) \xrightarrow{\sim} \det(\mathcal{E}_2)$, and we have already shown (in exterior power notes) that this holds when the bundles have constant rank, so the main problem is constructing the map. Disjoint clopen descent allows us to freely map into and out of an object so we may, in fact, assume that $\mathcal{E}_2$ has constant rank ℓ. Let $\{X_n\}$ be a clopen decomposition of X as above for the sheaf $\mathcal{E}_1$. Then by Corollary 2.2.7 of those notes, we obtain the canonical map

$$\Lambda^n \mathcal{E}_1|_{X_n} \otimes \Lambda^m \mathcal{E}_3|_{X_n} \to \Lambda^{m+n} \mathcal{E}_2|_{X_n}.$$

for each $n, m \geq 0$. In particular take $m = \ell - n$, since $\mathcal{E}_2$ has rank ℓ everywhere and on X_n, $\mathcal{E}_1$ has rank n, so $\mathcal{E}_2$ is forced to have rank $\ell - n$. Thus we may write the above map as

$$\det(\mathcal{E}_1)|_{X_n} \otimes \det(\mathcal{E}_3)|_{X_n} \to \det(\mathcal{E}_2)|_{X_n},$$

which is an isomorphism as the three terms now have constant rank on X_n. Thus we may ascend from $\{X_n\}$ to X providing the desired isomorphism. This gives a determinant map $\det\colon K_0^{vb}(X) \to \mathrm{Pic}(X)$. We also sometimes consider the joint homomorphism

$$(\det, \mathrm{rk})\colon K_0^{vb}(X) \to \mathrm{Pic}(X) \times H^0(X, \mathbb{Z}).$$

Finally, note that there is always a natural map $K_0^{vb}(X) \to K_0(X)$ in the Noetherian case which sends $[\mathcal{E}] \mapsto [\mathcal{E}[0]]$; it is well-defined as a short exact sequence of bundles gives an exact triangle in $\mathrm{Perf}(X)$. Note moreover that $K_0^{vb}(X)$ has an obvious product structure which is just given by regular tensor product: This works because all objects of $\mathrm{Vect}(X)$ are flat sheaves. Hence $K_0^{vb}(X)$ is a ring, and by definition the map into $K_0(X)$ is a ring map.

Lemma 3.3.11. *Let X be a Noetherian scheme with the resolution property. Then the natural map*

$$K_0^{vb}(X) \to K_0(X)$$

is an isomorphism (of rings) with inverse given by $[\mathcal{E}^\bullet] \mapsto \sum_i (-1)^i [\mathcal{E}^i]$ whenever $\mathcal{E}^\bullet$ is a bounded complex of vector bundles.

Corollary 3.3.12. *Let X be a regular Noetherian separated scheme. Then the maps*

$$K_0^{vb}(X) \xrightarrow{\sim} K_0(X) \xrightarrow{\sim} G_0(X) \xrightarrow{\sim} K_0(\mathrm{Coh}(X))$$

are all isomorphisms (e.g. of $K_0(X)$-modules).

Proof. The right-hand map is always an isomorphism, the middle map is an isomorphism by Prop. 3.3.7, and the first map is an isomorphism by the previous lemma and Prop. 3.1.56. □

Let X be a Noetherian scheme with the resolution property. Then we define $(\det, \mathrm{rk})\colon K_0(X) \to \mathrm{Pic}(X) \times H^0(X, \mathbb{Z})$ to be the unique homomorphism for which composition with $K_0^{vb}(X) \to K_0(X)$ gives $(\det, \mathrm{rk})$ defined previously. By the lemma, this is computed on an object $[E] \in K_0(X)$ by first representing $E = \mathcal{E}^\bullet$ by a bounded complex of vector bundles, and then

$$\det(E) = \bigotimes_{i \in \mathbb{Z}} \det(\mathcal{E}^i)^{(-1)^i} = \sum_{i \in \mathbb{Z}} (-1)^i [\det(\mathcal{E}^i)]$$

where in the second equality we use additive notation for the group $\mathrm{Pic}(X)$. Note that this is a homomorphism $K_0(X) \to \mathrm{Pic}(X)$ of additive groups. Alternatively, there is a multiplicative homomorphism $\mathrm{Pic}(X) \to K_0(X)^\times$ which simply sends $\mathcal{L} \mapsto [\mathcal{L}]$, clearly well-defined, and by the description above has det as a (set-theoretic) retraction. In particular $\mathrm{Pic}(X) \hookrightarrow K_0(X)^\times$.

Example 3.3.13. Let A be a Noetherian ring. Then $\operatorname{Spec} A$ is affine and hence has the resolution property, so $K_0(\operatorname{Spec} A)$ is equivalent to K_0 of the category of vector bundles over $\operatorname{Spec} A$, i.e. finite projective modules over A. This is usually called $K_0(A)$, i.e. the Grothendieck group of the ring itself.

Here are some general relations that we can always write down in $K_0(X)$. First of all, let $\mathcal{E}$ be a vector bundle of rank n and consider a map $\mathcal{E} \to \mathcal{O}_X$, i.e. a section of $\mathcal{E}^\vee$. Recall that this is a regular section if the corresponding Koszul complex

$$0 \to \det(\mathcal{E}) \to \Lambda^{n-1}\mathcal{E} \to \cdots \to \Lambda^2\mathcal{E} \to \mathcal{E} \to \mathcal{O}_X \to 0,$$

treated as concentrated in degrees $[-n, 0]$, is exact everywhere except for $\mathcal{O}_X$. Letting $Z \coloneqq V(\mathrm{im}\,(\mathcal{E} \to \mathcal{O}_X))$, this gives a relation

$$[\mathcal{O}_Z] = 1 - [\mathcal{E}] + [\Lambda^2\mathcal{E}] - \cdots + (-1)^n \det(\mathcal{E})$$

in $K_0(X)$. Let's further suppose that this data is given by line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$ with sections $s_i \in \mathcal{L}_i^\vee$. Then we have already seen that the Koszul complex is a product of the one element complexes $\mathcal{L}_i \to \mathcal{O}_X$, meaning

$$[\mathrm{Kos}(s_1, \ldots, x_n)] = (1 - [\mathcal{L}_1]) \cdots (1 - [\mathcal{L}_n])$$

in $K_0(X)$. Further suppose that $s_1, \ldots, s_n$ forms a regular sequence, i.e. $V(s_1, \ldots, s_n)$ is a complete intersection. Then the Koszul complex represents $\mathcal{O}_{V(s_1,\ldots,s_n)}$, and we get

$$\begin{aligned}[\mathcal{O}_{V(s_1,\ldots,s_n)}] &= (1 - [\mathcal{L}_1]) \cdots (1 - [\mathcal{L}_n]) \\ &= [\mathcal{O}_{V(s_1)}] \cdots [\mathcal{O}_{V(s_n)}]\end{aligned}$$

by the corresponding proof for tensor products. This is a sign that the product in $K_0(X)$ should be interpreted as an intersection product. Of course, whenever D is an effective Cartier divisor, then we always have $[\mathcal{O}_D] = 1 - [\mathcal{O}_X(-D)]$.

3.3.2. Intersections in projective space. This section will be a good demonstration of the power of derived categories. We will make some trivial observations about projective space and deduce a good amount of nontrivial intersection-theoretic facts from it.

Example 3.3.14. Let $X = \mathbb{P}^n_k$ and write $H = [\mathcal{O}_{\mathbb{P}^n}(1)]$ for the 'hyperplane class'. We know that $x_0, \dots, x_n \in H^0(X, \mathcal{O}_X(1))$ form a regular sequence and that $V(x_0, \dots, x_n) = \varnothing$, so the Koszul complex is acyclic. The above discussion shows that $[\mathcal{O}_{V(x_i)}] = 1 - [\mathcal{O}_X(-1)] = 1 + H$, and that

$$0 = [\mathcal{O}_{V(x_0)}] \cdots [\mathcal{O}_{V(x_n)}] = (1+H)^{n+1}.$$

Therefore, we obtain a ring homomorphism $\mathbb{Z}[H]/(1+H)^{n+1} \to K_0(\mathbb{P}^n)$.

Let us show our first real theorem, which is that this map is an isomorphism. Note that H is a unit in $K_0(X)$, and

$$H^\ell = [\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes \ell}] = [\mathcal{O}_{\mathbb{P}^n}(\ell)]$$

for all $\ell \in \mathbb{Z}$. Moreover in $\mathbb{Z}[x]/(1+x)^{n+1}$, x is a unit since

$$x = (1+x) - 1 \mid (1+x)^{n+1} - 1 = -1,$$

i.e. $x \mid -1$. Therefore the image of this map will contain all finite expressions of the form $\sum_{i \in \mathbb{Z}} a_i H^i$. The only other input is the following simple computation:

Example 3.3.15. Let k be a field and $X = \mathbb{P}^n_k$. Since X is proper over k, we have an Euler characteristic $\chi(X, \alpha)$ for $\alpha \in K_0(X)$. We claim that

$$\chi(X, \mathcal{O}_X(d)) = \frac{(d+1)\cdots(d+n)}{n!} \equiv \binom{d+n}{n},$$

where we are viewing the RHS as a polynomial in d. Indeed, this is pretty obvious from the explicit description of the cohomology of $\mathbb{P}^n$, but we can also show it inductively without assuming that result. Indeed, consider the hyperplane sequence

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to \mathcal{O}_{\mathbb{P}^{n-1}_k}(d) \to 0,$$

then we obtain

$$\chi(X, \mathcal{O}_X(d)) - \chi(X, \mathcal{O}_X(d-1)) = \chi(\mathbb{P}^{n-1}_k, \mathcal{O}(d))$$

Theorem 3.3.16. *Let k be a field and write $H = [\mathcal{O}_{\mathbb{P}^n_k}(1)] \in K_0(\mathbb{P}^n_k)$. The natural map*

$$\mathbb{Z}[x]/(1+x)^{n+1} \to K_0(\mathbb{P}^n_k) \qquad x \mapsto H$$

is an isomorphism of rings.

Proof. This map is surjective: to see this, note that for all $\ell \in \mathbb{Z}$. Since $\mathbb{P}^n$ is regular, $K_0(\mathbb{P}^n) = K_0(\mathrm{Coh}(\mathbb{P}^n))$. Moreover, we know that every coherent sheaf $\mathcal{F}$ admits a finite resolution by sums of line bundles (Cor. 3.1.61), from which it immediately follows that $[\mathcal{F}]$ is equivalent to $\sum_i a_i H^i$ □

APPENDIX A

Supplementary material

A.1. Descent

This section is mostly for fun: at the end we will prove fpqc-locality on the target for various properties of morphisms. This is an important result, especially in preparation for defining algebraic spaces and stacks, and faithfully flat descent is useful in its own right.

By *descent*, we will mean five related things:

(a) *Effective descent for modules on an open cover* means that we are given an open cover $\{U_i\}$ of X, and want to know how $\mathrm{Mod}(\mathcal{O}_X)$ is obtained from the categories $\mathrm{Mod}(\mathcal{O}_{U_i})$.

(b) *Effective descent for quasicoherent sheaves* means that we are given a morphism $f\colon X \to Y$ of schemes, and want to know how $\mathrm{QCoh}(Y)$ is obtained from $\mathrm{QCoh}(X)$.

(c) *Descent for morphisms* means we are given a morphism $U \to X$, and want to know how $\mathrm{Hom}(X, Y)$ is obtained from $\mathrm{Hom}(U, Y)$ for a scheme Y.

(d) *Effective descent for schemes* means we are given a scheme X (possibly over a base) with 'descent data', and want to see how it defines a scheme Y with some kind of cover $X \to Y$.

(e) *Descent for properties of morphisms* means that we are given a map $X \to Y$ and a descent morphism $Y' \to Y$, and want to know when a property of the morphism $X \times_Y Y' \to Y'$ 'descends' to $X \to Y$.

We will address all five questions in this section. (e) is particularly interesting, where we will generalize the fact that many properties of morphisms are local on the target to showing that they are fpqc-local on the target. Much of the discussion in this section could be considered standard introductory algebraic geometry, which is why we include it in an appendix.

Remark A.1.1. The word 'descent' refers to the following idea: We are given a kind of "cover" $p\colon Y' \to Y$ where data on Y' is easier to understand; data on Y can usually be pulled back to Y', (e.g. sheaves via pullback, Y-schemes via fiber product), and we want to know whether some data or property on Y' descends back down to Y. The basic example is where R is a ring, $(f_1, \ldots, f_r) = (1)$ are elements generating the unit ideal, and we have the surjective map

$$\coprod_{i=1}^{r} \operatorname{Spec} R_{f_i} \to \operatorname{Spec} R$$

on spectra. The terminology is in contrast to 'ascent', which records when a property has good stability under pullback of sheaves or base change.

A.1.1. Elementary results. We first recall the most basic form of descent, which is descent for an open cover. Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{U} = \{U_\alpha\}$ an open cover of X. Denote by $j_\alpha \colon U_\alpha \hookrightarrow X$ the corresponding inclusions. We want to define a category $\mathrm{Mod}(\mathcal{O}_\mathcal{U})$ as the limit of the diagram:

$$\prod_\alpha \mathrm{Mod}(\mathcal{O}_{U_\alpha}) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Mod}(\mathcal{O}_{U_\alpha \cap U_\beta}) \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\; \prod_{\alpha,\beta,\gamma} \mathrm{Mod}(\mathcal{O}_{U_\alpha \cap U_\beta \cap U_\gamma})$$

Let us spell this out: An object of $\mathrm{Mod}(\mathcal{O}_\mathcal{U})$ is a family $\{\mathcal{F}_\alpha : \alpha\}$ for $\mathcal{F}_\alpha \in \mathrm{Mod}(\mathcal{O}_{U_\alpha})$ equipped with a descent datum: this means a collection of isomorphisms

$$g_{\alpha\beta} \colon \mathcal{F}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \mathcal{F}_\beta|_{U_{\alpha\beta}}$$

satisfying the *cocycle condition* on triple overlaps, which is that the diagram

$$\begin{array}{ccc} \mathcal{F}_\alpha|_{U_{\alpha\beta\gamma}} & \xrightarrow{g_{\alpha\beta}} & \mathcal{F}_\beta|_{U_{\alpha\beta\gamma}} \\ & \searrow^{g_{\alpha\gamma}} & \downarrow{\scriptstyle g_{\beta\gamma}} \\ & & \mathcal{F}_\gamma|_{U_{\alpha\beta\gamma}} \end{array}$$

should commute. A morphism in this category is a collection of morphisms $\mathcal{F}_\alpha \to \mathcal{G}_\alpha$ commuting with the $g_{\alpha\beta}$'s. We will have various definitions like this, so at some point we will just say 'obvious category of descent data' to mean the corresponding construction.

There is a canonical map $j^* \colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_\mathcal{U})$ given by sending $\mathcal{F}$ to $\{\mathcal{F}|_{U_\alpha}\}$, where the descent data above are just identity maps, and the cocycle condition is tautological. A morphism is sent to its restrictions, which commute with identity maps.

Proposition A.1.2. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{U} = \{U_\alpha\}$ an open cover. The natural map*

$$j^* \colon \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_\mathcal{U})$$

is an equivalence of categories between $\mathrm{Mod}(\mathcal{O}_X)$ and the category of descent data. A quasi-inverse is given on objects by

$$((\mathcal{F}_\alpha), (g_{\alpha\beta})) \mapsto \mathrm{Eq}\Big(\prod_\alpha j_{\alpha,*}\mathcal{F}_\alpha \rightrightarrows \prod_{\alpha,\beta} j_{\alpha\beta,*}\mathcal{F}_\beta|_{U_{\alpha\beta}}\Big)$$

where the lower map is induced by restriction, and the upper one is restriction twisted by the $g_{\alpha\beta}$.

Proof. We begin with full faithfulness: if $f, g \colon \mathcal{F} \to \mathcal{G}$ are morphisms of sheaves with $j^*(f) = j^*(g)$, then $f|_{U_\alpha} = g|_{U_\alpha}$ for all α, so $f = g$. If we are given maps $f_\alpha \colon \mathcal{F}|_{U_\alpha} \to \mathcal{G}|_{U_\alpha}$ which agree on overlaps, then they glue to a map $f \colon \mathcal{F} \to \mathcal{G}$. This follows from the fact that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf.

Essential surjectivity (i.e. 'effectivity'): Let $((\mathcal{F}_\alpha),(g_{\alpha\beta}))$ be a descent datum, and let $\mathcal{F}$ be the equalizer written in the proposition statement. On the (α,β) component the top map is given by

$$\prod_\alpha j_{\alpha,*}\mathcal{F}_\alpha \to j_{\alpha,*}\mathcal{F}_\alpha \to j_{\alpha\beta,*}\mathcal{F}_\alpha|_{U_{\alpha\beta}} \xrightarrow{g_{\alpha\beta}} j_{\alpha\beta,*}\mathcal{F}_\beta|_{U_{\alpha\beta}},$$

and the bottom map is given simply by

$$\prod_\alpha j_{\alpha,*}\mathcal{F}_\alpha \to j_{\beta,*}\mathcal{F}_\beta \to j_{\alpha\beta,*}\mathcal{F}_\beta|_{U_{\alpha\beta}}.$$

Therefore, if $V \subset X$ is an open set then

$$\mathcal{F}(V) = \{(s_\alpha) \in \prod_\alpha \mathcal{F}_\alpha(V_\alpha \cap U) : g_{\alpha\beta}(s_\alpha|_{U_{\alpha\beta}}) = s_\beta|_{U_{\alpha\beta}\cap V}\ \forall\alpha,\beta\}$$

as limits are computed pointwise in sheaf categories.

Consider the projection $\mathcal{F} \to j_{U_\alpha,*}\mathcal{F}_\alpha$, which is adjoint to $\mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$; we claim this latter map is an isomorphism. Let $W \subset U_\alpha$, and consider sections on W. Then $\mathcal{F}(W) \to \mathcal{F}_\alpha(W)$ is injective: if $(s_\beta)_\beta \mapsto 0$, then $s_\alpha = 0$. All other $U_\beta \cap W$ are contained in $U_\alpha \cap W = W$, so it follows that $0 = g_{\alpha\beta}(s_\alpha|_{U_\beta\cap W}) = s_\beta|_{U_\beta\cap W} = s_\beta$ for all β, hence $s_\beta = 0$ for all β.

It is also surjective: if $t \in \mathcal{F}_\alpha(W)$, let $s_\beta = g_{\alpha\beta}(t)$ where $s_\beta \in \mathcal{F}_\beta(W \cap U_\beta)$. This is a compatible family, because for all β,γ,

$$\begin{aligned} g_{\beta\gamma}(s_\beta|_{U_{\alpha\beta}\cap W}) &= g_{\beta\gamma}g_{\alpha\beta}(t)|_{U_{\alpha\beta\gamma}\cap W} \\ &= g_{\alpha\gamma}(t)|_{U_{\alpha\beta\gamma}\cap W} \\ &= s_\gamma|_{U_{\alpha\beta\gamma}\cap W} \end{aligned}$$

by the cocycle condition. Note also that $U_{\alpha\beta\gamma} \cap W = U_{\beta\gamma} \cap W$ because W is already contained in U_α, so this proves the claim.

To complete the proof, we have to show that $j^*(\mathcal{F})$ is isomorphic to the descent data $((\mathcal{F}_\alpha),(g_{\alpha\beta}))$. The diagrams

$$\begin{array}{ccc} \mathcal{F}|_{U_{\alpha\beta}} & \longrightarrow & \mathcal{F}_\alpha|_{U_{\alpha\beta}} \\ & \searrow & \downarrow{\scriptstyle g_{\alpha\beta}} \\ & & \mathcal{F}_\beta|_{U_{\alpha\beta}} \end{array}$$

commute for all α,β by definition of the equalizer. Therefore the family of maps $\mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ described above is a morphism of descent data. Since each of these was an isomorphism, we are done. □

Remark A.1.3. Consider the special case $\mathcal{O}_X = \underline{\mathbb{Z}}$, which gives open-cover descent for abelian sheaves on a space. Then it is clear from the equalizer description that if the sheaves $\mathcal{F}_\alpha$ are given additional structure which is preserved by pushforwards, and the descent data are morphisms respecting this additional structure, then it will also descend. For instance, if $((\mathcal{F}_\alpha),(g_{\alpha\beta}))$ is a descent datum where F_α is a sheaf of rings and $g_{\alpha\beta}$ are ring-sheaf isomorphisms, then j_* exists on the level of sheaves of rings, so the resulting equalizer $\mathcal{F}$ is a sheaf of rings. The

restriction maps $\mathcal{F}|_{U_\alpha} \to \mathcal{F}_\alpha$ in the proof are also morphisms of sheaves of rings, so the essential surjectivity follows. Moreover, morphisms descend too: if we have compatible morphisms $f_\alpha \colon \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ which are sectionwise ring homomorphisms, then the constructed map $f \colon \mathcal{F} \to \mathcal{G}$ between sheaves of rings that is a priori a map of abelian sheaves, is also a ring homomorphism. This is because the restriction $f|_{U_\alpha}$ is a morphism of sheaves of rings, and the equations defining the ring axioms can be checked locally.

We will now use this to prove open-cover descent for ringed spaces over a fixed ringed space. Let $(Y, \mathcal{O}_Y)$ be a ringed space, **RS** denote the category of ringed spaces, and $\mathbf{RS}_Y$ denote the slice category over Y. Given an open cover $\{U_\alpha\} = \mathcal{U}$ of Y we define the category of descent data $\mathrm{Desc}(\mathbf{RS}_\mathcal{U})$ as follows:

(a) Objects: for each α a ringed space X_α with a morphism $f_\alpha \colon X_\alpha \to U_\alpha$, along with descent data $\varphi_{\alpha\beta}$ which are isomorphisms
$$\varphi_{\alpha\beta} \colon f_\alpha^{-1}(U_{\alpha\beta}) \xrightarrow{\sim} f_\beta^{-1}(U_{\alpha\beta})$$
of ringed spaces satisfying the cocycle condition: $\varphi_{\beta\gamma} \circ \varphi_{\alpha\beta} = \varphi_{\alpha\gamma}$ over triple intersections $U_{\alpha\beta\gamma}$.

(b) Morphisms: Families of morphisms of ringed spaces $X_\alpha \to X'_\alpha$ over U_α such that the induced diagrams
$$\begin{array}{ccc} f_\alpha^{-1}(U_{\alpha\beta}) & \longrightarrow & (f'_\alpha)^{-1}(U_{\alpha\beta}) \\ \cong\downarrow & & \downarrow\cong \\ f_\beta^{-1}(U_{\alpha\beta}) & \longrightarrow & (f'_\beta)^{-1}(U_{\alpha\beta}) \end{array}$$
commute for all α, β.

We make the same corresponding definitions for $\mathbf{LRS}_Y$ and $\mathrm{Desc}(\mathbf{LRS}_\mathcal{U})$ for locally ringed spaces. Once again, we obtain a canonical map from the original category to the category of descent data: we send $f \colon X \to Y$ to $f_\alpha = f^{-1}(U_\alpha) \to U_\alpha$, and get a tautological cocycle condition.

THEOREM A.1.4. *Let $(Y, \mathcal{O}_Y)$ be a (resp. locally) ringed space and $\mathcal{U} = \{U_\alpha\}$ an open cover of Y. The natural functor*
$$\mathbf{RS}_Y \to \mathrm{Desc}(\mathcal{U}, \mathbf{RS}) \qquad (\textit{resp.} \quad \mathbf{LRS}_Y \to \mathrm{Desc}(\mathcal{U}, \mathbf{LRS}))$$
is an equivalence of categories.

To prove this, we will prove a slightly more general absolute gluing statement and deduce the open-cover descent theorem from this. Fix an index set I, and define a category $\mathrm{Cov}_I(\mathbf{RS})$ as follows: an object is a ringed space $(X, \mathcal{O}_X)$ equipped with an I-indexed open cover $\{U_\alpha : \alpha \in I\}$. A morphism $f \colon (X, \{U_\alpha\}) \to (Y, \{V_\alpha\})$ is a morphism f of ringed spaces such that $f(U_\alpha) \subset V_\alpha$. We make the corresponding definitions of $\mathrm{Cov}_I(\mathbf{LRS})$.

Next, again given an index set I, there is a category of gluing data $\mathrm{Glue}_I(\mathbf{RS})$ defined as follows: its objects consist of a collection X_α of ringed spaces, equipped

with open subspaces $X_{\alpha\beta} \subset X_\alpha$ for each α, β and isomorphisms

$$\phi_{\alpha\beta} \colon X_{\alpha\beta} \xrightarrow{\sim} X_{\beta\alpha}$$

of ringed spaces satisfying the cocycle condition $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$. Its morphisms are maps $f_\alpha \colon X_\alpha \to Y_\alpha$ which preserve the subspaces $X_{\alpha\beta}$ and commute with $\phi_{\alpha\beta}$. We again make the corresponding definition for $\mathrm{Glue}_I(\mathbf{LRS})$. There is a natural functor

$$\mathrm{Cov}_I(\mathbf{RS}) \to \mathrm{Glue}_I(\mathbf{RS})$$

which sends X equipped with $\{U_\alpha\}$ to the collection $X_\alpha \coloneqq U_\alpha$ itself, where $X_{\alpha\beta} \coloneqq U_\alpha \cap U_\beta$ and the gluing isomorphisms are the identity maps.

Lemma A.1.5. *Let I be an index set. The natural functors*

$$\mathrm{Cov}_I(\mathbf{RS}) \to \mathrm{Glue}_I(\mathbf{RS}) \qquad \mathrm{Cov}_I(\mathbf{LRS}) \to \mathrm{Glue}_I(\mathbf{LRS})$$

are equivalences of categories.

Proof. The proof is the same in both cases. We begin with full faithfulness. Let $(X, \{U_\alpha\})$ and $(Y, \{V_\alpha\})$ be objects of the source category. If f, g are two morphisms whose images in the target agree, then we have in particular that $f|_{U_\alpha} = g|_{U_\alpha}$ for all α, which implies that $f = g$. Conversely, a morphism between the images consists of maps $f_\alpha \colon U_\alpha \to V_\alpha$ whose restrictions to $U_\alpha \cap U_\beta$ agree. In other words, the corresponding maps $f_\alpha \colon U_\alpha \to Y$ agree on an open cover and hence glue to a morphism (which maps U_α to V_α) in the source category.

Now let $(X_\alpha, X_{\alpha\beta}, g_{\alpha\beta})$ be a gluing datum. Define a topological space X by the coequalizer

$$\coprod_{\alpha,\beta} X_{\alpha\beta} \rightrightarrows \coprod_{\alpha} X_\alpha \to X,$$

where the top map is induced by $X_{\alpha\beta} \xrightarrow{g_{\alpha\beta}} X_{\beta\alpha} \hookrightarrow X_\beta$, and the bottom map is induced simply by $X_{\alpha\beta} \hookrightarrow X_\alpha$ (followed by the inclusions). Therefore X is the quotient of $\coprod_\alpha X_\alpha$ by the relation generated by $x \sim g_{\alpha\beta}(x)$. It follows from standard topology that each composite $j_\alpha \colon X_\alpha \to X$ is an open embedding, and $j_\alpha(X_\alpha) \cap j_\beta(X_\beta)$ is canonically identified with $X_{\alpha\beta}$ (and $X_{\beta\alpha}$).

We next construct a sheaf of rings on X. Consider the sheaves of rings $\mathcal{O}_{X_\alpha}$ on X_α for each α, viewed as an open cover of X. The isomorphism $g_{\alpha\beta}$ induces an isomorphism

$$\mathcal{O}_{X_\alpha}|_{j_\alpha(X_\alpha) \cap j_\beta(X_\beta)} \xrightarrow{\sim} \mathcal{O}_{X_\beta}|_{j_\alpha(X_\alpha) \cap j_\beta(X_\beta)}$$

satisfying the cocycle condition, because $g_{\alpha\beta}$ did. By Remark A.1.3 this data glues to a sheaf of rings $\mathcal{O}_X$ on X and an isomorphism of the corresponding descent data. This induces an isomorphism $\mathcal{O}_X|_{X_\alpha} \xrightarrow{\sim} \mathcal{O}_{X_\alpha}$ which is compatible with the $g_{\alpha\beta}$, and hence compatible on overlaps. It follows that j_α is an open inclusion of ringed spaces. Finally, the ringed space $(X, \mathcal{O}_X)$ equipped with the cover $\{j_\alpha(X_\alpha)\}$ is identified under the natural functor with the gluing datum we started with, by construction. □

Using this, let us prove the descent theorem for (locally) ringed spaces.

Proof of (Thm. A.1.4). The proofs in the two cases are the same. We again begin with full faithfulness. Let $f, g\colon X_1 \to X_2$ be morphisms over Y with structure maps p_1 and p_2. Their images agreeing in the descent data category means that the two restrictions of f and g,

$$f^{-1}p_2^{-1}(U_\alpha) = p_1^{-1}(U_\alpha) \to p_2^{-1}(U_\alpha)$$

agree. Since opens of the form $p_2^{-1}(U_\alpha)$ cover X_2 for various α, this implies that $f = g$. Next consider the same spaces X_i and p_i over Y, and now assume we have a morphism of descent data between their images. This means that we have maps $f_\alpha\colon p_1^{-1}(U_\alpha) \to p_2^{-1}(U_\alpha)$ over U_α satisfying the following compatibility diagram:

$$\begin{array}{ccc} p_1^{-1}(U_{\alpha\beta}) & \xrightarrow{f_\alpha} & p_2^{-1}(U_{\alpha\beta}) \\ \| & & \| \\ p_1^{-1}(U_{\alpha\beta}) & \xrightarrow{f_\beta} & p_2^{-1}(U_{\alpha\beta}) \end{array}$$

In other words, f_α and f_β agree on overlaps. Since $p_1^{-1}(U_\alpha) = f_\alpha^{-1}p_2^{-1}(U_\alpha) \to p_2^{-1}U_\alpha$, the $\{f_\alpha\}$ glue over the open cover $\{p_2^{-1}(U_\alpha)\}_\alpha$ giving a map $f\colon X_1 \to X_2$. This proves fullness.

Now let $(X_\alpha, f_\alpha, \varphi_{\alpha\beta})$ be a descent datum over the cover $U_\alpha \subset Y$. Set $X_{\alpha\beta} \coloneqq f_\alpha^{-1}(U_{\alpha\beta})$, so that $\varphi_{\alpha\beta}\colon X_{\alpha\beta} \xrightarrow{\sim} X_{\beta\alpha}$ and satisfies the cocycle condition. This is precisely a gluing datum in $\mathrm{Glue}_I(\mathbf{RS})$. Moreover, the maps $f_\alpha X_\alpha \to U_\alpha$ give a morphism of gluing data to the image of $(Y, \{U_\alpha\})$ under the functor of Lemma A.1.5.

Therefore by gluing there exists a ringed space $(X, \mathcal{O}_X)$ with an open cover $\{V_\alpha\}$ such that $V_\alpha \xrightarrow{\sim} X_\alpha$, so that $V_\alpha \cap V_\beta \xrightarrow{\sim} X_{\alpha\beta}$. By full faithfulness there is a unique map $X \to Y$ such that $X_\alpha \xrightarrow{\sim} V_\alpha \to Y$ factors through U_α and coincides with the given map f_α. It follows that the image of $X \to Y$ is isomorphic to the descent data we started with. □

Copying the notations of the previous few results, we can specialize to the case of schemes without any difficulty.

Corollary A.1.6.

1. *Let I be an index set. Then the natural functor*

$$\mathrm{Cov}_I(\mathbf{Sch}) \to \mathrm{Glue}_I(\mathbf{Sch})$$

is an equivalence.

Now let S be a scheme and $\mathcal{U}$ be an open cover of S.

2. *The natural functor*

$$\mathbf{Sch}_S \to \mathrm{Desc}(\mathcal{U}, \mathbf{Sch})$$

is an equivalence of categories.

3. *The natural functor*

$$\mathrm{QCoh}(S) \to \mathrm{QCoh}(\mathcal{U}) \coloneqq \mathrm{Desc}(\mathcal{U}, \mathrm{QCoh})$$

is an equivalence of categories.

PROOF. (1) and (2) follow immediately from the corresponding results for locally ringed spaces once we observe that a locally ringed space which has an open cover by schemes is a scheme. (3) Again full faithfulness doesn't change, and descent for $\mathrm{Mod}(\mathcal{O}_X)$ will produce an $\mathcal{O}_X$-module locally isomorphic to quasicoherent sheaves, which is therefore quasicoherent. □

Remark A.1.7. Here is another often-useful description of $\mathrm{Mod}(\mathcal{O}_\mathcal{U})$ if $\mathcal{U} = \mathcal{B}$ is a basis of opens for the topology on X. Namely, there is a category whose objects are collections $\mathcal{F}_\alpha \in \mathrm{Mod}(\mathcal{O}_{B_\alpha})$ for each α, equipped with the following descent data:

(a) For each inclusion $B_\alpha \subset B_\beta$, an isomorphism $g_{\alpha\beta} \colon \mathcal{F}_\alpha \xrightarrow{\sim} \mathcal{F}_\beta|_{B_\alpha}$.
(b) This data should satisfy the cocycle condition for inclusions $B_\alpha \subset B_\beta \subset B_\gamma$, i.e. that the diagram

$$\begin{array}{ccc} \mathcal{F}_\alpha & \xrightarrow{g_{\alpha\beta}} & \mathcal{F}_\beta|_{B_\alpha} \\ & \searrow^{g_{\alpha\gamma}} & \downarrow{\scriptstyle g_{\beta\gamma}|_{B_\alpha}} \\ & & \mathcal{F}_\gamma|_{B_\alpha} \end{array}$$

commutes.

Morphisms are defined in the obvious way. There is an obvious functor $\mathrm{Mod}(\mathcal{O}_\mathcal{B})$ to this category sending $\mathcal{F}_\alpha$ to itself, with the cocycles being special cases of the ones in $\mathrm{Mod}(\mathcal{O}_\mathcal{B})$, since if $B_\alpha \subset B_\beta$, then $B_{\alpha\beta} = B_\alpha$.

This functor is an equivalence of categories: full faithfulness is trivial because the overlaps $B_\alpha \cap B_\beta$ are covered by basis elements B_γ. Hence the map from $\mathrm{Mod}(\mathcal{O}_X)$ is also fully faithful. If $(\mathcal{F}_\alpha, g_{\alpha\beta})$ is a descent datum of the form above, then it determines a usual descent datum for the following reason: for all α, β, the full faithfulness part applies to the basis $\mathcal{B}$ restricted to $B_\alpha \cap B_\beta$. This in particular gives isomorphisms $\mathcal{F}_\alpha|_{B_{\alpha\beta}} \xrightarrow{\sim} F_\beta|_{B_{\alpha\beta}}$ extending the original ones.

Now let us begin our discussion of fpqc morphisms of schemes. An *fpqc morphism* is a faithfully flat quasicompact morphism of schemes, i.e. a flat quasicompact surjection. This is of course stable under composition, arbitrary base change, and is local on the target, as each of the three properties are like this independently. We often separate this into flat and quasicompact surjection, because flatness is a source-local property, and descent statements sometimes only need quasicompact surjectivity.

Example A.1.8. The canonical example of an fpqc morphism $Y' \to Y$ is the case where $U_1, \ldots, U_r$ is a finite open cover of Y, each $U_i \hookrightarrow Y$ is a quasicompact open immersion, and $Y' = \bigsqcup_i U_i$. Then this is clearly flat and surjective, and quasicompactness follows from the assumption. If Y is qcqs and $\{U_i : i \in I\}$ is an open cover, then it is easy to see that $\bigsqcup_i U_i \to Y$ is quasicompact iff only finitely many U_i are nonempty, and each U_i is quasicompact.

Definition A.1.9. A property $\mathcal{P}$ of morphisms is *fpqc-local on the target* if the following property holds: Given a cartesian square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where $Y' \to Y$ (and hence $X' \to X$) is fpqc, the morphism $X \to Y$ is in $\mathcal{P}$ iff $X' \to Y'$ is in $\mathcal{P}$.

Example A.1.10. If a property $\mathcal{P}$ of morphisms is fpqc-local on the target, then the property $\Delta(\mathcal{P})$, i.e. $\{f : \Delta_f \in \mathcal{P}\}$, is also fpqc-local on the target. Indeed, factor the cartesian square above through the diagonal map via:

$$\begin{array}{ccccc} X' & \longrightarrow & X' \times_{Y'} X' & \longrightarrow & Y' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X & \xrightarrow{\Delta_f} & X \times_Y X & \longrightarrow & Y \end{array}$$

Then the middle map is fpqc, and fpqc-locality applied to the left-hand square gives the result.

Example A.1.11. Suppose we have a base change diagram as above, labeled $\overline{f}, g, f, \overline{g}$ around the square as usual, and $g\colon Y' \to Y$ is a quasicompact surjection. If $V \subset Y$ is quasicompact open, then $\overline{g}^{-1}f^{-1}(V) = \overline{f}^{-1}g^{-1}(V)$, so the RHS is quasicompact, and by base change $\overline{g}$ is a quasicompact surjection, so $\overline{g}^{-1}f^{-1}(V)$ surjects onto $f^{-1}(V)$. Thus $f^{-1}(V)$ is quasicompact. Applying this to the diagonal as above shows that quasi-separatedness also descends along a quasicompact-surjective map. In particular, qcqs can be checked after quasicompact-surjective base change, and hence is fpqc local on the target.

Example A.1.12. The simplest fpqc-local property is surjectivity, which actually descends under any surjective base change. Indeed if we have a base change square as above with $\overline{f}$ surjective, then since g is surjective we find that $f\overline{g}$ is surjective, so $Y = f(\overline{g}(X')) = f(X)$ as $\overline{g}$ is surjective.

We want to ask the same questions as above for all of our other favorite properties of morphisms, but even for affine morphisms, or even simply the property of being an isomorphism, the answer is not clear. This is part of the motivation for proving faithfully flat descent. Instead, in the rest of this section, we will focus on proving more elementary properties.

Here is the basic method for reducing an fpqc-locality proof to the affine case. This works if the property is local on both the source and the target and stable under (at least faithfully flat) base change. Given a base change square as above where $\overline{f}\colon X' \to Y'$ satisfies $\mathcal{P}$, choose an affine open $V \subset Y$ and $U \subset f^{-1}(V)$.

Then we form the restricted bottom square in the diagram below.

$$\begin{array}{ccc} U' & \longrightarrow & \bigsqcup_i W_i \\ \downarrow & \lrcorner & \downarrow \\ \overline{g}^{-1}(U) & \longrightarrow & g^{-1}(V) \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & V \end{array}$$

Then the right-hand map is still fpqc, and $U \to V$ is a map of affine schemes. Now choose an affine open cover $W_1, \ldots, W_n$ of the quasicompact scheme $g^{-1}(V)$, and form the flat surjection $\bigsqcup_i W_i \to g^{-1}(V)$ as above. This is faithfully flat but may not be qc; in any case, the right-hand composite is faithfully flat between affine schemes, and the top map is a morphism of affine schemes which also satisfies $\mathcal{P}$. The affine case will imply that $U \to V$ has $\mathcal{P}$, and then locality on the source and target completes the proof.

Example A.1.13. Being flat is fpqc local on the target. Indeed, this satisfies the conditions above so we may reduce to the affine case, which is showing that if $A \to B$ is a ring map, $A \to A'$ is a faithfully flat extension and $A' \to B \otimes_A A'$ is flat, then $A \to B$ is flat. Indeed, if $0 \to M \to N$ is exact of A-modules, then by flatness of $B \otimes_A A'$ over A

$$0 \to M \otimes_A B \otimes_A A \to M \otimes_A B \otimes_A A'$$

is exact, so by faithful flatness $0 \to M \otimes_A B \to N \otimes_A B$ is exact. Note that now we have shown quasicompact, flat, and surjective are all fpqc-local, so being fpqc is fpqc-local.

Example A.1.14. A morphism being locally of finite type/presentation is fpqc-local on the target. Once again, these two properties are local on both source and target and stable under base change, so we may reduce to the affine case. Suppose $A \to A'$ is faithfully flat, and B is an A-algebra.

1. Assume $A' \to B \otimes_A A'$ is of finite type, and choose generators over A'; let $\{b_i\}$ be the finite set of elements of B appearing in summations defining the generators, and let $B_0 \coloneqq A[b_i : i] \subset B$. Then B_0 is a finite type A-algebra, and the map $B_0 \otimes_A A' \to B \otimes_A A'$ is surjective. It is also injective since A' is flat, so $(B/B_0) \otimes_A A' = 0$ by flatness. Thus $B/B_0 = 0$.
2. Now assume $A' \to B \otimes_A A'$ is of finite presentation. By (1), $A \to B$ is of finite type so choose an exact sequence

   $$0 \to I \to P \coloneqq A[x_0, \ldots, x_n] \to B \to 0$$

   over A, and tensor up to A'. By flatness of A' and the assumption, $I \otimes_A A'$ is a finitely generated ideal of $P' = P \otimes_A A'$. Since $P \to P'$ is faithfully flat, and $I \otimes_P P' = I \otimes_A A'$, this implies that I is finite over P.

The last assertion of the previous example was simply that finite generation of modules descends under faithfully flat extension. Indeed, if $A \to A'$ is faithfully flat, and M is an A-module for which $A' \otimes_A M$ is finite over A', then M was finite

over A. This is because we can take the submodule of M generated by elements appearing in sums generating $A' \otimes_A M$ over A', so that we obtain a map $A^n \to M$ whose base change to A' is surjective. Therefore $A^n \to M$ is surjective. The same proof shows that if $f\colon X \to Y$ is faithfully flat and $\mathcal{F}$ is a quasicoherent sheaf on Y, then we can check finite type (or presentation) of $\mathcal{F}$ by checking that $f^*\mathcal{F}$ is finite type/presentation.

Our next result is an important topological one, which is used in various proofs of fpqc-descent for properties of morphisms. Recall that a morphism $f\colon X \to Y$ of schemes is *submersive* if the underlying map $|f|\colon |X| \to |Y|$ is a topological quotient map.

Remark A.1.15. We'll use the following elementary property: if $f\colon X \to Y$ is a morphism of affine schemes and $f(X)$ is stable under specialization, then $f(X)$ is closed. Indeed, this corresponds to $\phi\colon A \to B$; let $\mathfrak{p} \in \overline{f(X)} \subset \operatorname{Spec} A$. For $a \notin \mathfrak{p}$, $\mathfrak{p} \in D(a)$ so $D(a) \cap f(X) \neq \varnothing$, meaning $B_a \neq 0$. This means that $B_{\mathfrak{p}} \neq 0$, so choose a prime $\mathfrak{q}$ of $B_{\mathfrak{p}}$, then $\mathfrak{q} \cap A$ is prime of A contained in $\mathfrak{p}$. Moreover it also equals $(\mathfrak{q} \cap B) \cap A$, hence lies in the image of f. Since $f(X)$ is stable under specialization, $\mathfrak{p} \in f(X)$.

Lemma A.1.16. *An fpqc morphism of schemes is (universally) submersive.*

Proof. Let $f\colon X \to Y$ be fpqc. Then f is surjective, so assume that $Z \subset |Y|$ is a subset such that $f^{-1}(Z) \subset X$ is closed. Checking that Z is closed may be done locally on the target, so assume Y is affine. Then X is quasicompact, so let $X' = \bigsqcup_i U_i \to X$ be a faithfully flat map from an affine scheme. The further preimage in X' of Z is still closed, so we reduce to the case where X and Y are affine.

Let $\phi\colon A \to B$ be faithfully flat, and assume $Z \subset \operatorname{Spec} A$ such that $f^{-1}(Z) = V(I)$ is closed in $\operatorname{Spec} B$, i.e. $Z = f(V(I))$ by surjectivity. Then Z is the image of $V(I) \to \operatorname{Spec} A$, so by the remark above we may prove it is stable under specialization. If $\mathfrak{p} \in Z$ and $\mathfrak{p} \subset \mathfrak{p}'$, first choose $\mathfrak{q}' \in \operatorname{Spec} B$ such that $f(\mathfrak{q}') = \mathfrak{p}'$ by surjectivity. Since flat maps satisfy going down, there exists $\mathfrak{q} \subset \mathfrak{q}'$ in B such that $f(\mathfrak{q}) = \mathfrak{p}$. Thus $\mathfrak{q} \in f^{-1}(Z) = V(I)$, so $\mathfrak{q}' \in V(I)$, so $\mathfrak{p}' \in f(V(I)) = Z$. This shows that Z is stable under specialization. □

We remark that in the case that fpqc is actually fppf, the above property is obvious because f is an open surjection. We now prove some basic topological consequences. Observe that if $\mathcal{P}$ descends under fpqc base change, it follows that 'universally $\mathcal{P}$' is fpqc-local. Indeed, it is stable under base change, and if W is a Y-scheme, we can base change the whole cartesian square to obtain:

$$\begin{array}{ccccc} X' \times_Y W & \longrightarrow & Y' \times_Y W & \longrightarrow & Y' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X \times_Y W & \longrightarrow & W & \longrightarrow & Y \end{array}$$

Thus $Y' \times_Y W \to W$ is fpqc, and moreover if $X' \to Y'$ is universally $\mathcal{P}$, then

$$X' \times_Y W \to X' \times_{Y'} (Y' \times_Y W) \to Y' \times_Y W$$

is universally $\mathcal{P}$, in particular $\mathcal{P}$. Thus $X \times_Y W \to W$ is $\mathcal{P}$ by the assumption. Here are some applications:

Lemma A.1.17. *Being $\mathcal{P}$ descends under fpqc base change for:*

$$\mathcal{P} \in \{\textit{open map}, \textit{closed map}, \textit{injective on sets}\}.$$

Therefore universally open, universally closed, and radicial are all properties which are fpqc-local on the target.

Proof. Consider a cartesian square with g fpqc:

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose $\overline{f}$ is open/closed, and let $T \subset X$ be an open/closed subset. Since g is submersive, it suffices to show that $g^{-1}(f(T))$ is open/closed, and we claim that this equals $\overline{f}(\overline{g}^{-1}(T))$ which will complete the proof. In one direction, by surjectivity of $\overline{g}$ we have

$$f(T) = f\overline{g}\overline{g}^{-1}T = g\overline{f}(\overline{g}^{-1}T) \implies g^{-1}f(T) \supset \overline{f}\overline{g}^{-1}T.$$

On the other hand, for $t \in T$, and $y' \in Y'$ such that $g(y') = f(t)$, we can always choose $x' \in X'$ mapping to t and y'. So $\overline{f}(x') = y \in \overline{f}\overline{g}^{-1}T$.

It remains to show that injectivity descends. Indeed this descends through any surjective base change; if $x_1, x_2 \in X$ and $f(x_1) = f(x_2) = y$, we can choose y' lying above y and then points x_1', x_2' lying above (x_1, y') and (x_2, y'). Then these map to the same point under $\overline{f}$, hence are equal, so it follows that $x_1 = x_2$. □

Example A.1.18. It follows that being separated is fpqc-local on the target. Indeed, since being an immersion is stable under base change and an immersion which is a closed map is a closed immersion, it follows that separatedness is equivalent to the diagonal being universally closed. Thus this follows from the lemma and Example A.1.10. As a consequence, being proper is also fpqc-local on the target, because we checked this for quasicompact, locally of finite type, separated, and universally closed. By contrast, being projective is not fpqc-local since it isn't even Zariski-local.

Example A.1.19. We give two proofs that being unramified is fpqc-local on the target. First, locally of finite type is fpqc-local, and by the argument above the diagonal being an open immersion is equivalent to the diagonal being radicial. Thus unramified is fpqc-local. Second proof: if we take the base change square above with g fpqc, then $g^*\Omega_{X/Y} = \Omega_{X'/Y'}$, so if this is zero then $\Omega_{X/Y} = 0$ because g is faithfully flat. Either way, this also implies being étale is fpqc-local because étale = flat + finite presentation + unramified.

A.1.2. Faithfully flat descent. Here is the rough idea of this section: we consider presheaves $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ where $\mathcal{C}$ is an appropriate category of schemes equipped with an fpqc topology, and we wish to see whether or not F is an fpqc sheaf. This is true in the literal sense for only one example, which is the functor

$$\mathrm{Hom}_{\mathbf{Sch}_S}(-, Y)\colon \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$$

for a Y-scheme S. This example will prove that big topologies like the étale topology, fppf topology, and fpqc topology are all subcanonical, which is of fundamental importance for the theory of algebraic spaces. In the other examples, $\mathcal{D}$ will be some category of categories, and F a pseudofunctor. Since we have avoided the language of ∞-categories thus far, we will make similar ad-hoc constructions as in the open cover descent of the previous section.

We begin with QCoh-descent. Let $f\colon X \to Y$ be a morphism of schemes. Write π_1, π_2 for the two projections $X \times_Y X \to X$, and write

$$p_{ij}\colon X \times_Y X \times_Y X \to X \times_Y X \qquad p_i\colon X \times_Y X \times_Y X \to X$$

for the corresponding projections as well. We define the *category of descent data associated to* f, written

$$\mathrm{Desc}(f) = \mathrm{Desc}(X/Y) = \mathrm{Desc}(f, \mathrm{QCoh})$$

as follows:

(a) Objects are pairs $(\mathcal{F}, \theta)$ where $\mathcal{F} \in \mathrm{QCoh}(X)$ and θ is an isomorphism

$$\theta\colon \pi_1^*\mathcal{F} \xrightarrow{\sim} \pi_2^*\mathcal{F} \qquad \text{in } \mathrm{QCoh}(X \times_Y X)$$

satisfying the cocycle condition

$$(p_{23}^*\theta) \circ (p_{12}^*\theta) = p_{13}^*\theta$$

in $\mathrm{QCoh}(X \times_Y X \times_Y X)$. Observe that $p_{ij}^*\mathcal{F}\colon p_i^*\mathcal{F} \xrightarrow{\sim} p_j^*\mathcal{F}$ (for $i < j$), so this makes sense.

(b) Morphisms $(\mathcal{F}, \theta) \to (\mathcal{F}', \theta')$ are maps $\phi\colon \mathcal{F} \to \mathcal{F}'$ in $\mathrm{QCoh}(X)$ making the diagram

$$\begin{array}{ccc} \pi_1^*\mathcal{F} & \xrightarrow{\theta} & \pi_2^*\mathcal{F} \\ {\scriptstyle \pi_1^*\phi}\downarrow & & \downarrow{\scriptstyle \pi_2^*\phi} \\ \pi_1^*\mathcal{F}' & \xrightarrow{\theta'} & \pi_2^*\mathcal{F}' \end{array}$$

commute in $\mathrm{QCoh}(X \times_Y X)$. Composition is given by composition of the underlying maps and is therefore clearly well-defined and associative.

In this scenario, there is a canonical map

$$f^*\colon \mathrm{QCoh}(Y) \to \mathrm{Desc}(X/Y)$$

defined by sending $\mathcal{G}$ to $f^*\mathcal{G}$ equipped with the canonical isomorphism $\pi_1^*f^*\mathcal{G} \xrightarrow{\sim} \pi_2^*f^*\mathcal{G}$ induced by $f\pi_1 = f\pi_2$. The question we are trying to answer is whether this is an isomorphism. We will prove that this is the case when f is an fpqc morphism of schemes.

The construction of this category is contravariantly (pseudo-)functorial in the chosen morphism, and the canonical functor is natural with respect to this pseudofunctor. More precisely, whenever we are given a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

there is a canonical functor $(\overline{g}, g)^*\colon \mathrm{Desc}(X/Y) \to \mathrm{Desc}(X'/Y')$ making the diagram

$$\begin{array}{ccc} \mathrm{Desc}(X'/Y') & \xleftarrow{\overline{f}^*} & \mathrm{QCoh}(Y') \\ {\scriptstyle \overline{g}^*}\uparrow & & \uparrow{\scriptstyle g^*} \\ \mathrm{Desc}(X/Y) & \xleftarrow{f^*} & \mathrm{QCoh}(Y) \end{array}$$

commute up to natural isomorphism. This functor is defined as follows: Denote the data corresponding to π and p as $\overline{\pi}$ and $\overline{p}$ for $\overline{f}$. Commutativity induces maps

$$\overline{g}^{(r)}\colon X' \times_{Y'} \cdots \times_{Y'} X' \to X \times_Y \cdots \times_Y X$$

between the fiber products, and these maps intertwine the projections. Therefore, given a descent datum $(\mathcal{F}, \theta)$ in $\mathrm{Desc}(X/Y)$, we can consider the corresponding data

$$\overline{g}^*\mathcal{F} \qquad \overline{g}^{(2)*}\theta\colon \overline{g}^{(2)*}\pi_1^*\mathcal{F} = \overline{\pi}_1^*\overline{g}^*\mathcal{F} \to \overline{g}^{(2)*}\pi_2^*\mathcal{F} = \overline{\pi}_2^*\overline{g}^*\mathcal{F}$$

in $\mathrm{Desc}(X'/Y')$. The cocycle condition still holds, because applying $\overline{g}^{(3)*}$ to the equation $p_{23}^*\theta \circ p_{12}^*\theta = p_{13}^*\theta$ gives the cocycle condition for $\overline{g}^*\mathcal{F}$, since

$$\overline{g}^{(3)*}p_{ij}^* = \overline{p}_{ij}^*\overline{g}^{(2)*}.$$

Morphisms behave properly because we can apply $\overline{g}^{(2)}$ to the defining commutative square, and the same intertwining relation provides a morphism in the target category. It is then obvious that the diagram above commutes.

Remark A.1.20. In the definition, there is not much generality lost by considering one vs. many morphisms to a fixed scheme Y. Indeed, if we start with a collection $f_\alpha\colon X_\alpha \to Y$ of morphisms, then consider the corresponding descent category $\mathrm{Desc}(\coprod_\alpha X_\alpha/Y)$ for the induced map $f = (f_\alpha)$ from the coproduct. We have the distributivity relation

$$(\coprod_\alpha X_\alpha) \times_Y (\coprod_\alpha X_\alpha) = \coprod_{\alpha,\beta} X_\alpha \times_Y X_\beta.$$

Denote by $\pi_1^{\alpha\beta}\colon X_\alpha \times_Y X_\beta \to X_\alpha$ and $\pi_2^{\alpha\beta}\colon X_\alpha \times_Y X_\beta \to X_\beta$ the projections. Since $\mathrm{QCoh}(\coprod_\alpha X_\alpha) = \prod_\alpha \mathrm{QCoh}(X_\alpha)$, a descent datum can be viewed as a collection of quasicoherent sheaves $\{\mathcal{F}_\alpha\}$ on X_α, equipped with a collection of isomorphisms

$$\theta_{\alpha\beta}\colon \pi_1^{\alpha\beta *}\mathcal{F}_\alpha \xrightarrow{\sim} \pi_2^{\alpha\beta *}\mathcal{F}_\beta \qquad \text{in } \mathrm{QCoh}(X_\alpha \times_Y X_\beta)$$

satisfying the following cocycle condition:

$$\begin{array}{ccc} p^{\alpha\beta\gamma *}\mathcal{F} & \xrightarrow{p_{12}^{\alpha\beta\gamma *}\theta_{\alpha\beta}} & p^{\alpha\beta\gamma *}\mathcal{F} \\ & \searrow^{p_{13}^{\alpha\beta\gamma *}\theta_{\alpha\gamma}} & \downarrow{\scriptstyle p_{23}^{\alpha\beta\gamma *}\theta_{\beta\gamma}} \\ & & p^{\alpha\beta\gamma *}\mathcal{F} \end{array}$$

In light of the notation, and given the equivalence of the perspectives, it is clear why we stick to only one morphism.

Example A.1.21. Using the above remark, we can rephrase the QCoh-descent result for schemes from the previous section. Namely, let $\{U_\alpha\}$ be an open cover of a scheme X. Then $U_\alpha \times_X U_\beta = U_\alpha \cap U_\beta$, and the projections from the fiber products become inclusions, so pulling back along them is just restriction. Therefore the remark above shows that $\mathrm{Desc}(\coprod_\alpha U_\alpha/X)$ is canonically identified with $\mathrm{QCoh}(\{U_\alpha\})$. In particular, it follows from Cor. A.1.6 that the natural map

$$\mathrm{QCoh}(X) \to \mathrm{Desc}(\coprod_\alpha U_\alpha/X)$$

is an equivalence of categories. Note that nothing we have said in this or the previous remark relies on the cover being finite, however $\coprod_\alpha U_\alpha$ will usually not be affine even when the U_α are affine if the cover is genuinely infinite.

Our major goal of this section is to prove fpqc descent for quasicoherent sheaves. The main step in proving this is the affine case, so let us discuss this separately. Suppose $\phi\colon A \to B$ is a ring homomorphism; in this case we write $\mathrm{Desc}(\phi)$ for the category defined above, but we can be slightly more explicit about the descent data.

Namely, it consists of a B-module M and an isomorphism

$$\theta\colon M \otimes_A B \xrightarrow{\sim} B \otimes_A M$$

of $B \otimes_A B$-modules making the diagram of $B \otimes_A B \otimes_A B$-modules

$$\begin{array}{ccc} M \otimes_A B \otimes_A B & \xrightarrow{\theta \otimes B} & B \otimes_A M \otimes_A B \\ & \searrow^{\theta_{13}} & \downarrow{\scriptstyle B \otimes \theta} \\ & & B \otimes_A B \otimes_A M \end{array}$$

commute (where θ_{13} is the obvious insertion of 1 in the center). The canonical functor is now given by

$$\phi^*\colon \mathrm{Mod}(A) \to \mathrm{Desc}(\phi) \qquad N \mapsto B \otimes_A N$$

with the descent datum

$$B \otimes_A N \otimes_A B \xrightarrow{\sim} B \otimes_A B \otimes_A N \qquad b_0 \otimes x \otimes b_1 \mapsto b_0 \otimes b_1 \otimes x.$$

Of course this is indeed $B \otimes_A B$-linear as $N \otimes_A B = B \otimes_A N$ as B-modules.

Definition A.1.22. Let ϕ be a ring homomorphism. ϕ is a *descent morphism* if the functor

$$\phi^*\colon \mathrm{Mod}(A) \to \mathrm{Desc}(\phi)$$

is fully faithful, and an *effective descent morphism* if it is an equivalence.

Now suppose (M,θ) is an effective descent datum for ϕ. Motivated by the descent results of the previous section, we form the equalizer between the canonical map and the descent cocycle. So, define an A-module $\phi_!(M,\theta)$ by the equalizer

$$\phi_!(M,\theta) \to M \underset{c}{\overset{\rho}{\rightrightarrows}} B \otimes_A M,$$

where $\rho(m) = \theta(m \otimes 1)$ and $c(m) = 1 \otimes m$. Both of these maps are (split) monomorphisms over A due to the multiplication map $B \otimes_A M \to M$ sending $b \otimes m \mapsto bm$ (and twisting by θ^{-1}). By the universal property and extension-restriction adjunction, for any A-module N we have identifications

$$\begin{aligned}\mathrm{Hom}_A(N, \phi_!(M,\theta)) &= \{\alpha\colon N \to M : \theta(\alpha(n)\otimes 1) = 1 \otimes \alpha(n)\}_A \\ &= \{\beta\colon N \otimes_A B \to M : \theta(\beta(1\otimes n)\otimes 1) = 1 \otimes \beta(1\otimes n)\}_B.\end{aligned}$$

Using the $B\otimes_A B$-structure on $B\otimes_A M$, we can multiply both sides of the condition with $b_0 \otimes b_1$ and it becomes equivalent to

$$(*) \qquad \theta(\beta(b_0\otimes n)\otimes b_1) = b_0 \otimes \beta(b_1 \otimes n)$$

for all $b_0, b_1 \in B$. On the other hand, viewing $N \otimes_A B = \phi^*(N)$ as an object of the descent category, observe that a morphism $\phi^*(N) \to (M,\theta)$ is precisely a map $\beta\colon N \otimes_A B \to M$ making the diagram

$$\begin{array}{ccc} B \otimes_A N \otimes_A B & \xrightarrow{\cong} & B \otimes_A B \otimes_A N \\ {\scriptstyle \beta\otimes B}\downarrow & & \downarrow{\scriptstyle B\otimes\beta} \\ M \otimes_A B & \xrightarrow{\theta} & B \otimes_A M \end{array}$$

commute. This commutativity expresses the same condition $(*)$, so we conclude that

$$\mathrm{Hom}_A(N, \phi_!(M,\theta)) \cong \mathrm{Hom}_{\mathrm{Desc}(\phi)}(\phi^*(N), (M,\theta)),$$

i.e. $\phi_!$ extends to a functor $\phi_!\colon \mathrm{Desc}(\phi) \to \mathrm{Mod}(A)$ and we have an adjunction $\phi^* \dashv \phi_!$ where the counit $B \otimes_A \phi_!(M,\theta) \to M$ is induced by the equalizer map. Importantly, the adjunction gives specific maps for which checking full faithfulness/equivalence is equivalent to checking isomorphisms.

Lemma A.1.23. *Keeping the notation above, if ϕ is flat, then the counit*

$$\phi^*\phi_! \to \mathrm{Id}_{\mathrm{Desc}(\phi)}$$

is an isomorphism. In particular, in this case $\phi_!$ is fully faithful and ϕ^* is essentially surjective.

PROOF. Since B is flat over A, we may tensor the defining equalizer

$$B \otimes_A \phi_!(M,\theta) \to B \otimes_A M \underset{B\otimes c}{\overset{B\otimes\rho}{\rightrightarrows}} B \otimes_A B \otimes_A M$$

and it remains an equalizer. Consider the injection $\rho\colon M \to B \otimes_A M$; we claim its image coincides with the equalizer E of the maps above. Indeed, if $\rho(m) = \sum_i b_i \otimes m_i$, then the cocycle condition says that $\sum_i b_i \otimes 1 \otimes m_i$ equals the composite

$$m \mapsto \sum_i b_i \otimes m_i \mapsto \sum_i b_i \otimes \theta(m_i \otimes 1) = \sum_i b_i \otimes \rho(m_i).$$

These two expressions are the image of $\rho(m)$ under $B\otimes\rho$ and $B\otimes c$, thus $\rho(M) \subset E$.

On the other hand, if we have an arbitrary element $x = \sum_i b_i \otimes m_i \in E$, then by definition this means $\sum_i b_i \otimes 1 \otimes m_i = \sum_i b_i \otimes \rho(m_i)$. Now apply the multiplication map $B \otimes_A B \otimes_A M \to B \otimes_A M$ to both sides (i.e. $b_0 \otimes b_1 \otimes m \mapsto b_0 b_1 \otimes m$). The LHS becomes $\sum_i b_i \otimes m_i = x$, and the RHS becomes $\sum_i b_i \rho(m_i)$ via the B-module action on the left component of $B \otimes_A M$. Then

$$x = \sum_i b_i \rho(m_i) = \sum_i (b_i \otimes 1) \cdot \theta(m_i \otimes 1) = \rho(\sum_i b_i m_i \otimes 1) = \rho(\mu(x)),$$

where $\mu\colon B \otimes_A M \to M$ is multiplication. This shows that ρ induces an isomorphism $M \xrightarrow{\sim} E \subset B \otimes_A M$ and the inverse is μ.

To conclude, observe that the diagram on the left of

$$\begin{array}{ccc} \phi_!(M,\theta) & \longrightarrow & M \\ & \searrow & \| \\ & & M \end{array} \qquad\qquad \begin{array}{ccc} B \otimes_A \phi_!(M,\theta) & \longrightarrow & B \otimes_A M \\ & \overset{\varepsilon}{\searrow} & \downarrow{\scriptstyle \mu} \\ & & M \end{array}$$

commutes tautologically, so by adjunction the one on the right involving μ does. If we restrict $B \otimes_A M$ to E, then μ becomes an isomorphism and the upper map becomes an isomorphism, so ε is an isomorphism. □

Now let's analyze the unit map of the adjunction $\phi^* \dashv \phi_!$. For an A-module N, this is given by the canonical map to the equalizer of $B \otimes_A N \rightrightarrows B \otimes_A B \otimes_A N$, where now the two maps are simply $b\otimes n \mapsto 1\otimes b\otimes n$ and $b\otimes 1\otimes n$. Alternatively we can phrase this by saying that the counit at N is an isomorphism iff the sequence

$$(**) \qquad 0 \to N \to B \otimes_A N \xrightarrow{d} B \otimes_A B \otimes_A N$$

is exact, where $d(b \otimes n) = 1 \otimes b \otimes n - b \otimes 1 \otimes n$. However, since the first map is $n \mapsto 1 \otimes n$, the sequence is a complex.

Remark A.1.24. Note that from the above exact sequence, we observe that if the unit is an isomorphism (i.e. ϕ^* is fully faithful = ϕ is a descent morphism), then ϕ has the property that $\phi \otimes_A N$ is injective for all N. We have earlier called this property $A \to B$ being a pure extension (or a universally injective map of A-modules). A stronger result than the one we will prove here is that in fact a morphism is of effective descent iff it is universally injective. So in particular there are no descent morphisms which aren't effective descent.

Proposition A.1.25. *Let $\phi\colon A \to B$ be a faithfully flat ring homomorphism. Then the adjunction*

$$\phi^*\colon \operatorname{Desc}(\phi) \rightleftarrows \operatorname{Mod}(A) : \phi_!$$

forms an equivalence of categories, so ϕ is an effective descent morphism.

Proof. Let N be an A-module. We may check that $(**)$ is exact after applying $B \otimes_A (-)$ producing the complex of B-modules

$$0 \to B \otimes_A N \xrightarrow{u} B \otimes_A B \otimes_A N \xrightarrow{B\otimes d} B \otimes_A B \otimes_A B \otimes_A N.$$

The first map, $u\colon b \otimes n \mapsto 1 \otimes b \otimes n$, is split by

$$s\colon B \otimes_A B \otimes_A N \to B \otimes_A N \qquad b_0 \otimes b_1 \otimes n \mapsto b_0 b_1 \otimes n,$$

and hence is injective. Next we define a map

$$r\colon B \otimes_A B \otimes_A B \otimes_A N \to B \otimes_A B \otimes_A N$$

by $r(b_0 \otimes b_1 \otimes b_2 \otimes n) = b_0 b_2 \otimes b_1 \otimes n$, which is well-defined at least as a map of A-modules because the expression is A-multilinear. Now

$$\begin{aligned} r(B \otimes d(b_0 \otimes b_1 \otimes n)) &= r\big(b_0 \otimes 1 \otimes b_1 \otimes n - b_0 \otimes b_1 \otimes 1 \otimes n\big) \\ &= b_0 b_1 \otimes 1 \otimes n - b_0 \otimes b_1 \otimes n, \end{aligned}$$

so $r \circ (B \otimes d) = u \circ s - \mathrm{Id}$ by the definitions. Therefore, if $(B \otimes d)x = 0$, then $u(s(x)) = x$, i.e. $x \in \operatorname{im} u$. This shows that the unit is an isomorphism, and we already finished the rest of the proof in the previous lemma. □

Now we move on to proving fpqc-descent for quasicoherent sheaves. We continue using the notation from earlier: $f\colon X \to Y$ is an fpqc morphism, π_i are the projections from $X \times_Y X$ and p_i are the projections from $X \times_Y X \times_Y X$.

Theorem A.1.26. *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Then the natural functor*

$$f^*\colon \operatorname{QCoh}(Y) \to \operatorname{Desc}(X/Y)$$

is an equivalence of categories.

Proof. First assume that the base Y is affine. Then X is quasicompact, so we may choose a finite affine cover $\{U_i\}$, and consider the composite

$$U = \coprod_i U_i \xrightarrow{q} X \xrightarrow{f} Y,$$

which is a faithfully flat map of affine schemes. We can package the maps in the somewhat trivial commutative diagram

$$\begin{array}{ccccc} Y & = & Y & \longleftarrow & X \\ \uparrow & & \uparrow & & \uparrow \\ X & \longleftarrow & U & = & U \end{array}$$

which we have proved induces a commutative diagram

$$\begin{array}{ccccc} \mathrm{QCoh}(Y) & = & \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ {\scriptstyle ?}\downarrow & & \cong\downarrow & & \downarrow\cong \\ \mathrm{Desc}(X/Y) & \xrightarrow{q^*} & \mathrm{Desc}(U/Y) & \xrightarrow{(1,f)^*} & \mathrm{Desc}(U/X) \end{array}$$

of categories. The right-hand map is an equivalence by Example A.1.21 and the middle map is one by Prop. A.1.25. This diagram contains the major argument of the proof.

To prove that the left-hand map is an equivalence, it suffices to show that $\mathrm{Desc}(X/Y) \to \mathrm{Desc}(U/Y)$ is fully faithful. Faithfulness is clear: A morphism of descent data is determined by the underlying map of sheaves $\mathcal{F} \to \mathcal{F}'$, and this map is determined by its pullback to U, since further pullbacks provide $\mathcal{F}|_{U_i} \to \mathcal{F}'|_{U_i}$ for all i.

For fullness, let α be a morphism $q^*(\mathcal{F}, \theta) \to q^*(\mathcal{F}', \theta')$ in $\mathrm{Desc}(U/Y)$. We have to check that $q^*\mathcal{F} \to q^*\mathcal{F}'$ is induced by a morphism $\beta\colon \mathcal{F} \to \mathcal{F}'$, and that β commutes with θ and θ'. The former is clear because we can simply apply $(1, f)^*$ to pull α into $\mathrm{Desc}(U/X)$, and then it is induced by a unique map $\mathcal{F} \to \mathcal{F}'$ in $\mathrm{QCoh}(X)$. The second part follows right away because by assumption (and the fact that q intertwines the projections), the pullback of the commutativity constraint along $U \times_Y U \to X \times_Y X$ is commutative. But this map is faithfully flat, as the composition

$$U \times_Y U \to U \times_Y X \to X \times_Y X,$$

and hence pulling back along it is a faithful functor. This finishes the case that Y is affine.

Now we do the general case. Observe that whenever $V \subset Y$ is an open subset, we have a commutative diagram:

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \longrightarrow & \mathrm{QCoh}(V) \\ \downarrow & & \downarrow \\ \mathrm{Desc}(X/Y) & \longrightarrow & \mathrm{Desc}\left(f^{-1}(V)/V\right) \end{array}$$

When V is affine, the right-hand map is an equivalence. *Faithfulness:* If $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(Y)$ and $\alpha\colon \mathcal{F} \to \mathcal{G}$, its image in $\mathrm{Desc}(X/Y)$ determines $\alpha|_V\colon \mathcal{F}|_V \to \mathcal{G}|_V$ for each affine open $V \subset Y$ by commutativity of the diagram and the equivalence above. Thus it determines α. *Fullness:* Now suppose we're given a map $\alpha\colon (f^*\mathcal{F}, \mathrm{can}) \to (f^*\mathcal{G}, \mathrm{can})$ in $\mathrm{Desc}(X/Y)$. Once again, for each affine open $V \subset Y$ we can push this to a map

$$(f_V^*(\mathcal{F}|_V), \mathrm{can}) \to (f_V^*(\mathcal{G}|_V), \mathrm{can})$$

in $\mathrm{Desc}(f^{-1}(V)/V)$, which is the pullback of a unique map $\beta_V\colon \mathcal{F}|_V \to \mathcal{G}|_V$. Repeat this for all such V. Now if $V' \subset V$ is a smaller affine open, then $\beta_{V'} = \beta_V|_{V'}$. This follows from the faithfulness applied to V'. It follows that $\{\beta_V : V\}$ glues to a unique map $\beta\colon \mathcal{F} \to \mathcal{G}$, and another application of faithfulness gives $f^*\beta = \alpha$.

For essential surjectivity, we will apply Remark A.1.7. Let $(\mathcal{F}, \theta)$ be a descent datum; for each affine open $V \subset Y$, choose an isomorphism

$$\beta_V \colon (\mathcal{F}|_{f^{-1}(V)}, \theta|_{f^{-1}(V)}) \xrightarrow{\sim} (f_V^* \mathcal{G}_V, \mathrm{can})$$

in $\mathrm{Desc}(f^{-1}(V)/V)$ for $\mathcal{G}_V \in \mathrm{QCoh}(V)$. If $W \subset V$ is a smaller affine open, then the isomorphism

$$(f_W^* \mathcal{G}_W, \mathrm{can}) \xrightarrow{\beta_W^{-1}} (\mathcal{F}|_{f^{-1}(W)}, \theta|_{f^{-1}(W)}) \xrightarrow{\beta_V|_{f^{-1}(W)}} (f_V^* \mathcal{G}_V, \mathrm{can})$$

lifts to an isomorphism $\alpha_{WV} \colon \mathcal{G}_W \xrightarrow{\sim} \mathcal{G}_V|_W$ by full faithfulness. Note that (suppressing restriction notation) we have

$$f^*(\alpha_{VU} \circ \alpha_{WV}) = (\beta_U \beta_V^{-1})(\beta_V \beta_W^{-1}) = f^*(\alpha_{WU}),$$

and f^* is faithful since f is faithfully flat. Therefore $\{G_V, \alpha_{VU}\}$ form a cocycle in the category of Remark A.1.7, and hence glue to a sheaf $\mathcal{G}$ with isomorphisms $\mathcal{G}|_V \xrightarrow{\sim} \mathcal{G}_V$ commuting with the descent data. Thus these glue to a well-defined map $(f^*\mathcal{G}, \mathrm{can}) \to (\mathcal{F}, \theta)$ which fits into a commutative diagram with the β_V's over all V. We conclude that this map is an isomorphism in $\mathrm{Desc}(X/Y)$. □

Our first corollary of this result is that schemes are sheaves in the fpqc topology. This is foundational for the theory of algebraic spaces and stacks.

Definition A.1.27. Let $\mathcal{C}$ be a category with pullbacks. A map $f \colon X \to Y$ is an *effective epimorphism* if the natural diagram

$$X \times_Y X \underset{\mathrm{pr}_1}{\overset{\mathrm{pr}_2}{\rightrightarrows}} X \xrightarrow{f} Y$$

is a coequalizer.

Example A.1.28. Let $\mathcal{C} = \mathbf{Set}$ and suppose $f \colon X \to Y$ is surjective. Then $X \times_Y X = \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$, and if $g \colon X \to Z$ is a map, it agrees on the projections iff $g(x_1) = g(x_2)$ whenever $f(x_1) = f(x_2)$. Thus a map sending $y \mapsto g(x)$ for any $x \in f^{-1}(x)$ is well-defined and unique, so f is an effective epimorphism. One similarly shows that in an abelian category, the epimorphisms and effective epimorphisms coincide. The condition says that the only obstruction to descending a map out of X to a map out of Y is the 'formal one', that the map must agree on the fibers of f.

Example A.1.29. Let X be a scheme and $\{U_\alpha\}$ be a (possibly infinite) open cover of X. Then $\coprod_\alpha U_\alpha \to X$ is an effective epimorphism. Indeed, the standard diagram

$$\coprod_{\alpha,\beta} U_\alpha \cap U_\beta \rightrightarrows \coprod_\alpha U_\alpha \longrightarrow X$$

is a coequalizer. This provides the intuition that effective descent morphisms may be effective epimorphisms.

We also make the corresponding definition for effective monomorphism. In general, we expect descent for algebraic data to be like a sheaf condition, which should be an effective monomorphism. On the other hand, descent for full spaces involves gluing the spaces, which we expect to be an effective epimorphism.

Remark A.1.30. In general, whenever we are considering a descent functor $\mathcal{C}(T) \to \mathrm{Desc}(T, \mathcal{C})$, where $\mathcal{C}$ is some kind of fibered category (interpreted loosely, because of our ad hoc definition of descent data), full faithfulness usually corresponds to the existence of effective mono/epimorphisms, or something akin to it. For example, let $f\colon X \to Y$ be a map and consider the functor $\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}}(X/Y)$. For $\mathcal{F}, \mathcal{G}$ on Y, full faithfulness is the statement that for each map $f^*\mathcal{F} \to f^*\mathcal{G}$ in $\mathrm{QCoh}(X)$ whose two pullbacks to $\mathrm{QCoh}(X \times_Y X)$ agree, there is a unique map $\mathcal{F} \to \mathcal{G}$ inducing $f^*\mathcal{F} \to f^*\mathcal{G}$. In other words,

$$\mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{QCoh}(X)}(f^*\mathcal{F}, f^*\mathcal{G}) \rightrightarrows \mathrm{Hom}_{\mathrm{QCoh}(X \times_Y X)}(p^*\mathcal{F}, p^*\mathcal{G})$$

is an equalizer, where $p\colon X \times_Y X \to Y$ is the unique map. But now we can rewrite this by adjunction, as saying that for all $\mathcal{F} \in \mathrm{QCoh}(Y)$, applying $\mathrm{Hom}_{\mathrm{QCoh}(Y)}(\mathcal{F}, -)$ to

$$(*) \qquad \mathcal{G} \to f_* f^* \mathcal{G} \rightrightarrows p_* p^* \mathcal{G}$$

produces an equalizer. Thus $(*)$ is an equalizer in $\mathrm{QCoh}(Y)$ for each $\mathcal{G} \in \mathrm{QCoh}(Y)$. This is similar to an effective monomorphism, but not exactly. In fact, what is really happening is that descent is the statement that $f^*\colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is an effective monomorphism in a stacky sense. Indeed,

$$\mathrm{QCoh}(Y) \to \mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(X \times_Y X)$$

almost exhibits f^* as an effective monomorphism in the sense of 2-categories. But, we know that for descent of objects we have to check the cocycle condition, which takes place in $\mathrm{QCoh}(X \times_Y X \times_Y X)$.

Theorem A.1.31. *Let $f\colon X \to Y$ be an fpqc morphism of S-schemes. Then f is a (universal) effective epimorphism in $\mathbf{Sch}_S$. In particular, the diagram*

$$\mathrm{Hom}_S(Y, Z) \longrightarrow \mathrm{Hom}_S(X, Z) \rightrightarrows \mathrm{Hom}_S(X \times_Y X, Z)$$

is an equalizer of sets for all S-schemes Z.

Proof. It suffices to treat the case that $S = \operatorname{Spec} \mathbb{Z}$. Indeed, assuming this case, an S-morphism $g\colon X \to Z$ coequalizing the maps already induces a unique morphism $\overline{g}\colon Y \to Z$, and we just have to check it's an S-morphism. But the absolute result implies that f is an epimorphism, and the two composites $X \to Y \to Z \to S$ and $X \to Y \to S$ agree by assumption, so the two maps $Y \to S$ agree.

Now we prove the main result. First assume $Z = \operatorname{Spec} R$ is affine. By adjunction, we have to show

$$\mathrm{Hom}(R, \Gamma(Y, \mathcal{O}_Y)) \to \mathrm{Hom}(R, \Gamma(X, \mathcal{O}_X)) \rightrightarrows \mathrm{Hom}(R, \Gamma(X \times_Y X, \mathcal{O}_{X \times_Y X}))$$

is an equalizer for all rings R, which will follow as long as

$$\Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X) \rightrightarrows \Gamma(X \times_Y X, \mathcal{O}_{X \times_Y X})$$

is an equalizer of rings. This is clear e.g. from the equalizer of Hom-sets in Remark A.1.30 by taking $\mathcal{F} = \mathcal{G} = \mathcal{O}_Y$.

Now consider the general case. Let $g\colon X \to Z$ be a map such that $g\pi_1 = g\pi_2$. For each $y \in Y$, choose $x \in X$ such that $f(x) = y$ and define $\widetilde{g}\colon |Y| \to |Z|$ by $\widetilde{g}(y) = g(x)$. This is independent of the choice of x, because if x, x' lie over y, then there exists $w \in X \times_Y X$ lying over (x, x'), so $g\pi_1(w) = g\pi_2(w)$ means that $g(x) = g(x')$. Thus $\widetilde{g}f = g$ as maps. This choice of $\widetilde{g}$ is already unique because $|f|$ was surjective.

Moreover, if $U \subset Z$ is open, then

$$g^{-1}(U) = f^{-1}\widetilde{g}^{-1}(U)$$

is open, so $\widetilde{g}^{-1}(U)$ is open since f is submersive (Lemma A.1.16). Thus $\widetilde{g}$ is continuous. It is already clear at this stage that $\widetilde{g}$ will be a unique morphism; indeed choose an affine open $V \subset Z$, then $f^{-1}\widetilde{g}^{-1}(V) \to \widetilde{g}^{-1}(V)$ is fpqc and the map $\widetilde{g}^{-1}(V) \to V$ coequalizes the two projection maps, so $\widetilde{g}^{-1}(V) \to V$ is uniquely determined by the case of affine target. Thus $\widetilde{g}$ is unique if it exists.

On the other hand, this same observation provides a unique morphism $h_V\colon \widetilde{g}^{-1}(V) \to V$ which on underlying spaces is a restriction of $\widetilde{g}$. It also follows from uniqueness that if $V' \subset V$ is a smaller affine open, then $h_{V'}\colon \widetilde{g}^{-1}(V') \to V'$ agrees with the restriction of h_V. Thus for V and W any two affine opens, h_V and h_W agree on $\widetilde{g}^{-1}(V \cap W)$ because the intersection is covered by smaller affine opens.

We conclude by Cor. A.1.6 that there is a unique morphism $h\colon Y \to Z$ such that $|h| = \widetilde{g}$ whose restrictions agree with $\widetilde{g}^{-1}(V) \to V$. We already know $hf = g$ on the underlying spaces, and equality as morphisms may be checked on the affine open cover of Z. This completes the proof. □

In the spirit of the previous result, we now change our fibered category of interest from QCoh to **Sch** or some full subcategory. The goal is to answer the following question: If $X \to Y$ is a map, this induces a base change map

$$\mathbf{Sch}_Y \to \mathbf{Sch}_X \qquad Y' \mapsto Y' \times_Y X,$$

and thus there is an associated category of descent data $\mathrm{Desc}_{\mathbf{Sch}}(X/Y)$ and a map $\mathbf{Sch}_Y \to \mathrm{Desc}_{\mathbf{Sch}}(X/Y)$ where a Y-scheme is sent to its pullback. The question is when this is fully faithful and/or an equivalence.

Here is the formal definition. Let $\mathcal{C} \subset \mathbf{Sch}$ be a full *fibered* subcategory of schemes, by which we mean a full replete subcategory $\mathcal{C}_Y \subset \mathbf{Sch}_Y$ for each Y such that if $Y' \to Y \in \mathcal{C}_Y$ and $X \to Y$ is a map, then $Y' \times_Y X \to X \in \mathcal{C}_X$. I.e. it is the same data as a class of morphisms stable under base change. For a map $f\colon X \to Y$, we define the category $\mathrm{Desc}_{\mathcal{C}}(X/Y)$ as follows:

(a) Objects are pairs (X', η) consisting of an $X' \in \mathcal{C}_X$ and an isomorphism

$$\eta\colon X' \times_{\pi_1} X^{(2)} \xrightarrow{\sim} X' \times_{\pi_2} X^{(2)}$$

of $X^{(2)} = X \times_Y X$-schemes, satisfying the cocycle condition

$$(\eta \times_{p_{23}} X^{(3)}) \circ (\eta \times_{p_{12}} X^{(3)}) = \eta \times_{p_{13}} X^{(3)}$$

over $X^{(3)} = X \times_Y X \times_Y X$.

(b) Morphisms $(X_1', \eta_1) \to (X_2', \eta_2)$ are maps $h\colon X_1' \to X_2'$ in $\mathcal{C}_X$ making the diagram

$$\begin{array}{ccc} X_1' \times_{\pi_1} X^{(2)} & \xrightarrow{h\times_{\pi_1} 1} & X_2' \times_{\pi_1} X^{(2)} \\ {\scriptstyle\eta_1}\downarrow & & \downarrow{\scriptstyle\eta_2} \\ X_1' \times_{\pi_2} X^{(2)} & \xrightarrow[h\times_{\pi_2} 1]{} & X_2' \times_{\pi_2} X^{(2)} \end{array}$$

commute in $\mathcal{C}_{X^{(2)}}$.

Note that if $\mathcal{C} \subset \mathcal{C}'$, then we also have $\mathrm{Desc}_{\mathcal{C}}(X/Y) \subset \mathrm{Desc}_{\mathcal{C}'}(X/Y)$. As usual, we have the natural map

$$\mathcal{C}_Y \to \mathrm{Desc}_{\mathcal{C}}(X/Y)$$

defined by sending $Y' \mapsto Y' \times_Y X$ equipped with the canonical descent data $\mathrm{Id}\colon Y' \times_Y X^{(2)} = Y' \times_Y X^{(2)}$

Lemma A.1.32. *Let $f\colon X \to Y$ be a morphism of schemes. Then the natural functor*

$$\mathbf{Sch}_Y \to \mathrm{Desc}_{\mathbf{Sch}}(X/Y)$$

is fully faithful if and only if f is a universal effective epimorphism.

Proof. Let Y_1, Y_2 be schemes over Y. A map between the corresponding canonical data in $\mathrm{Desc}_{\mathbf{Sch}}(X/Y)$ is a map $Y_1 \times_Y X \to Y_2 \times_Y X$ over X such that both pullbacks to maps

$$Y_1 \times_Y X \times_Y X \to Y_2 \times_Y X \times_Y X$$

coincide. In other words, the map is fully faithful iff

$$Y(Y_1, Y_2) \to X(Y_1 \times_Y X, Y_2 \times_Y X) \rightrightarrows (X \times_Y X)(Y_1 \times_Y X \times_Y X, Y_2 \times_Y X \times_Y X)$$

is an equalizer for all Y-schemes Y_1 and Y_2 (where we have suppressed Hom to make the equation fit on the page). By adjunction, this is equivalent to

$$\mathrm{Hom}_Y(Y_1, Y_2) \to \mathrm{Hom}_Y(Y_1 \times_Y X, Y_2) \rightrightarrows \mathrm{Hom}_Y(Y_1 \times_Y X \times_Y X, Y_2)$$

being an equalizer, and hence

$$Y_1 \times_Y X \times_Y X \rightrightarrows Y_1 \times_Y X \to Y_1$$

being a coequalizer in $\mathbf{Sch}_Y$ for all Y_1. In other words, full faithfulness is equivalent to $Y_1 \times_Y X \to Y_1$ being an effective epimorphism in $\mathbf{Sch}_Y$ for all $Y_1 \in \mathbf{Sch}_Y$. We have seen that this is equivalent to being an effective epimorphism in $\mathbf{Sch}$ (see the beginning of the proof of Thm. A.1.31). This is what it means for $X \to Y$ to be a universal effective epimorphism. □

Remark A.1.33. Therefore, it follows that $\mathbf{Sch}_Y \to \mathrm{Desc}_{\mathbf{Sch}}(X/Y)$ is fully faithful if $X \to Y$ is fpqc. This is called 'fpqc descent for morphisms'. The question of essential surjectivity is much more subtle, and it is in fact not true that fpqc implies effective descent for schemes. In fact, this even fails for étale covers and projective morphisms. However, one case which works well is for affine morphisms. The intuition is that affine schemes over Y correspond to quasicoherent $\mathcal{O}_Y$-algebras, and we can descend the underlying quasicoherent sheaf, so all we

have to do is descend the algebra structure. We will show this now, and we will in fact extend this result to the quasi-affine case.

If X is a scheme, we write $\mathrm{CAlg}_{\mathrm{qc}}(X)$ for the category of quasicoherent $\mathcal{O}_X$-algebras on X. Recall that relative Spec gives the canonical correspondence

$$\mathrm{Hom}_X(W, \underline{\mathrm{Spec}}_X(\mathcal{A})) \cong \mathrm{Hom}_{\mathrm{CAlg}(X)}(\mathcal{A}, f_*\mathcal{O}_W)$$

for an X-scheme $g\colon W \to X$ and $\mathcal{A} \in \mathrm{CAlg}_{\mathrm{qc}}(X)$, although $f_*\mathcal{O}_W$ may not be quasicoherent. Finally, we know that this restricts to an equivalence of categories between $\mathrm{CAlg}_{\mathrm{qc}}(X)$ and affine schemes over X. The first step is to discuss descent for $\mathrm{CAlg}_{\mathrm{qc}}(X)$ over an arbitrary scheme.

As usual, if $f\colon X \to Y$ is a morphism, we make the corresponding definition $\mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}}(X/Y)$ for the category of descent data comparable to $\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)$, except that now morphisms should be algebra morphisms, and the quasicoherent sheaf on X should be an $\mathcal{O}_X$-algebra. We get the canonical map $\mathrm{CAlg}_{\mathrm{qc}}(Y) \to \mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}}(X/Y)$ as usual.

Proposition A.1.34. *Let $f\colon X \to Y$ be an fpqc morphism of schemes. The natural functor*

$$\mathrm{CAlg}_{\mathrm{qc}}(Y) \to \mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}}(X/Y)$$

is an equivalence of categories.

This has a relatively direct proof, but to emphasize the categorical perspective let us give an alternative proof, which is to show that for any morphism, the functor

$$\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}}(X/Y)$$

satisfies the following properties:

(a) $\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)$ is naturally a symmetric monoidal category, and the functor above is symmetric monoidal.
(b) The natural functor in the proposition is given by taking commutative unital algebra objects with respect to the monoidal functor above.

(b) will be clear once we show that the category of commutative algebra objects of $\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)$ are indeed $\mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}}(X/Y)$. Then the result will follow from fpqc descent for quasicoherent sheaves. First we prove (a); given a pair of descent data $(\mathcal{F}_i, \theta_i)$ for $i = 1, 2$, their tensor product is $\mathcal{F}_1 \otimes \mathcal{F}_2$ equipped with

$$\theta_1 \otimes \theta_2\colon \pi_1^*(\mathcal{F}_1 \otimes \mathcal{F}_2) \to \pi_2^*(\mathcal{F}_1 \otimes \mathcal{F}_2),$$

well-defined since π_i^* is monoidal. The cocycle condition follows from the fact that the p_{ij}^* are monoidal, and when tensoring morphisms the commutativity constraint follows from the fact that π_i^* are monoidal as well. The unit is $\mathcal{O}_X$, which is pulled back from Y and hence has canonical descent data. The associativity and unitality axioms follow from the same data on $\mathrm{QCoh}(X)$ and the fact that pullback is a monoidal functor.

Consider the canonical functors

$$\mathrm{QCoh}(Y) \to \mathrm{Desc}_{\mathrm{QCoh}}(X/Y) \to \mathrm{QCoh}(X).$$

It follows by construction that the second map is monoidal, and the first map is also because f^* is monoidal. Therefore we obtain induced maps

$$\mathrm{CAlg}_{\mathrm{qc}}(Y) \to \mathrm{CAlg}(\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)) \to \mathrm{CAlg}_{\mathrm{qc}}(Y)$$

on commutative algebra objects. By the right-hand functor, if $(\mathcal{F}, \theta)$ comes with a commutative algebra object structure, it follows that $\mathcal{F}$ is an object of $\mathrm{CAlg}_{\mathrm{qc}}(X)$. It is also clear that the map θ is a morphism in $\mathrm{CAlg}(\mathrm{QCoh}(X \times_Y X))$, hence is a morphism of quasicoherent $\mathcal{O}_X$-algebras. A morphism is in particular required to be an algebra homomorphism between quasicoherent algebras, and this is all the data required to be a morphism in CAlg. By construction, we have identified

$$\mathrm{CAlg}(\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)) = \mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}}(X/Y)$$

as we sought to do. This finishes the proof of Prop. A.1.34.

Remark A.1.35. Here is a second, more ∞-operadic way to see this. First, for any ∞-category $\mathcal{C}$, the functor $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ creates limits. In particular, write $\mathbf{Cat}_\infty$ for the ∞-category of ∞-categories, and

$$\mathrm{CMon}(\mathbf{Cat}_\infty) = \mathrm{SymMonCat}_\infty$$

for the ∞-category of symmetric monoidal categories. Thus the functor $\mathrm{SymMonCat}_\infty \to \mathbf{Cat}_\infty$ creates limits. In particular, the limit diagram

$$\mathrm{QCoh}(X) \rightrightarrows \mathrm{QCoh}(X \times_Y X) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathrm{QCoh}(X \times_Y X \times_Y X)$$

defining $\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)$ exists in $\mathrm{SymMonCat}_\infty$, and the natural map from $\mathrm{QCoh}(Y)$ exists in this category as well. Moreover, there is also a functor $\mathrm{CAlg}\colon \mathrm{SymMonCat}_\infty \to \mathbf{Cat}_\infty$ sending a symmetric monoidal category to its category of commutative unital algebras. CAlg also preserves limits, so applying this to the above diagram shows that

$$\mathrm{CAlg}(\mathrm{Desc}_{\mathrm{QCoh}}(X/Y)) = \mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}}(X/Y)$$

as desired.

More generally, our proof above also shows that almost all algebraic structures defined on top of quasicoherent sheaves behave well with respect to descent. For example, it is now easy to see that properties on sheaves like dualizability, invertibility, etc., will all descend well.

Now, again let $\mathcal{C} \subset \mathbf{Sch}$ be a fibered subcategory of schemes. If $f\colon X \to Y$ is a morphism, we will say $\mathcal{C}$ satisfies f-effective descent if the natural map

$$\mathcal{C}_Y \to \mathrm{Desc}_{\mathcal{C}}(f)$$

is an equivalence of categories. Note that full faithfulness restricts along $\mathcal{C}$, meaning that if $\mathcal{C} \subset \mathcal{C}'$ and $\mathcal{C}'$ satisfies full faithfulness (i.e. descent for morphisms), then $\mathcal{C}$ does too. In particular, for all $\mathcal{C}$ we have full faithfulness against universal effective epimorphisms, such as fpqc maps.

Example A.1.36. Suppose $g\colon Y' \to Y$ is a universal effective epimorphism (e.g. fpqc) and $\mathcal{C}$ satisfies g-effective descent. Then membership in $\mathcal{C}$ is g-local, meaning that if

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ \downarrow{\scriptstyle \overline{g}} & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

is cartesian and $X' \in \mathcal{C}_{Y'}$, then $X \in \mathcal{C}_Y$. Indeed, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_Y & \xrightarrow{\simeq} & \mathrm{Desc}_{\mathcal{C}}(Y'/Y) \\ \downarrow & & \downarrow \\ \mathbf{Sch}_Y & \hookrightarrow & \mathrm{Desc}_{\mathbf{Sch}}(Y'/Y) \end{array}$$

The image of X under the lower map is $(X \times_Y Y', \mathrm{can})$, and since $(X', \overline{f}) \cong X \times_Y Y'$ and $\mathcal{C}_{Y'}$ is replete, this lies in $\mathrm{Desc}_{\mathcal{C}}(Y'/Y)$. Thus there is $X_1 \in \mathcal{C}_Y$ such that $(X_1 \times_Y Y', \mathrm{can}) \cong (X \times_Y Y', \mathrm{can})$, so by full faithfulness on the bottom we conclude $X_1 \cong X$ in $\mathbf{Sch}_Y$, i.e. $X \in \mathcal{C}_Y$ by repleteness.

Example A.1.37. Let $\mathcal{C} = \mathbf{Sch}^{\simeq}$, i.e. $\mathcal{C}_Y = \mathbf{Sch}_Y^{\simeq}$ is the groupoid at Y. Then for any $f\colon X \to Y$, $\mathcal{C}_Y \to \mathrm{Desc}(X/Y)$ is essentially surjective. Indeed, if (X', θ) is effective descent data for $X' \in \mathbf{Sch}_X^{\simeq}$ and

$$\theta\colon X' \times_{\pi_1} (X \times_Y X) \xrightarrow{\sim} X' \times_{\pi_2} (X \times_Y X),$$

is an isomorphism over $X \times_Y X$, then in fact $(X', \theta) \cong (X, \mathrm{can})$. Indeed, if not we could pull back along the isomorphism to induce another descent datum (X, θ'), but since the functors $(-) \times_{\pi_i} (X \times_Y X)$ preserve terminal objects, there is a unique morphism can between the images of X. Thus $\theta' = \mathrm{can}$ and the claim follows. As a consequence, we deduce from Example A.1.36 that being an isomorphism is fpqc-local on the target.

Remark A.1.38. This is a sort of converse to the previous example. Suppose $g\colon Y' \to Y$ is a map and $\mathcal{C}$ satisfies g-effective descent. Then a fibered subcategory $\mathcal{C}' \subset \mathcal{C}$ automatically satisfies the full faithfulness part of g-descent. Suppose that for $\mathcal{C}$-objects membership in $\mathcal{C}'$ is g-local, defined just as in the example above. Then $\mathcal{C}'$ also satisfies g-effective descent. Indeed, descent for $\mathcal{C}$ gives a cartesian square like the one above where $(X', \overline{f}) \in \mathcal{C}'_{Y'}$, so g-locality shows that $(X, f) \in \mathcal{C}'_Y$, which is effectivity.

In the remainder of this section, we will show that fpqc effective descent is satisfied for open immersions, affine morphisms, and quasi-affine morphisms. We denote the corresponding fibered subcategories by

$$\mathrm{OpenImm} \qquad \mathrm{Aff} \qquad \mathrm{QAff}$$

in what follows. Note that even given the relations between the categories above, there aren't formal relationships between effective descent for each of the categories above; in any case, we will use the first two to prove the third.

Proposition A.1.39. *Let $f\colon X \to Y$ be an fpqc morphism of schemes. The natural functor*

$$\mathrm{Aff}_Y \to \mathrm{Desc}_{\mathrm{Aff}}(X/Y)$$

is an equivalence of categories.

Proof. We already know it is fully faithful. Let (X', θ) be a descent datum where X' is an affine X-scheme and θ is an isomorphism $X' \times_{\pi_1} (X \times_Y X) \xrightarrow{\sim} X' \times_{\pi_2} (X \times_Y X)$. We know that $X' \xrightarrow{\sim} \underline{\mathrm{Spec}}_X(h_*\mathcal{O}_X)$, and by stability under base change

$$X' \times_{\pi_1} (X \times_Y X) \xrightarrow{\sim} \underline{\mathrm{Spec}}_{X \times_Y X}(\pi_i^* h_* \mathcal{O}_{X'})$$

Thus, the inverse of the descent data corresponds to an isomorphism

$$\widetilde{\theta}^{-1}\colon \pi_1^* h_* \mathcal{O}_{X'} \xrightarrow{\sim} \pi_2^* h_* \mathcal{O}_{X'},$$

and this map satisfies the cocycle condition because of the cocycle condition for θ. Thus $(h_*\mathcal{O}_{X'}, \widetilde{\theta}^{-1})$ is an object of $\mathrm{Desc}_{\mathrm{CAlg}_{\mathrm{qc}}(X)}(X/Y)$, and hence descends to a quasicoherent $\mathcal{O}_Y$-algebra $\mathcal{A}$ such that $f^*\mathcal{A} = h_*\mathcal{O}_{X'}$ agreeing with the descent data. Thus $X' = \mathrm{Spec}_X(f^*\mathcal{A}) = \underline{\mathrm{Spec}}_Y(\mathcal{A}) \times_Y X$, also agreeing with the descent data θ. □

Lemma A.1.40. *Let $f\colon X \to Y$ be an fpqc morphism of schemes. The natural functor*

$$\mathrm{OpenImm}_Y \to \mathrm{Desc}_{\mathrm{OpenImm}}(X/Y)$$

is an equivalence of categories.

Proof. We already know it's fully faithful, so let (X', θ) be a descent datum where $j\colon X' \to X$ is an open immersion. Then θ is an isomorphism $\pi_1^{-1}(X') \cong \pi_2^{-1}(X')$, but since this is required to be an $X \times_Y X$-morphism, this just means

$$\pi_1^{-1}(X') = \pi_2^{-1}(X') \qquad \text{in } X \times_Y X.$$

Now consider $f(X') \subset Y$, and note $X' \subset f^{-1}(f(X'))$. If $x \in X$ such that $f(x) \in f(X')$, let $x' \in X'$ such that $f(x) = f(x')$, and choose a point z lying over (x, x'). Then $z \in \pi_2^{-1}(X')$, so $\pi_1(z) = x \in X'$. Thus $f^{-1}(f(X')) = X'$, and since f is submersive (Lemma A.1.16) this means $f(X')$ is open. It follows that the open immersion $f(X') \hookrightarrow Y$ pulls back to the open immersion $X' \hookrightarrow X$ with its unique descent datum. □

Now we come to the final result of this section. Recall that a qcqs morphism $g\colon Y' \to Y$ is quasi-affine iff the induced map $Y' \to \underline{\mathrm{Spec}}_Y(g_*\mathcal{O}_{Y'})$ is a quasicompact open immersion. We will also use flat base change, which means taking H^0 of the flat base change along a qcqs morphism. The key construction is as follows: let S be a base scheme and consider the fully faithful adjunction

$$\mathrm{Aff}_S(-)\colon (\mathbf{Sch}_S)_{\mathrm{qcqs}} \rightleftarrows \mathrm{Aff}_S : \iota,$$

where we define $\mathrm{Aff}_S(X) = \underline{\mathrm{Spec}}_S(p_*\mathcal{O}_X)$ where $p\colon X \to S$ is the structure map. The unit gives the canonical map $X \to \mathrm{Aff}_S(X)$, which is affine iff X is affine over S, and a quasicompact immersion iff X is quasi-affine over S.

Formation of this map commutes with flat base change. Indeed, suppose $g\colon S' \to S$ is flat. By flat base change and since p is qcqs, $\overline{g}^* p_* \mathcal{O}_X = \overline{p}_* \mathcal{O}_{X \times_S S'}$, which implies

$$\underline{\mathrm{Spec}}_S(p_*\mathcal{O}_X) \times_S S' = \underline{\mathrm{Spec}}'_S(\overline{g}^* p_* \mathcal{O}_X) = \underline{\mathrm{Spec}}_{S'}(\overline{p}_* \mathcal{O}_{X \times_S S'}),$$

which means that

$$(X \to \mathrm{Aff}_S(X)) \times_S S' = \big(X \times_S S' \to \mathrm{Aff}_{S'}(X \times_S S')\big).$$

This has the following consequence: if $\mathcal{C}$ is a fibered subcategory of qcqs schemes over a base and $f\colon X \to Y$ is flat, then there's a natural map

$$\mathrm{Aff}_X(-)\colon \mathrm{Desc}_{\mathcal{C}}(X/Y) \to \mathrm{Desc}_{\mathrm{Aff}}(X/Y)$$

sending (X', θ) to

$$\mathrm{Aff}_X(X') \qquad \widetilde{\theta}\colon \mathrm{Aff}_{X^{(2)}}(X' \times_{\pi_1} X^{(2)}) \to \mathrm{Aff}_{X^{(2)}}(X' \times_{\pi_2} X^{(2)}),$$

where the RHS makes sense because $\pi_i\colon X^{(2)} \to X$ is flat and so the source and target are equal to $\mathrm{Aff}_X(X') \times_{\pi_i} X^{(2)}$. The cocycle condition also follows from the same observation. Moreover, we retain a natural map

$$(X', \theta) \to (\mathrm{Aff}_X(X'), \mathrm{Aff}_{X^{(2)}}(\theta))$$

in $\mathrm{Desc}_{\mathcal{C}}(X/Y)$. This lets us descend constructions relating to the affine hull easily.

THEOREM A.1.41. *Let $f\colon X \to Y$ be an fpqc morphism. The natural functor*

$$\mathrm{QAff}_Y \to \mathrm{Desc}_{\mathrm{QAff}}(X/Y)$$

is an equivalence of categories.

PROOF. We have to show essential surjectivity. Let (X', θ) be a descent datum. Apply the functor and natural map constructed above to obtain a morphism

$$(X', \theta) \to (\mathrm{Aff}_X(X'), \widetilde{\theta}),$$

where we know that $X' \to \mathrm{Aff}_X(X')$ is a quasicompact open immersion. By Prop. A.1.39 we conclude that the RHS is isomorphic to $(Y' \times_Y X, \mathrm{can})$ for Y' an affine Y-scheme. Thus at this stage we have the filled-in part of the diagram:

$$\begin{array}{ccccc} X' & \longrightarrow & Y' \times_Y X & \longrightarrow & X \\ \vdots & & \downarrow & \lrcorner & \downarrow \\ Y'' & \dashrightarrow & Y' & \longrightarrow & Y \end{array}$$

Now consider the quasicompact open immersion $j\colon (X', \theta) \to (Y' \times_Y X, \mathrm{can})$. By definition this means the diagram

$$\begin{array}{ccc} \pi_1^* X' & \xrightarrow{\quad\theta\quad} & \pi_2^* X' \\ \downarrow & & \downarrow \\ Y' \times_Y X \times_Y X & = & Y' \times_Y X \times_Y X \end{array}$$

commutes, so we can view $(X', \overline{\theta})$ as an object of $\mathrm{Desc}_{\mathbf{Sch}}(Y' \times_Y X / Y')$. Then X' is an open immersion over $Y' \times_Y X$, so by descent for OpenImm, we can find

$Y'' \to Y'$ as in the commutative rectangle above, making the left-hand square a pullback, where $Y'' \to Y'$ is an open immersion. It is also quasicompact because that is fpqc-local on the target. By adjunction, we obtain a map

$$(X', \theta) \to (Y'' \times_Y X, \text{can}) \qquad \text{in } \text{Desc}_{\text{QAff}}(X/Y).$$

This is an isomorphism by the pasting lemma and the diagram above. □

A.1.3. Fpqc descent and algebraic spaces. Now we turn back to the question of showing that properties of morphisms are fpqc-local on the target. In the previous section, we have developed relatively strong techniques to do this, which immediately give us a few new properties.

Example A.1.42. By Example A.1.36, the three preceding results imply that being an open immersion, affine morphism, or quasi-affine morphism, are all fpqc local on the target. We also used this to observe that being an isomorphism is fpqc-local on the target. Here are some immediate deductions which are purely formal:

(a) Being a monomorphism is fpqc-local; indeed a map is a monomorphism iff its diagonal is an isomorphism.
(b) Being integral is fpqc-local, because integral = affine + universally closed.
(c) Being finite is fpqc-local, because finite = integral + locally of finite type.
(d) Being semiseparated is fpqc-local, because this is affineness of the diagonal.

For properties (b) and (c), we obtain the stronger result that these subcategories satisfy fpqc-effective descent, by Remark A.1.38.

Remark A.1.43. There are easier proofs of these results that don't rely on fpqc-descent. For example, for isomorphism, one can use that radicial, surjective, and universally open are local, so the base map is a homeomorphism. Then isomorphisms of maps of quasicoherent sheaves also descend so we can conclude by flat base change. We can also give a proof of affine by noting that qcqs descends, and then we can construct the relative Spec directly and use that isomorphisms descend.

Proposition A.1.44. *Let $g\colon Y' \to Y$ be an fpqc morphism. The natural functor*

$$\mathcal{C}_Y \to \text{Desc}_{\mathcal{C}}(Y'/Y)$$

is an equivalence for:

(a) $\mathcal{C}$ the fibered category of closed subschemes.
(b) $\mathcal{C}$ the fibered category of quasicompact immersions.

In particular, being a closed immersion or quasicompact immersion is fpqc-local on the target.

Proof. It suffices to show that being a quasicompact immersion is fpqc-local. Indeed, by Thm. A.1.41 and Remark A.1.38 this will prove (b). By a second application of Remark A.1.38 and the fact that closed immersion = universally closed + closed immersion, we will deduce (a) and the second conclusion.

Let $f\colon X \to Y$ be a map, and suppose that $\overline{f}\colon X' := X \times_Y Y' \to Y'$ is a quasicompact immersion. Thus f is qcqs by Example A.1.11, so we may form the quasicoherent ideal sheaf $\mathcal{I}$ by the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to f_*\mathcal{O}_X$$

on Y. Applying the exact functor g^*, we deduce $g^*\mathcal{I} = \ker(\mathcal{O}_{Y'} \to \overline{f}_*\mathcal{O}_{X'})$ by flat base change. By base change for a closed immersion, we can split our cartesian square into the following one:

$$\begin{array}{ccccc} X' & \longrightarrow & V(g^*\mathcal{I}) & \hookrightarrow & Y' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & V(\mathcal{I}) & \hookrightarrow & Y \end{array}$$

Since $\overline{f}$ is a quasicompact immersion, $X' \to V(g^*\mathcal{I})$ is a (quasicompact) open immersion, so $X \to V(\mathcal{I})$ is a quasicompact open immersion by Example A.1.42. This completes the proof. □

Now let us show some general facts about universal effective epimorphisms. We will see that unlike epimorphisms and effective epimorphisms, they behave quite well. Of course, the class of universal effective epimorphisms is tautologically closed under base change.

Proposition A.1.45. *Let $\mathcal{C}$ be a category with pullbacks, and $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms.*

(a) If gf is a universal effective epimorphism, then g is a universal effective epimorphism.
(b) If f is a universal effective epimorphism and g is a (resp. universal) effective epimorphism, then gf is a (resp. universal) effective epimorphism.
(c) A split epimorphism is a universal effective epimorphism.

In particular the class of universal effective epimorphisms contains all isomorphisms, is stable under base change, and is closed under composition.

Proof. Consider the cartesian diagram:

$$\begin{array}{ccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

It implies that: in (a), we only have to show that g is an effective epimorphism, and in (b), we can ignore the 'resp. universal' part. Moreover, in both cases the hypotheses already imply epimorphism, so we just need to show the existence part of the universal property.

(a) We will use the following commutative diagram:

$$\begin{array}{ccccc}
 & & X \times_Z X & \xrightarrow[q_2]{} & X \\
 & & {\scriptstyle u}\downarrow & & \downarrow \\
X \times_Z Y & \longrightarrow & Y \times_Z Y & \xrightarrow[\pi_2]{} & Y \\
{\scriptstyle p_1}\downarrow & \lrcorner & {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z
\end{array}$$

Let $a\colon Y \to A$ satisfy $a\pi_1 = a\pi_2$. Since $\pi_i u = fq_i$, applying u gives $afq_1 = afq_2$. Since gf is effective epi, there exists $h\colon Z \to A$ such that $hgf = af$. The middle row composite is the projection p_2, so $gp_2 = gfp_1$. Then

$$hgp_2 = hgfp_1 = afp_1 = a\pi_1(f,1) = a\pi_2(f,1) = ap_2.$$

Since $X \to Z$ is a universal effective epi, p_2 is an epi, hence $hg = a$.

(b) Consider the commutative diagram

$$\begin{array}{ccccccc}
X \times_Y X & \longrightarrow & X \times_Z X & \longrightarrow & Y \times_Z Y \\
\downarrow{\scriptstyle p_i} & & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle q_i} \\
X & = & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
\end{array}$$

of projection maps. Let $a\colon X \to A$ be such that $a\pi_1 = a\pi_2$. Then $ap_1 = ap_2$, so there exists (a unique) $h\colon Y \to A$ such that $hf = a$. The middle map $e\colon X \times_Z X \to Y \times_Z Y$ is an epimorphism, because it is the composite of

$$X \times_Z X \xrightarrow{1\times f} X \times_Z Y \xrightarrow{f\times 1} Y \times_Z Y,$$

each of which is an epimorphism because f is a universal effective epimorphism. But

$$hq_i e = hf\pi_i = a\pi_i,$$

so $hq_1 e = hq_2 e$, thus $hq_1 = hq_2$. Since g is an effective epimorphism, there exists $j\colon Z \to A$ such that $jg = h$, so $j(gf) = a$.

(c) By (a), we reduce to showing that an identity map 1_X is an effective epimorphism. Then both maps $X \times_X X \rightrightarrows X$ are isomorphisms, so this follows from the fact that $X \rightrightarrows X \to X$ is a coequalizer in any category. □

Lemma A.1.46. *Being a universal effective epimorphism of schemes is Zariski-local on the target and fpqc-local on the target.*

Proof. Fpqc locality is easy: if $g\colon Y' \to Y$ is fpqc, and $\overline{f}\colon X' \to Y'$ is the base change of $X \to Y$ and is a universal effective epimorphism, then because $Y' \to Y$ is a universal effective epimorphism (Thm. A.1.31), the composite $X' \to Y' \to Y$ is a universal effective epimorphism by Prop. A.1.45. But this equals $X' \to X \to Y$, so part (a) of the same result implies $X \to Y$ is one too.

Let $f\colon X \to Y$ be a map, $\{U_i\}$ an open cover of Y, and assume that $f^{-1}(U_i) \to U_i$ is a universal effective epimorphism for all i. Thus $f^{-1}(U_i \cap U_j) \to U_i \cap U_j$ is a

universal effective epimorphism for all i, j. For both projections the diagram

$$\begin{array}{ccccc} X_{U_i} \times_U X_{U_i} & \xrightarrow{p_i} & X_{U_i} & \longrightarrow & U_i \\ \downarrow & & \downarrow & & \downarrow \\ X \times_Y X & \xrightarrow{\pi_i} & X & \longrightarrow & Y \end{array}$$

commutes, so assume we are given a map $h\colon X \to Z$ such that $h\pi_1 = h\pi_2$. Then $h|_{X_{U_i}} p_1 = h|_{X_{U_i}} p_2$, so there exists a unique map $g_i\colon U_i \to Y$ such that

$$\begin{array}{ccc} X_{U_i} & \longrightarrow & U_i \\ \downarrow & & \downarrow g_i \\ X & \xrightarrow{h} & Z \end{array}$$

commutes. In particular the restriction $g_i|_{U_i \cap U_j}$ is also the unique map making the same diagram commute for U_i replaced with $U_i \cap U_j$. Thus the $\{g_i\}$ are compatible on overlaps and glue to $g\colon Y \to Z$, which satisfies $gf = h$ since this is checked locally. Since h is determined by its restrictions to $\{U_i\}$, it is unique satisfying this property. □

Next let us discuss how to descend other kinds of data under an fpqc map. For example, suppose $f\colon X \to Y$ is fpqc. We know that f^* is conservative, so we can descend the property of being an isomorphism in $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$. We remark that the functor $Lf^*\colon D_{\mathrm{qc}}(Y) \to D_{\mathrm{qc}}(X)$ is also faithful conservative; indeed, by flatness, if $A \to B$ is a map and $Lf^*A \to Lf^*B$ is an isomorphism, then $f^*H^p(A) \to f^*H^p(B)$ is an isomorphism, hence $H^p(A) \to H^p(B)$ is an isomorphism. Thus we have a good chance at describing descent properties of the derived category as well.

Example A.1.47. Let $f\colon X \to Y$ be fpqc, and suppose $\mathcal{F} \in \mathrm{QCoh}(Y)$. If $f^*\mathcal{F}$ is of finite type (resp. finite presentation), then $\mathcal{F}$ is: this is local on Y, so assume Y is affine, and then we can further pull back along $\coprod_i U_i \to X$ to assume that X is affine. Let $A \to B$ be faithfully flat and M be an A-module.

(a) Suppose $B \otimes_A M$ is finite over B. Choose generators over B, write them as sums of tensor products, and extract elements $m_1, \ldots, m_r \in M$ involved in this expression. Then the map $A^r \to M$ is surjective, for $B^r \to B \otimes_A M$ is surjective.

(b) Suppose $B \otimes_A M$ is finitely presented over B. Then M is finite over A by (a), so choose an exact sequence $0 \to K \to A^r \to M \to 0$. Then $B \otimes_A K$ is finite over B, so K is finite over A.

Example A.1.48. Let $f\colon X \to Y$ be fpqc. If $f^*\mathcal{F}$ is flat, then $\mathcal{F}$ is flat. Indeed, $0 = \mathcal{T}or_*(f^*\mathcal{F}, f^*(-)) = f^*\mathcal{T}or_*(\mathcal{F}, -)$, so $\mathcal{T}or_*(\mathcal{F}, -) = 0$. Thus, if $f^*\mathcal{F}$ is a vector bundle, then $\mathcal{F}$ is a vector bundle, because then $\mathcal{F}$ is finitely presented and flat, hence finite locally free.

Here are two nice applications of fpqc descent for properties of morphisms taken from the Stacks project. We recall that being radicial is equivalent to having

surjective diagonal. Indeed, suppose $f\colon X \to Y$ is radicial, and let $z \in X \times_Y X$. Then $\pi_1\colon X \times_Y X \to X$ is injective, so let $x = \pi_1(z)$. Then the composite

$$X \xrightarrow{\Delta_f} X \times_Y X \xrightarrow{\pi_1} X$$

is the identity, so x must map to z because $\pi_1(z) = \pi_1(\Delta_f(x)) = x$. Conversely, let Δ_f be surjective and suppose that $f(x_1) = f(x_2)$. Then we can choose z lying over (x_1, x_2), so there exists x such that $\Delta_f(x) = z$, which implies that $x = x_1$ and $x = x_2$.

Remark A.1.49. This observation also implies that radicial is the right notion of injectivity. Indeed, if X and Y are finite type schemes over an algebraically closed field k, then a map $X \to Y$ is surjective iff $X(k) \to Y(k)$ is surjective. Thus $X \to Y$ is radicial iff $X(k) \to X(k) \times_{Y(k)} X(k)$ is surjective, iff $X(k) \to Y(k)$ is injective.

Example A.1.50. An fpqc monomorphism is an isomorphism. Indeed, if f is fpqc, consider the base change square where f is fpqc:

$$\begin{array}{ccc} X \times_Y X & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Y \end{array}$$

Since $X \times_Y X \to X$ is a section of an isomorphism, it is an isomorphism, so $X \to Y$ is an isomorphism because this is fpqc-local on the base. More generally, this applies to any universal effective epimorphism, since we saw in Example A.1.37 that isomorphisms are local for any universal effective epimorphism. This gives some kind of theorem saying 'epi+mono=iso'.

Lemma A.1.51. *A radicial étale morphism is an open immersion.*

Proof. Let $f\colon X \to Y$ be radicial étale. Since it is an open map, $f(X) \hookrightarrow Y$ is an open immersion, and moreover $X \to f(X)$ is still radicial étale because these are local on the target. Thus we may assume f is surjective. Then f is fppf and in particular fpqc; moreover, Δ_f is surjective by radicial-ness and an open immersion by étale-ness, thus it is an isomorphism. Therefore f is an isomorphism by the previous example. □

Let us return to properties of morphisms. Now, we will have more of a focus on derived-category properties, such as quasi-perfectness. We expect fpqc morphisms to behave well mostly because of flatness as explained before, which also implies that $Lf^* = f^*$ preserves cohomological amplitude. It also reflects it, since $H^p(Lf^*(A)) = f^*H^p(A)$ is 0 iff $H^p(A) = 0$.

Lemma A.1.52. *Let $f\colon X \to Y$ be an fpqc morphism of schemes. Let $A \in D_{\mathrm{qc}}(Y)$.*

(a) If $Lf^*(A)$ is locally of finite tor-dimension, then A is locally of finite tor-dimension.
(b) If $Lf^*(A)$ is m-pseudocoherent, then A is m-pseudocoherent.
(c) If $Lf^*(A)$ is perfect, then A is perfect.

PROOF. (c) follows from (a) and (b). (*a*) By restricting Y we may assume X is quasicompact, so $Lf^*(A)$ has tor amplitude in $[a,b]$. Then for all $\mathcal{F} \in \mathrm{QCoh}(Y)$, $Lf^*(A) \otimes^{\mathbf{L}} Lf^*(\mathcal{F}) = Lf^*(A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F})$, and $Lf^*(\mathcal{F}) = f^*(\mathcal{F})$, so this has amplitude in $[a,b]$. Since Lf^* reflects amplitude this means $A \otimes^{\mathbf{L}}_{\mathcal{O}_Y} \mathcal{F}$ has amplitude in $[a,b]$.

(*b*) The question is local on Y, so assume it's affine. Then X is quasicompact, so let a be such that $Lf^*(A)$ and A are in $D^{\leq a}_{\mathrm{qc}}$. If $a < m$, then A is m-pseudocoherent. Otherwise, $H^a(Lf^*A) = f^*H^a(A)$ is of finite type, hence $H^a(A)$ is of finite type. Since Y is affine, we may choose a surjection $\mathcal{O}_Y^r \twoheadrightarrow H^a(A)$, and a complex $\mathcal{F}^\bullet$ of quasicoherent sheaves representing A in $\mathrm{Ch}^{\leq a}$. Thus $\mathcal{O}_Y^r \twoheadrightarrow H^a(A)$ lifts along the surjection $\mathcal{F}^a \to H^a(A)$ to a map $\mathcal{O}_Y^r[-a] \to A$. Let C be the cone, and apply Lf^* to obtain

$$\mathcal{O}_X^r[-a] \to Lf^*A \to Lf^*C \to \mathcal{O}_X^r[-a+1]$$

exact. Thus $Lf^*C \in D^{<a}$ is m-pseudocoherent, so by induction C is m-pseudocoherent. Thus A is m-pseudocoherent. □

Corollary A.1.53. *Let $g\colon Y \to Y'$ be an fpqc morphism and consider a cartesian square:*

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

If $\overline{f}$ is quasi-perfect, resp. quasi-proper, resp. locally of finite tor dimension, then f is quasi-perfect, resp. quasi-proper, resp. locally of finite tor dimension.

PROOF. Observe that $L\overline{f}^* Lg^* = L\overline{g}^* Lf^*$, and locally on Y the LHS is bounded, thus the RHS is as well. Since $L\overline{g}^*$ reflects amplitude, this means Lf^* is locally bounded.

If f is quasi-perfect (resp. quasi-proper), then first of all $\overline{f}$ is qcqs so f is qcqs by Example A.1.11. By tor-independent base change, we have

$$Lg^* Rf_* E = R\overline{f}_* L\overline{g}^* E$$

for all $E \in D_{\mathrm{qc}}(X)$. If $\overline{f}$ is quasi-perfect (resp. quasi-proper) and E is perfect (resp. pseudocoherent), then the RHS is perfect (resp. pseudocoherent). Thus Rf_*E is perfect (resp. pseudocoherent) by Lemma A.1.52. □

A strong result which would eventually be nice to add here is that the entire derived category satisfies fpqc effective descent. But, the real way to formulate and prove this is using derived ∞-categories and limits of ∞-categories, which we have not discussed.

Example A.1.54. General immersions are *not* fpqc-local on the target. Here is a simple example: Let $X = \coprod_p \operatorname{Spec} \mathbb{Z}/p\mathbb{Z}$ over primes $p > 0$ and consider the natural map $X \to \operatorname{Spec} \mathbb{Z}$. If this were an immersion, then its image $\operatorname{Spec} \mathbb{Z} - \{(0)\}$ would be locally closed, but since it's missing only one point, this would force it to be either closed or open. $\{(0)\}$ is not maximal, and if $\operatorname{Spec} \mathbb{Z} - \{(0)\} = V(I)$ then $I \subset \bigcap_p (p) = 0$.

Next, consider the map $\mathbb{Z} \to \prod_p \mathbb{Z}_{(p)}$. This is flat because it is a torsion-free abelian group. The fiber over a maximal ideal $(q) \subset \mathbb{Z}$ is the LHS in

$$\mathbb{Z}/q\mathbb{Z} \otimes_{\mathbb{Z}} \prod_p \mathbb{Z}_{(p)} = \prod_p \mathbb{Z}_{(p)}/q\mathbb{Z}_{(p)} = \mathbb{Z}/q\mathbb{Z},$$

since q is a unit in $\mathbb{Z}_{(p)}$ if $p \neq q$. Therefore $\mathbb{Z} \to \prod_p \mathbb{Z}_{(p)}$ is faithfully flat by the fiberwise criterion, hence corresponds to an fpqc map $Y' \to \operatorname{Spec} \mathbb{Z}$. Consider the fpqc base change

$$X' = X \times Y' = \coprod_{p>0} \operatorname{Spec} \mathbb{Z}/p\mathbb{Z} \times Y' = \coprod_{p>0} \operatorname{Spec} \mathbb{Z}/p\mathbb{Z}$$

by the preceding computation. Now let e_p be the pth coordinate idempotent in $\prod_p \mathbb{Z}_{(p)}$, and consider the open set $U = \bigcup_p D(e_p)$. Localizing at e_p forces $e_p = 1$, hence gives $\mathbb{Z}_{(p)}$, so the open immersion $U \to Y'$ is equal to $\coprod_p \operatorname{Spec} \mathbb{Z}_{(p)} \hookrightarrow Y'$. The map $X' \to Y'$ factors through U as the disjoint union of $\operatorname{Spec} \mathbb{Z}/p\mathbb{Z} \to \operatorname{Spec} \mathbb{Z}_{(p)}$, which is a closed immersion. Thus $X' \to U \to Y'$ is an immersion, but $X \to Y$ is not.

This failure is fixed by moving to the fppf topology. In general, there are quite a few things which work better in the fppf topology, especially because an fppf map is open, and because much of the nontrivial theory of algebraic spaces requires finite presentation. This will motivate the definition of algebraic space that we give.

Definition A.1.55.

(a) An *fppf* morphism of schemes is a morphism which is faithfully flat and locally of finite presentation.
(b) A *fppf covering family* of a scheme Y is a family of maps $\{Y_i' \to Y : i \in I\}$ which are flat and locally of finite presentation, and $\coprod_i Y_i' \to Y$ is surjective.
(c) A property $\mathcal{P}$ of morphisms is *fppf-local* on the target if the following condition holds: for every fppf covering family $\{Y_i' \to Y\}$, a map $X \to Y$ satisfies $\mathcal{P}$ if and only if $X \times_Y Y_i' \to Y_i'$ satisfies $\mathcal{P}$ for each i.

It is clear that a family of maps $\{Y_i' \to Y\}$ is an fppf covering iff $\coprod_i Y_i' \to Y$ is an fppf map. Moreover, we have the following important characterization of locality:

Lemma A.1.56. *Let $\mathcal{P}$ be a property of morphisms of schemes. Then the following are equivalent:*

(a) $\mathcal{P}$ is fppf-local on the target.
(b) $\mathcal{P}$ is Zariski local on the target, and for each fppf map $Y' \to Y$, a map $X \to Y$ satisfies $\mathcal{P}$ iff $X \times_Y Y' \to Y'$ satisfies $\mathcal{P}$.
(c) $\mathcal{P}$ is Zariski local on the target, and for each fppf map $Y' \to Y$ between affine schemes, a map $X \to Y$ satisfies $\mathcal{P}$ iff $X \times_Y Y' \to Y'$ satisfies $\mathcal{P}$.

PROOF. $(a) \Rightarrow (c)$ because a Zariski covering is also an fppf-covering. $(b) \Rightarrow (a)$ Given an fppf covering family $\{Y_i' \to Y\}$, assume that $X \times_Y Y_i' \to Y_i'$ satisfies $\mathcal{P}$ for all i. Then $\coprod_i Y_i' \to Y$ is an fppf map, and by Zariski-locality,

$$X \times_Y (\coprod_i Y_i') = \coprod_i X \times_Y Y_i' \to \coprod_i Y_i'$$

satisfies $\mathcal{P}$. Thus $X \to Y$ satisfies $\mathcal{P}$.

Now $(c) \Rightarrow (b)$: let $g\colon Y' \to Y$ be fppf and $f\colon X \to Y$ be a map. Let $U \subset Y$ be an affine open, and $\{V_i\}$ be an affine open cover of $g^{-1}(U) \subset Y'$. Since g is open and surjective, $\{g(V_i)\}$ is an open cover of U, hence covered by a finite set $g(V_1), \ldots, g(V_r)$. Thus the composite

$$V \coloneqq \coprod_{i=1}^{r} V_i \to g^{-1}(U) \to U$$

is an fppf morphism of affine schemes. Moreover, since $\mathcal{P}$ is Zariski-local, $\overline{f}^{-1}(V_i) \to V_i$ satisfies $\mathcal{P}$ for each i, so $\coprod_i \overline{f}^{-1}(V_i) \to V$ satisfies $\mathcal{P}$. We can factor the base change square to obtain a cartesian square:

$$\begin{array}{ccc} \coprod_i \overline{f}^{-1}(V_i) & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow \\ f^{-1}(U) & \longrightarrow & U \end{array}$$

By our assumption, this implies $f^{-1}(U) \to U$ satisfies $\mathcal{P}$. Since $\mathcal{P}$ is Zariski local and U was an arbitrary affine open, $X \to Y$ satisfies $\mathcal{P}$. □

Note that as a special case, if $\mathcal{P}$ is fpqc-local, then in particular (c) holds in the lemma above, so we conclude that $\mathcal{P}$ is also fppf-local. This is nice because it works for infinite covers as well. We will from here on out ignore all set-theoretic issues (which are slightly more important than usual, as we will use large sites).

Lemma A.1.57. *Let S be a scheme. The collection of fppf-covering families (consisting of S-morphisms) forms a Grothendieck pretopology on the category $\mathbf{Sch}_S$. We denote the corresponding site by $\mathbf{Sch}_S^{\mathrm{fppf}}$.*

PROOF. (1) The identity map is fppf. (2) If $\{Y_i' \to Y\}$ is a covering family, then $\coprod_i Y_i' \to Y$ is fppf, so for any $V \to Y$,

$$(\coprod_i Y_i') \times_Y Z = \coprod_i (Y_i' \times_Y Z) \to Z$$

is fppf, and thus $\{Y_i' \times_Y Z \to Z\}$ is an fppf covering of Z. (3) Assume $\{Y_i' \to Y\}$ is an fppf cover and $\{Y_{ij}'' \to Y_i'\}$ is an fppf cover of Y_i'. Then the maps

$$\coprod_i \coprod_j Y_{ij}'' \to \coprod_i Y_i' \to Y$$

are each fppf, so the composite is fppf. □

The site $\mathbf{Sch}_S^{\text{fppf}}$ is called the *big fppf site over* S. We denote the corresponding category of presheaves and sheaves of sets by

$$\mathrm{PSh}(\mathbf{Sch}_S^{\text{fppf}}) \qquad \mathrm{Sh}(\mathbf{Sch}_S^{\text{fppf}}).$$

If F is a sheaf, then it preserves arbitrary products. Indeed, for $\{X_i\}$, let $X = \prod_i X_i$, then $\{X_i \to X\}$ is a Zariski open cover, so this follows from the sheaf condition. Conversely, let $F \in \mathrm{PSh}(\mathbf{Sch}_S^{\text{fppf}})$ and suppose F preserves products. Then F is a sheaf iff

$$F(Y) \to F(Y') \rightrightarrows F(Y' \times_Y Y')$$

is an equalizer for each fppf map $Y' \to Y$. This is clear by our assumption, and the fact that coproducts commute with pullback. Our next step is to show that the representable presheaves are sheaves, i.e. the topology is subcanonical.

Lemma A.1.58. *An fppf morphism of schemes over S is a (universal) effective epimorphism in $\mathbf{Sch}_S$. In particular, the presheaves*

$$\mathrm{Hom}_{\mathbf{Sch}_S}(-, X)\colon \mathbf{Sch}_S^{\text{fppf}} \to \mathbf{Set}$$

are sheaves for the fppf topology for each S-scheme X.

Proof. The second statement follows because $\mathrm{Hom}_{\mathbf{Sch}_S}(-, X)$ preserves products, so by the observation above we may check the sheaf condition for one fppf morphism. By definition, this holding for all X is equivalent to the fppf morphism being an effective epimorphism.

Being an effective epimorphism is base independent, so we may assume $S = \operatorname{Spec}\mathbb{Z}$. By Lemma A.1.46, being a universal effective epimorphism is Zariski-local and fpqc-local on the target, hence it is fppf-local on the target. Thus if $f\colon X \to Y$ is fppf, it suffices to show that the base change $X \times_Y X \to X$ along itself is a universal effective epimorphism. But it is a split epimorphism, so we conclude by Prop. A.1.45. □

In particular, we obtain a fully faithful embedding $\mathbf{Sch}_S \hookrightarrow \mathrm{Sh}(\mathbf{Sch}_S^{\text{fppf}})$, and in this context we usually identify the category of S-schemes with its essential image in fppf sheaves. Likewise, if F is a sheaf (or presheaf) and T is an S-scheme, we identify $F(T) = \mathrm{Hom}(T, F)$, and call both the T-points of F. We also call a map $T \to F$ where T is a scheme a 'scheme over F'.

Remark A.1.59. Recall that in general, sheaves on a slice site form the slice of sheaves on the site. Therefore, given a morphism $S' \to S$ of schemes, we have

$$\mathrm{Sh}(\mathbf{Sch}_S^{\text{fppf}})/S' = \mathrm{Sh}(\mathbf{Sch}_S^{\text{fppf}}/S') = \mathrm{Sh}(\mathbf{Sch}_{S'}^{\text{fppf}})$$

Explicitly, the identification is as follows: if $F \to S'$ is a morphism of sheaves, then we define $F_{S'} \in \mathrm{Sh}(\mathbf{Sch}_{S'}^{\text{fppf}})$ on an S'-scheme $X \xrightarrow{u} S'$ by the universal property

$$\mathrm{Hom}_{\mathrm{Sh}(\mathbf{Sch}_{S'}^{\text{fppf}})}(X, F_{S'}) = F \times_{S'} X,$$

meaning that $F_{S'}(X) = F(X) \times_{S'(X)} \{u\}$, so the X-points of $F_{S'}$ are the X-points of F, viewed as an S-scheme, whose image in $\mathrm{Hom}(X, S')$ is u. Conversely, given $G \in \mathrm{Sh}(\mathbf{Sch}_{S'}^{\mathrm{fppf}})$, we define

$$F(X) = \coprod_{u\colon X \to S'} G(X, u)$$

where u runs over S-maps. Therefore we can be a bit careless about the base.

Definition A.1.60. Let $F \to G$ be a morphism in $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$.

(a) $F \to G$ is *representable by schemes* if for each scheme $T \to G$ over G, the pullback $F \times_G T$ is a scheme.

(b) Let $\mathcal{P}$ be a property of morphisms of schemes which is stable under base change and fppf-local on the target. $F \to G$ is *representable and satisfies* $\mathcal{P}$ if it is representable, and for each scheme $T \to G$, the morphism $F \times_G T \to T$ of schemes satisfies $\mathcal{P}$.

For instance, a morphism of schemes is always representable by schemes because schemes have pullbacks.

Example A.1.61. Here are the first basic observations about the definitions above.

(a) If $F \to G$ is representable by schemes, then for all sheaves H over G, $F \times_G H \to H$ is representable. Indeed, if $T \to H$ is a scheme, then $F \times_G H \times_H T = F \times_G T$. Thus if $F \to G$ is representable and satisfies $\mathcal{P}$ in the sense above, then this notion is stable under base change.

(b) Let U be a scheme and $U \to G$ a representable fppf morphism. Suppose that $F \to G$ is representable by schemes. If $F \times_G U \to U$ satisfies $\mathcal{P}$ then $F \to G$ is 'representable and satisfies $\mathcal{P}$' in the sense of the definition above. Indeed, for all schemes $T \to G$, we have a base change diagram:

$$\begin{array}{ccccc} F \times_G U \times_G T & \longrightarrow & F \times_G T & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U \times_G T & \longrightarrow & T & \longrightarrow & G \end{array}$$

Since $U \to G$ is representable fppf, $U \times_G T \to T$ is an fppf morphism of schemes, and likewise $F \times_G T$ is a scheme. By assumption, the morphism $F \times_G U \to U$ of schemes has $\mathcal{P}$, so $F \times_G \times_G T \to U \times_G T$ has $\mathcal{P}$. Since $\mathcal{P}$ is fppf-local on the target, this means $F \times_G T \to T$ has $\mathcal{P}$. Since T was arbitrary, we conclude $F \to G$ has representable $\mathcal{P}$.

Thus these properties really only require checking on one fppf cover.

Earlier we saw that $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}}) = \mathrm{Sh}(\mathbf{Sch}^{\mathrm{fppf}})/S$, so in particular S is a terminal object. We will in general write products in this category using $(-) \times_S (-)$ even though the S is implicit. Note that a sheaf F is a scheme if and only if there is a representable map $F \to T$ for T an S-scheme, iff $F \to S$ is representable.

Next, we explain an important application of descent for schemes, which we have not seen too much of yet. First of all, Lemma A.1.58 shows that if $f\colon X \to Y$

is an fppf morphism of schemes, then $\mathbf{Sch}_Y \to \mathrm{Desc}_{\mathbf{Sch}}(X/Y)$ is fully faithful. Effective descent can tell us whether a given fppf sheaf is a scheme.

Lemma A.1.62. *Let $\mathcal{C}$ be a fibered subcategory of schemes satisfying g-effective descent. Suppose we are given a cartesian diagram*

$$\begin{array}{ccc} X' & \longrightarrow & T' \\ \downarrow & \lrcorner & \downarrow g \\ F & \longrightarrow & T \end{array}$$

in $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$ *where X', T', and T' are schemes, g is fppf, and $X' \in \mathcal{C}_{T'}$. Then F is a scheme and $F \in \mathcal{C}_T$.*

Proof. Consider the equivalence $\mathcal{C}_T \to \mathrm{Desc}_{\mathcal{C}}(T'/T)$, and let $\pi_i \colon T' \times_T T' \to T'$ be the two projections. Identify $X' = g^*F$, then X' is equipped with canonical descent datum $\pi_1^* g^* F = \pi_2^* g^* F$ because $g\pi_1 = g\pi_2$. Thus there exists a scheme $X \in \mathcal{C}_T$ such that $h \colon X \times_T T' \xrightarrow{\sim} X'$ commuting with the canonical descent datum, which means that the two pullbacks

$$X \times_T T' \times_T T' \to F \times_T T' \times_T T'$$

are equal. Therefore by projecting, the two maps

$$X \times_T (T' \times_T T') = X' \times_X X' \to F$$

are equal. Moreover the map $X' \to X$ is an fppf cover, being a pullback of an fppf cover. Working in T-schemes, the sheaf condition says that

$$\mathrm{Hom}_{\mathbf{Sch}_T}(X, F) \to \mathrm{Hom}_{\mathbf{Sch}_T}(X', F) \rightrightarrows \mathrm{Hom}_{\mathbf{Sch}_T}(X' \times_X X', F)$$

is an equalizer. Thus the map $X' \to F$ descends to a unique $X \to F$ which induces the isomorphism $X \times_T T' \xrightarrow{\sim} F \times_T T'$. Since X and F are fppf sheaves, we can detect isomorphisms after passing to the cover $\{T' \to T\}$. □

Example A.1.63. Here is a simple instance of the lemma above. Let $F \to G$ be a morphism of sheaves, and suppose that for each *affine scheme* T' with a map $T' \to G$, the pullback $F \times_G T'$ is a scheme. Then in fact $F \to G$ is representable. To see this, let T be a scheme and consider the pullback $F \times_G T \to T$. Let $\{U_i\}$ be an affine open cover of T and $U = \coprod_i U_i \to T$ the corresponding fppf map. Then

$$(F \times_G T) \times_T U = F \times_G U = \coprod_i F \times_G U_i$$

is a scheme, and all schemes satisfy effective descent along $U \to T$ by open-cover descent. Therefore $F \times_G T$ is a scheme.

Lemma A.1.64. *Let $F \to G$ be a morphism in* $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$. *The following are equivalent:*

(a) The diagonal $F \to F \times_G F$ is representable by schemes.

(b) For all schemes $U, V \to F$, the natural map

$$U \times_F V \to U \times_G V$$

is representable by schemes.

Moreover, the following are equivalent:

(c) The diagonal $F \to F \times_S F$ is representable by schemes.
(d) For all schemes $U, V \to F$, $U \times_F V$ is a scheme.
(e) For all schemes U, any map $U \to F$ is representable by schemes.

In particular, if G has representable diagonal, then $F \to F \times_G F$ is representable iff F has representable diagonal.

Proof. Consider the natural base change diagram

$$\begin{array}{ccc} U \times_F V & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow \\ U \times_G V & \longrightarrow & F \times_G F \end{array}$$

which follows from Fubini's theorem for limits. Then $(a) \Rightarrow (b)$ follows immediately. Conversely, let $T \to F \times_G F$ be a scheme, which we can factor as $T \to T \times_G T \to F \times_G F$. This comes with two maps $T \to F$, so putting $U = V = T$ in the diagram above, we obtain the cartesian diagram:

$$\begin{array}{ccccc} P & \longrightarrow & T \times_F T & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ T & \longrightarrow & T \times_G T & \longrightarrow & F \times_G F \end{array}$$

Then P is a scheme by assumption, which is what we wanted.

$(d) = (e)$ These two are equivalent by definition of representable. $(c) = (d)$ Take $G = S$ in $(a) = (b)$, so that $U \times_S V$ is a scheme.

For the last statement, note that if G has representable diagonal then by (d), $U \times_G V$ is a scheme, so as in (a) the map from $U \times_F V$ being representable is equivalent to this sheaf being a scheme. □

Example A.1.65. A representable morphism $F \to G$ of fppf sheaves has representable diagonal. Indeed, let $T \to F \times_G F$ be a scheme, which is thus also a scheme over G, and consider $H = T \times_G F$, which is a scheme. The projection $H \to F$ induces $H \times_T H \to F \times_G F$. Note that $H \times_T H = T \times_G (F \times_G F)$, so

$$H \times_T H \underset{F \times_G F}{\times} F = T \times_G F = H.$$

Thus form the diagram:

$$\begin{array}{ccccc} P & \longrightarrow & H & \longrightarrow & F \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ T & \longrightarrow & H \times_T H & \longrightarrow & F \times_G F \end{array}$$

The left-hand square consists of schemes, so P is a scheme, completing the proof by pasting.

Definition A.1.66. Let X be an object of $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$. X is an *algebraic space over S* if the following two conditions hold:

(a) The diagonal $X \to X \times_S X$ is representable by schemes.

(b) There exists a scheme U and a representable étale surjection $U \to X$.

Such a scheme and map $U \to X$ is called *an étale atlas* for X. The full subcategory of $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$ spanned by the algebraic spaces over S is denoted $\mathbf{AlgSp}_S \subset \mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$.

Example A.1.67. Suppose X is an S-scheme. Then X is an algebraic space over S, because $X \times_S X$ is a scheme, and $\mathrm{Id}\colon X \to X$ can be taken as an étale atlas. Thus we have an inclusion $\mathbf{Sch}_S \subset \mathbf{AlgSp}_S$.

Our goal in this section is *not* to fully develop the theory of algebraic spaces, but just to say enough about them to have some working examples of how to conduct basic proofs. Here are the very basic few observations.

1. Let $f\colon X \to Y$ be a morphism of algebraic spaces over a base scheme S. Then the diagonal $\Delta_f\colon X \to X \times_Y X$ is always representable by schemes, by the previous lemma.
2. By the same lemma, a map from a scheme to an algebraic space is always representable by schemes. By Example A.1.61, we see that for any (representable) fppf morphism $V \to Y$ from a scheme, we can check properties of morphisms of representable morphisms $X \to Y$ after pullback to $X \times_Y V \to V$. In particular, we can always check properties on the atlas.
3. Let F be an fppf sheaf. If F admits a representable map $F \to X$ for X an algebraic space, then F is an algebraic space. Indeed, if $U \to X$ is an étale atlas, then $U \times_X F \to F$ is an étale atlas for F. Second, to see that F has representable diagonal, it suffices to note that any representable morphism has representable diagonal, which we saw in Example A.1.65

Lemma A.1.68. *Let $X \to Y \leftarrow Y'$ be morphisms of algebraic spaces over S. Then the fiber product $X' \coloneqq X \times_Y Y'$ in $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$ is an algebraic space over S. In particular $\mathbf{AlgSp}_S$ is closed under pullbacks.*

Proof. Let $U \to X$ and $V \to Y'$ be representable étale surjections from schemes. Consider the iterated pullback diagram:

$$\begin{array}{ccccc}
U \times_Y V & \longrightarrow & X'_V & \longrightarrow & V \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
X'_U & \longrightarrow & X' & \longrightarrow & Y' \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
U & \longrightarrow & X & \longrightarrow & Y
\end{array}$$

By base change, $X'_U \to X'$ and $X'_V \to X'$ are representable étale surjections, so the composite $U \times_Y V \to X'_V \to X'$ is a representable étale surjection. However since Y has representable diagonal, $U \times_Y V$ is a scheme (Lemma A.1.64). To see that X' has representable diagonal, observe that $X' \to X' \times_{Y'} X'$ is a pullback of $X \to X \times_Y X$. The latter is representable since $X \to Y$ is a morphism of algebraic

spaces, so the former is as well. Since Y' has representable diagonal, this implies X' does by Lemma A.1.64. □

The proof also tells us that given an étale atlas for X and Y', an étale atlas for the pullback is the pullback of the atlases over Y. The simplest properties of morphisms to extend to algebraic spaces are those of the diagonal, because it is always automatically representable. Let $f\colon X \to Y$ be a morphism of algebraic spaces. Then f is quasiseparated, locally separated, semiseparated, or separated, if $\Delta_f\colon X \to X \times_Y X$ is (representable and) quasicompact, an immersion, affine, or a closed immersion. As the definition indicates, the diagonals of morphisms of algebraic spaces need not be immersions.

Next, consider the case of an affine morphism. We want to say that $X \to Y$ is affine if for all affine schemes $T \to Y$, $X \times_Y T$ is an affine scheme. By Example A.1.63, this forces $X \to Y$ to already be representable, so an affine morphism of algebraic spaces is automatically representable by schemes. Thus, there is no ambiguity if we say that a morphism of algebraic spaces is a closed immersion, finite morphism, integral morphism, if and only if it is representable and satisfies this property.

A similar thing holds for an open immersion. We define an open immersion of algebraic spaces $j\colon X \to Y$ as a morphism such that for all schemes $T \to Y$, $X \times_Y T \to T$ is an open immersion of schemes. In particular we assume by default that it is representable. Using this we can define a reasonable notion of an underlying topological space associated to an algebraic space.

If X is an algebraic space, write

$$|X| \coloneqq \{\operatorname{Spec} K \to X : K \text{ a field}\}/\sim,$$

where $(\operatorname{Spec} K \to X) \sim (\operatorname{Spec} L \to X)$ if there is a common extension F and a commutative diagram:

$$\begin{array}{ccccc} & & \operatorname{Spec} K & & \\ & \nearrow & & \searrow & \\ \operatorname{Spec} F & & & & X \\ & \searrow & & \nearrow & \\ & & \operatorname{Spec} L & & \end{array}$$

In other words $|X| = \varinjlim_K X(K)$. Observe that if T is a scheme, then there is a canonical map from $|T|$ in the usual sense to $\{\operatorname{Spec} K \to T : K\}/\sim$ sending $t \mapsto i_t\colon \operatorname{Spec} \kappa(t) \to X$. It has a well-defined inverse by sending $\operatorname{Spec} K \to T$ to its image, thus we obtain a bijection.

Example A.1.69. Let us recall the argument involved in the purely scheme-theoretic statement above. We must show that if two maps

$$i_1, i_2\colon \operatorname{Spec} K_1, \operatorname{Spec} K_2 \to X$$

have the same image in X, then we can find a common extension. Indeed, if the image is x, then i_1 factors through $\operatorname{Spec} \kappa(x) \to X$, so we have a base change

diagram:

$$\begin{array}{ccccc} Z & \longrightarrow & Z' & \longrightarrow & \operatorname{Spec} K_1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec} K_2 & \longrightarrow & \operatorname{Spec} \kappa(x) & \longrightarrow & X \end{array}$$

Then $|Z'|$ is the preimage of x, which we know consists of the whole space $\operatorname{Spec} K_1$, so Z' is nonempty. On the other hand $Z \to Z'$ is surjective by base change, so Z is nonempty. Choosing a point of Z does the trick.

Let $f\colon X \to Y$ be a morphism of algebraic spaces. For each field K, this induces a map $X(K) \to Y(K)$, and by naturality a map $|f| : |X| \to |Y|$. A subset $U \subset |X|$ is open if for all schemes $T \to X$, $U \times_{|X|} |T| \subset |T|$ is open. In other words, we give $|X|$ the final topology with respect to the maps $|T| \to |X|$ induced by schemes over X. Note that if X was representable by a scheme, then this agrees with its topology via the identity map, and the fact that morphisms of schemes are continuous.

Example A.1.70. Many properties can be detected intrinsically on the underlying space $|X|$. For instance, if $f\colon X \to Y$ is a representable morphism of algebraic spaces, then $|f|$ is surjective iff f is representable surjective. ($\Leftarrow$) Let $y\colon \operatorname{Spec} K \to Y$ represent a point of $|Y|$, so by assumption $X_K \to \operatorname{Spec} K$ is a surjective morphism of schemes, and hence we may choose a point $x\colon \operatorname{Spec} L \to X_K \to X$ which maps to y. ($\Rightarrow$) Let $T \to Y$ be a scheme and consider the morphism of schemes $X \times_Y T \to T$. Given $y\colon \operatorname{Spec} K \to T \to Y$, choose $x\colon \operatorname{Spec} L \to X$ such that $|f|(x) = y$, meaning the diagram

$$\begin{array}{ccccc} \operatorname{Spec} L & & \longrightarrow & & X \\ \downarrow & & & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & T & \longrightarrow & Y \end{array}$$

commutes. By definition, $\operatorname{Spec} L \to X$ factors through $T \times_Y X$. Therefore, it is reasonable to define a morphism of algebraic spaces f to be surjective if $|f|$ is surjective.

Along the lines of this example, many of the standard topological facts from scheme theory pass immediately to algebraic spaces.

Proposition A.1.71. *Let $f\colon X \to Y$ be a morphism of algebraic spaces over S.*

(a) The induced map $|f| : |X| \to |Y|$ is continuous.

(b) Given a cartesian diagram

$$\begin{array}{ccc} X' & \xrightarrow{\overline{f}} & Y' \\ {\scriptstyle \overline{g}}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

of algebraic spaces, we have $|g|^{-1}|f|(C) = \left|\overline{f}\right| |\overline{g}|^{-1}(C)$ for any subset $C \subset |X|$.

(c) Assume f is representable by schemes.

i. f is a representable universally open map iff for all algebraic spaces $Y' \to Y$, the map $|X \times_Y Y'| \to |Y|$ is open.

ii. f is a representable universally closed map iff for all algebraic spaces $Y' \to Y$, the map $|X \times_Y Y'| \to |Y|$ is closed.

iii. f is a representable radicial map iff for all algebraic spaces $Y' \to Y$, the map $|X \times_Y Y'| \to |Y|$ is injective.

In all cases above, it suffices to check the condition for schemes Y'.

PROOF. (a) Let $U \subset |Y|$ be open. Then we have to show that as sets, $U \times_{|Y|} |X| \subset |X|$ is open, which means showing that

$$(U \times_{|Y|} |X|) \times_{|X|} |T| = U \times_{|Y|} |T|$$

is open for every scheme $T \to X$. This is true because then $T \to X \to Y$ is a scheme over Y.

(b) We omit the $|-|$ in the proof. By functoriality,

$$\overline{f}\overline{g}^{-1}(C) \subset g^{-1}g\overline{f}\overline{g}^{-1}(C) = g^{-1}f\overline{g}\overline{g}^{-1}(C) \subset g^{-1}f(C).$$

Conversely, consider a point $y' \in |Y'|$ such that $g(y') = f(x)$ for some $x \in C$. Then the composites of the corresponding maps $\operatorname{Spec} K \to Y'$ and $\operatorname{Spec} L \to X$ with f and g represent the same point in $|Y|$, so we may choose $\operatorname{Spec} M \to \operatorname{Spec} K \to Y$ equal to $\operatorname{Spec} M \to \operatorname{Spec} L \to Y$. Thus the maps $\operatorname{Spec} M \to Y'$ and $\operatorname{Spec} M \to X$ agree upon composition with f and g, hence represent a point $\operatorname{Spec} M \to X'$ which lies over y' and x.

(c) Since representable and $\mathcal{P}$ is stable under base change, it suffices to prove ($\Rightarrow$) forgetting about Y'. Note that in all cases, ($\Leftarrow$) for Y' a scheme (and hence the final statement) is immediate from the definition of 'representable and $\mathcal{P}$', and the fact that for a scheme, the underlying space agrees with its usual underlying space.

(i)/(ii) ($\Rightarrow$) Let $V \subset |X|$ be an open/closed subset. We want to show that for all schemes $T \to Y$, $|f|(V) \times_{|Y|} |T|$ is open/closed in $|T|$. Write $\overline{f}\colon X_T \to T$ for the base change of f, which is an open/closed map of schemes by assumption. By (b),

$$|f|(V) \times_{|Y|} |T| = \left|\overline{f}\right| (V \times_{|X|} |X_T|),$$

which is open/closed in $|T|$ since $V \times_{|X|} |X_T|$ is open-closed in $|X_T|$ since $|X| \to |X_T|$ is continuous.

(iii) ($\Rightarrow$) Suppose $x_1, x_2 \in |X|$ and $|f|(x_1) = |f|(x_2) = y$. Writing $y\colon \operatorname{Spec} K \to Y$, then the base change $X \times_Y \operatorname{Spec} K \to \operatorname{Spec} K$ is an injective morphism of schemes, so the domain has at most one point. Choosing a common extension $x_1, x_2\colon \operatorname{Spec} L \to X$, both x_1 and x_2 factor through $X \times_Y \operatorname{Spec} K$, so they represent the same point of X. □

In light of this result, we *define* the notions of universally open, closed, and radicial for morphisms of algebraic spaces via the underlying spaces as in the lemma. Thus the proposition shows that in the case of representable morphisms

(e.g. morphisms of schemes), this reduces to the classical case. For the final result of this section, we discuss open subspaces of an algebraic space.

Let X be an algebraic space over a scheme S. It follows from the previous result that given an open immersion $j\colon U \to X$, $|j|$ is an open map of topological spaces. Thus there is an obvious functor

$$\mathrm{OpenImm}\,/X \to \mathrm{Open}(|X|)$$

sending an open immersion to its image.

Proposition A.1.72. *Let X be an algebraic space over S. The natural functor*

$$\mathrm{im}\colon\ \mathrm{OpenImm}\,/X \to \mathrm{Open}(|X|)$$

is an equivalence of categories.

Proof. We construct an inverse as follows; let $\mathcal{P}(|X|)$ denote the set of all subsets of $|X|$, viewed as a constant presheaf. For each scheme T, the map

$$\mathrm{Hom}_{\mathbf{AlgSp}_S}(T, X) \to \mathcal{P}(|X|) \qquad f \mapsto |f|\,(|T|)$$

corresponds to a morphism $X \to \mathcal{P}(|X|)$ of presheaves. For each subset $V \subset |X|$, pulling back defines a subpresheaf X_V:

$$\begin{array}{ccc} X_V & \longrightarrow & \mathcal{P}(V) \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \longrightarrow & \mathcal{P}(|X|) \end{array}$$

This is a sheaf: if $\{T_i \to T\}$ is a fppf cover and $f \in X(T)$ satisfies $T_i \to T \xrightarrow{f} X \in X_V(T_i)$ for each i, then $\bigsqcup_i |T_i| \to |T| \to |X|$ has image in V, but the first map is surjective, so the latter map has image in V.

We claim that $X_V \hookrightarrow X$ is representable; indeed, let $f\colon T \to X$ be a scheme, and observe that for all schemes T',

$$\begin{aligned} \mathrm{Hom}(T', T \times_X X_V) &= \mathrm{Hom}(T', T \times_{\mathcal{P}(|X|)} \mathcal{P}(V)) \\ &= \{g\colon T' \to T : \mathrm{im}\,(T' \to T \to X) \subset V\} \\ &= \{g\colon T' \to T : \mathrm{im}\,(g) \subset |f|^{-1}\,(V)\} \\ &= \mathrm{Hom}(T', f^{-1}(V)), \end{aligned}$$

where $f^{-1}(V) \subset T$ is the open subscheme with underlying space $|f|^{-1}\,(V)$. This shows that $X_V \hookrightarrow X$ is representable, hence X_V is an algebraic space, and thus $X_V \hookrightarrow X$ is an open immersion.

These two constructions are mutually inverse. Indeed, let $x\colon \operatorname{Spec} K \to X$ represent a point of $|X|$; the computation above shows that pulling back along $X_V \hookrightarrow X$ is $x^{-1}(V)$, which is nonempty iff $x \in V$. On the other hand, Prop. A.1.72(b) shows that this is nonempty iff x is the image of $|X_V| \to |X|$. This proves that $X_V \to X$ has image V.

Conversely, let $j\colon U \to X$ be an open immersion with image V, and observe that the natural map $\mathrm{Hom}(T, U) \to \mathrm{Hom}(T, X)$ has image in $\mathrm{Hom}(T, X_V)$, so we

obtain a natural map $U \to X_V$. We claim this is an isomorphism: indeed, for all schemes $T \to X_V$ we have the following cartesian diagram:

$$\begin{array}{ccccc} U_T & \longrightarrow & T & = & T \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & X_V & \hookrightarrow & X \end{array}$$

The upper-right morphism is an isomorphism because $X_V \hookrightarrow X$ is a monomorphism. Then both $X_V \times_X T$ and $U \times_X T$ coincide with the open subscheme of T defined by $V \times_{|X|} |T|$, in the latter case by Prop. A.1.71(b). Thus $U_T \to T$ is an isomorphism, which concludes the proof. □

We conclude from the proof the following universal property of an open immersion $j\colon U \to X$ in $\mathbf{AlgSp}_S$: For each map $f\colon W \to X$ of algebraic spaces such that $\operatorname{im}|f| \subset |U|$, there is a unique map $W \to U$ factoring f through j. Indeed, this is by definition if W is a scheme, and if not then observe that $\operatorname{Hom}(T, W) \to \operatorname{Hom}(T, X)$ factors through $\operatorname{Hom}(T, U)$ by assumption, so $W \to X$ factors through U.

It is also clear from the proof that if $f\colon X \to Y$ is a morphism and $j\colon U \to Y$ is an open immersion, then $X \times_Y U = f^{-1}(U)$, i.e. the open subspace corresponding to $|f|^{-1}(|U|)$.

A.1.4. Properties of morphisms of algebraic spaces. In this section we will discuss how to extend more notions of morphisms of schemes to algebraic spaces. By the end, we will prove our first nontrivial result about algebraic spaces, which is that an fppf sheaf is an algebraic space iff it is an étale quotient of schemes. We will also be able to write down some examples of non-scheme algebraic spaces, though it will be quite difficult to find a truly 'nice' geometric one.

Let's begin with quasicompactness. An algebraic space X is *quasicompact* if $|X|$ is a compact topological space. A morphism $X \to Y$ of algebraic spaces is *quasicompact* if for every quasicompact algebraic space Y' and map $Y' \to Y$, the fiber product $X \times_Y Y'$ is quasicompact.

Lemma A.1.73. *An algebraic space X is quasicompact iff it admits a (representable) étale surjection from an affine scheme.*

Proof. One direction is clear because a continuous image of a compact space is compact. On the other hand, choose an étale surjection $p\colon U \to X$ from a scheme U. Note that p is universally open (Prop. A.1.71), $|p|$ is an open surjective map of spaces. Let $\{V_i\}$ be an affine open cover of U. Then $|p|(V_i)$ is an open cover of $|X|$, thus has a finite subcover $|p|(V_1), \ldots, |p|(V_r)$. Then the map

$$\coprod_{i=1}^{r} V_i \to U \to X$$

has affine source, and is a composition of representable étale maps, hence representable étale. It is also surjective on spaces, which we have shown is equivalent to being representable surjective. □

Proposition A.1.74. *Let $f\colon X \to Y$ be a morphism of algebraic spaces.*

(a) f is quasicompact iff for all affine schemes $T \to Y$, $X \times_Y T$ is quasicompact.
(b) If f is representable, then f is quasicompact iff f is representable quasicompact.
(c) Suppose $U \to Y$ is a surjective universally open morphism, and $X \times_Y U \to U$ is quasicompact. Then f is quasicompact.

Proof. For (a)/(c), let $Y' \to Y$ be a quasicompact algebraic space, and consider the base change diagram:

$$\begin{array}{ccccc}
V' & \longrightarrow & V & & \\
\downarrow & \lrcorner & \downarrow & & \\
X_U \times_Y Y' & \longrightarrow & U \times_Y Y' & \longrightarrow & U \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
X \times_Y Y' & \longrightarrow & Y' & \longrightarrow & Y \\
\downarrow & \lrcorner & \downarrow & & \\
X & \longrightarrow & Y & &
\end{array}$$

where $X_U = X \times_Y U$. Then $U \times_Y Y' \to Y'$ is surjective and universally open, so since Y' is quasicompact, we may choose an affine scheme V as above so that the composite $V \to Y'$ is surjective. If we are in the situation of (a), then this shows V' is quasicompact and $V' \to X \times_Y Y'$ is surjective, which completes the proof. In the situation of (c), we note that $X_U \to U$ is quasicompact, so $X_U \times_Y Y' \to U \times_Y Y'$ is quasicompact, hence V' is quasicompact. The same conclusion follows.

(b) If f is quasicompact, then $X \times_Y T \to T$ is also quasicompact (as algebraic spaces), so if T is a scheme, apply the definition to all quasicompact open subschemes $U \subset T$. Conversely, if f is representable quasicompact then $X \times_Y T \to T$ is quasicompact (as schemes) for all affine schemes T, which implies that $|X \times_Y T|$ is quasicompact. Thus we conclude by (a). □

In light of this, standard results about schemes tend to pass through. For instance, a morphism $X \to Y$ from a quasicompact algebraic space to a quasiseparated algebraic space is quasicompact, by the cancellation lemma (and (b) above). Thus a morphism of qcqs algebraic spaces is qcqs.

Proposition A.1.75. *Let $f\colon X \to Y$ be a morphism of algebraic spaces. The following are equivalent:*

(a) f is an isomorphism.
(b) $f(T)\colon X(T) \to Y(T)$ is an isomorphism for all S-schemes T.
(c) $X \times_Y T \to T$ is an isomorphism for all affine schemes $T \to Y$.
(d) There is an (representable) fppf map $U \to Y$ from a scheme such that $X \times_Y U \to U$ is an isomorphism.

PROOF. $(a) = (b)$ by definition, and these imply (c) and (d). Assuming (c), we have a cartesian diagram:

$$\begin{array}{ccccc} f(T)^{-1}(t) & \hookrightarrow & X_T(T) & \longrightarrow & X(T) \\ \cong\downarrow & \lrcorner & \cong\downarrow & \lrcorner & \downarrow \\ t & \hookrightarrow & T(T) & \longrightarrow & Y(T) \end{array}$$

This says that for each $t \in Y(T)$ the fiber has exactly one point. Thus $X(T) \to Y(T)$ is an isomorphism for all affine schemes T admitting a map to Y, though this still holds if not because then $X(T) = Y(T) = \varnothing$. Finally, every scheme T admits an affine open cover $\{T_i \to T\}$ which is an fppf covering, so $X(T) \xrightarrow{\sim} Y(T)$ for all schemes T.

(d) For all (e.g. affine) schemes $T \to Y$, we obtain a cartesian diagram

$$\begin{array}{ccc} X_U \times_Y T & \xrightarrow{\cong} & U \times_Y T \\ \downarrow & \lrcorner & \downarrow \\ X \times_Y T & \longrightarrow & T \end{array}$$

where T is a scheme, $U \times_Y T$ is a scheme since Y is an algebraic space, and $X_U \times_Y T$ is a scheme because it's isomorphic to $U \times_Y T$. Fppf maps are universal effective epimorphisms for schemes, so by Example A.1.37 isomorphisms satisfy effective fppf descent for schemes. Therefore by Lemma A.1.62 we conclude that $X \times_Y T$ is a scheme and $X \times_Y T \xrightarrow{\sim} T$ is an isomorphism. □

Next let us discuss properties of morphisms which are étale-local on the source and the target. A property $\mathcal{P}$ of morphisms of schemes is *étale-local on the source* if a morphism $f\colon T' \to T$ satisfies $\mathcal{P}$ iff given an étale cover $\{T'_i \to T'\}$ of the source, the maps $T'_i \to T$ all satisfy $\mathcal{P}$. We can of course make the same definition for being fppf local on the source.

Example A.1.76. Many familiar properties that are Zariski local on the source, closed under composition, and satisfied by étale (resp. flat and locally of finite presentation) morphisms, are étale (resp. fppf) local on the source. Indeed, this assumption already takes care of one direction of the iff. The next

Somewhat surprisingly, it is actually a non-trivial result of commutative algebra that the finite type/presentation hypotheses can be checked fppf-locally on the source. In particular, we will need the following lemma to do this; notably it relies on openness of the flat locus and the flat-local criterion, both in the Noetherian case.

Lemma A.1.77. *Let $R \to S$ be a flat (resp. faithfully flat) ring map, with a finite presentation*

$$S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_r).$$

There exists a finitely generated $\mathbb{Z}$-subalgebra $R_0 \subset R$ such that $f_i \in R_0[x_0, \ldots, x_n]$, and

$$S_0 \coloneqq R_0[x_0, \ldots, x_n]/(f_1, \ldots, f_r)$$

is flat (resp. faithfully flat) over R_0.

PROOF. Abbreviate $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{f} = (f_1, \ldots, f_r)$. Let $R_i \subset R$ range over the finitely generated $\mathbb{Z}$-subalgebras containing the coefficients in $\underline{f}$. Writing $S_i = R_i[\underline{x}]/(\underline{f})$, it follows that $R = \varinjlim_i R_i$ and $S = \varinjlim_i S_i$ by exactness of filtered colimits. TODO: Finish this annoying proof; rough steps:

1. $S_j = R_j \otimes_{R_i} S_i$ because polynomials. Fix $\mathfrak{q} \in \operatorname{Spec} S$, $\mathfrak{p} = \mathfrak{q} \cap R$, and $\mathfrak{q}_i = \mathfrak{q} \cap S_i$, $\mathfrak{p}_i = R \cap \mathfrak{p}_i$. $D_i = (R_i)_{\mathfrak{p}_i}$ and $E_i = (S_i)_{\mathfrak{q}_i}$, then $\varinjlim_i D_i = R_{\mathfrak{p}}$ and $\varinjlim_i E_i = S_{\mathfrak{q}}$.
2. $\mathfrak{m}_i \subset D_i$ maximal, $K_i = \operatorname{Tor}_1^{D_i}(E_i, D_i/\mathfrak{m}_i)$ is finitely generated over E_i by Noetherianity. Note $K_i = \ker(\mathfrak{m}_i \otimes_{D_i} E_i \to E_i)$. The generators map to zero in $\mathfrak{m}_i D \otimes_D E$, hence map to zero in $\mathfrak{m}_i D_j \otimes_{D_j} E_j$ for some $j \geq i$.
3. Then show $K_i \otimes_{E_i} E_j \to \operatorname{Tor}_1^{D_j}(E_j, D_j/\mathfrak{m}_i D_j)$ is surjective. By construction of j, this tor module vanishes. Moreover the closed fiber $E_j/\mathfrak{m}_i E_j$ is flat over $D_j/\mathfrak{m}_i D_j$, so apply flat local criterion to deduce that E_j is flat over D_j.
4. Apply openness of the flat locus, and more... Remark: there's a somewhat easier route if we just want to show that it is étale local, using a standard-étale presentation.

□

Proposition A.1.78. *Consider the following properties of morphisms of schemes.*

(a) Being flat is fppf-local on the source.
(b) Being locally of finite type (resp. locally of finite presentation) is fppf-local on the source.
(c) Being étale is étale-local on the source.
(d) Being smooth is étale-local on the source.

In particular, being flat and locally of finite presentation is fppf-local on the source.

PROOF. (a) By joint surjectivity of an fppf cover, this is due to the following fact: if $S \to R$ is a local ring map, $R \to R'$ is a (faithfully) flat local ring map, and $R' \to S$ is flat, then $S \to R$ is flat. If $M \hookrightarrow N$ and we want to check $M \otimes_S R \to N \otimes_S R$ is injective, we can do this after base change to R'.

(c) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ where f is faithfully flat étale and gf is étale. Since f is formally smooth,

$$0 \to f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z} \to \Omega^1_{X/Y} \to 0$$

is exact, thus by assumption $f^*\Omega^1_{Y/Z} = 0$. Since f is faithfully flat, $\Omega^1_{Y/Z} = 0$. Since being locally of finite presentation and flat are étale local on the source, we conclude that étale is étale-local on the source.

(b) Let the composite $X \to Y \to Z$ be locally of finite type/presentation with $X \to Y$ an fppf cover. Since everything is local on the target, we may assume Z is affine. Taking an affine open $U \subset Y$ and considering the composite $X \times_Y U \to U \to Z$, we reduce to the case that Y is affine. Finally, since Y is quasicompact and $X \to Y$ is open, we can precompose with a flat, locally of finite

presentation map $X' \to X$ from an affine scheme such that $X' \to X \to Y$ is fppf. Thus we may assume X is affine as well.

So consider $A \to B \to C$ ring maps whose composite is of finite type/presentation, and $B \to C$ is faithfully flat of finite presentation. We want to show that $A \to B$ is of finite type/presentation.

(a) Finite type case: The generators of C over A are also generators of B, so write $C = B[x_0, \ldots, x_n]/(f_1, \ldots, f_r)$ where the images of the x_i are A-generators. By the lemma, let $R_0 \subset B$ be a finitely generated $\mathbb{Z}$-subalgebra and a faithfully flat R_0-algebra $S_0 = R_0[\underline{x}]/(\underline{f})$, so $C = B \otimes_{R_0} S_0$. Writing $B_0 = A \otimes_{\mathbb{Z}} R_0$ and $C_0 = A \otimes_{\mathbb{Z}} S_0$, both finitely presented over A, we observe that $B_0 \to C_0$ is faithfully flat. Since $C_0 \to C$ is surjective by construction, we deduce that $B_0 \to B$ is surjective, thus B is finite type over A.

(b) Keep the notation above, and assume in addition that C is finitely presented over A. Write $B = B_0/K$, then it suffices to show that K is finitely generated. Then $C = C_0/KC_0$, and since C and C_0 are finitely presented over A we deduce that KC_0 is finitely generated. Choose generators of this module, and let $K' \subset K$ be the ideal generated by the finitely many elements of K involved in the generator expressions. Then $K'C_0 = KC_0$, so by faithful flatness we deduce $K' = K$ is finitely generated.

(d) Assume that $X \to Y \to Z$ are morphisms of schemes where $X \to Y$ is surjective étale and $X \to Z$ is smooth. Then $Y \to Z$ is flat and locally of finite presentation by (a) and (b), so it remains to show that $Y \to Z$ has geometrically regular fibers. For each $z \in Z$ and field extension $K \supset \kappa(z)$, we have the base change diagram:

$$\begin{array}{ccccc} X_z \times_{\kappa(z)} K & \longrightarrow & Y_z \times_{\kappa(z)} K & \longrightarrow & \operatorname{Spec} K \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z \end{array}$$

Then $(X_z)_K \to (Y_z)_K$ is a faithfully flat map of locally Noetherian schemes with regular source, so $(Y_z)_K$ is regular by Lemma 3.1.36. □

Here is how assuming locality on the source gives us a new kind of definition of morphism for algebraic spaces.

Lemma A.1.79. *Let $\mathcal{P}$ be a property of morphisms of schemes which is stable under base change and étale (resp. fppf) local on the source and target. Then the following conditions are equivalent for a morphism $f\colon X \to Y$ of algebraic spaces:*

(a) There exists a representable étale (resp. fppf) surjection $U \to X$ from a scheme such that $U \to X \to Y$ is representable $\mathcal{P}$.

(b) For every representable étale (resp. fppf) surjection $U \to X$ from a scheme, $U \to X \to Y$ is representable $\mathcal{P}$.

(c) There exist representable étale (resp. fppf) surjections $V \to Y$ and $U \to X \times_Y V$ from schemes such that $U \to V$ satisfies $\mathcal{P}$.

(d) For every representable étale (resp. fppf) surjections $V \to Y$ and $U \to X \times_Y V$ from schemes, $U \to V$ satisfies $\mathcal{P}$.

PROOF. $(b) \Rightarrow (a)$ and $(d) \Rightarrow (c)$. Assume (c), and suppose that $U \to X$ is as in (b). We have already shown that to check $U \to Y$ is representable $\mathcal{P}$ it suffices to check on a single étale (resp. fppf) cover of Y. Let $V \to Y$ and $W \to X_V$ be the maps in (c), and consider the diagram:

$$\begin{array}{ccccc} U \times_X W & \longrightarrow & W & & \\ \downarrow & \lrcorner & \downarrow & & \\ U \times_Y V & \longrightarrow & X_V & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & X & \longrightarrow & Y \end{array}$$

By our assumption, $U \times_X W \to U \times_Y V$ is an étale (resp. fppf) surjection of schemes, as is $U \times_Y V \to X_V$ and $U \times_X W \to W$. Since $W \to X_V \to V$ is $\mathcal{P}$, étale (resp. fppf) locality on the source implies that

$$(U \times_X W \to W \to X_V \to V) = (U \times_X W \to U \times_Y V \to X_V \to V)$$

is a $\mathcal{P}$ morphism of schemes. Another application of locality on the source implies that $U \times_Y V \to V$ satisfies $\mathcal{P}$, which proves (b).

Now assume (a), i.e. that $U \to X$ is an étale (resp. fppf) surjection for which $W \to X \to Y$ is representable $\mathcal{P}$. Assume $V \to Y$ and $W \to X \times_Y V$ are as in (d); we use the same diagram as above. We are given that $U \times_Y X_V \to V$ satisfies $\mathcal{P}$; since $W \to X_V$ and hence $U \times_X W \to U \times_Y V$ are étale (resp. fppf) surjections, locality implies that the composites

$$U \times_X W \to V$$

satisfy $\mathcal{P}$ in either direction, thus $U \times_X W \to W \to X_V \to V$ does. Thus by locality on the source, we conclude $W \to X_V \to V$ satisfies $\mathcal{P}$. □

Definition A.1.80. Let $\mathcal{P}$ be a property of morphisms satisfying the hypotheses of Lemma A.1.79. We say a morphism $f\colon X \to Y$ of algebraic spaces *satisfies* $\mathcal{P}$ if one of the equivalent conditions of the lemma holds with respect to $\mathcal{P}$.

Given this definition and the previous lemma, we can now make sense of an arbitrary morphism of algebraic spaces $f\colon X \to Y$ being e.g. smooth, étale, étale surjective, and so on.

Example A.1.81. Let $\mathcal{P}$ be one of the properties defined above for algebraic spaces, and assume that $\mathcal{P}_{\text{scheme}}$ is closed under composition. Then:

1. If $X \to Y$ is representable, then it is $\mathcal{P}$ iff it is representable $\mathcal{P}$. This is clear from (a),(b), and using locality on the source.
2. $\mathcal{P}$ is stable under base change. E.g., use Lemma A.1.79(a) and construct $U \times_Y Y' \to X \times_Y Y' \to Y'$ which is representable $\mathcal{P}$ and $U \times_Y Y' \to X \times_Y Y'$ is a representable étale surjection.

3. $\mathcal{P}$ is stable under composition. Indeed, let $X \to Y \to Z$ both individually be $\mathcal{P}$, choose $U \to Y \to Z$ representable $\mathcal{P}$ from a scheme U with $U \to Y$ an étale cover, then $X_U \to U$ satisfies $\mathcal{P}$ by (b). Thus choose $V \to X_U$ an étale cover with $V \to U$ satisfying $\mathcal{P}$ of schemes, then $V \to X \to Y \to Z$ is representable $\mathcal{P}$ and $V \to X$ is an étale cover by base change.

At this point, since we now have independent notions of fppf and étale, we can define the properties of being étale (resp. fppf) local on the source and target for morphisms of algebraic spaces. Thus some more remarks:

4. Let $\mathcal{Q}$ be a property of a pullback-stable class of morphisms of algebraic spaces which is local on the target with respect to representable étale/fppf surjections from schemes (for instance, representable $\mathcal{P}$ among the class of representable maps where $\mathcal{P}$ is étale/fppf local for schemes). Let $X \to Y$ be a morphism (in the relevant class) and $Y' \to Y$ an étale/fppf surjection for which $X \times_Y Y' \to Y'$ satisfies $\mathcal{P}$. Then by choosing $V \to Y'$ from a scheme, pulling back shows that $X \times_Y V \to V$ satisfies $\mathcal{Q}$, and $V \to Y' \to Y$ is representable étale/fppf from a scheme by (1), (3). Thus $X \to Y$ satisfies $\mathcal{Q}$, i.e. $\mathcal{Q}$ is étale/fppf-local on the target.
5. Let $\mathcal{P}$ be as in the rest of this example. Then $\mathcal{P}$ is étale/fppf-local on the target; suppose we are given a base change diagram on the right below

$$\begin{array}{ccccc} U' & \dashrightarrow & X' & \longrightarrow & Y' \\ \vdots & \lrcorner & \downarrow & \lrcorner & \downarrow \\ U & \dashrightarrow & X & \longrightarrow & Y \end{array}$$

   where $X' \to Y'$ satisfies $\mathcal{P}$ and $Y' \to Y$ is an étale/fppf surjection. Choose $U \to X$ a representable étale surjection from a scheme, then $U' \to Y'$ is representable $\mathcal{P}$ by Lemma A.1.79, so $U \to X \to Y$ is representable $\mathcal{P}$ by (4), hence $X \to Y$ is $\mathcal{P}$ by definition.
6. $\mathcal{P}$ is also étale/fppf-local on the source: Suppose $X' \to X$ is an étale/fppf surjection and $X' \to X \to Y$ satisfies $\mathcal{P}$; choose an étale surjection $U \to X'$ from a scheme so that $U \to X' \to Y$ is representable $\mathcal{P}$, i.e. $U \to X \to Y$ is representable $\mathcal{P}$, meaning $X \to Y$ is $\mathcal{P}$.

For instance, (4) applies to isomorphisms by Prop. A.1.75(d). It also applies to properties defined using the diagonal: if $X' \to Y'$ is a base change of $X \to Y$, then so too is $X' \to X' \times_{Y'} X'$ of $X \to X \times_Y X$, and moreover both are representable so it suffices to see that $X' \times_{Y'} X' \to X \times_Y X$ is étale/fppf if $Y' \to Y$ is. Indeed, the former is the base change of the latter.

Example A.1.82. We want to use Example A.1.81 to conclude that representable $\mathcal{P}$ is étale/fppf-local on the target, but the problem is that representability does not satisfy fppf-descent. Instead, let $\mathcal{C} \subset \mathbf{Sch}$ be a fibered subcategory of schemes satisfying fppf-effective descent. Then viewing $\mathcal{C}$ as a property of morphisms of schemes, $\mathcal{C}$ is fppf-local on the target.

Now suppose $X \to Y$ is a morphism of spaces and $U \to Y$ is an fppf cover from a scheme so that $X_U \to U$ is representable $\mathcal{C}$. For each scheme $T \to Y$,

$X_U \times_Y T \to U \times_Y T$ is representable $\mathcal{C}$, and is the base change of $X \times_Y T \to T$. Since $U \times_Y T \to T$ is an fppf cover of schemes, Lemma A.1.62 implies that $X \times_Y T$ is a scheme and $X \times_Y T \to T$ is $\mathcal{C}$. Thus (4) above now does show that representable $\mathcal{C}$ is indeed fppf-local on the target for algebraic spaces. We will see below that fpqc-effective descent implies fppf-effective descent, so this in particular applies to affine morphisms, quasiaffine morphisms, open immersions, etc.

Example A.1.83. There's a short list of properties of morphisms of algebraic spaces we defined that aren't covered by one of the two examples above, which are the ones depending on the underlying space. First, note that quasicompact is already fppf-local by Prop. A.1.74(d) and (4) above. Surjectivity is surjective-local hence fppf-local. By the diagonal case, this means that radicial is fppf-local. Moreover, a representable fppf surjection from a scheme is in particular universally open and hence induces a quotient map on spaces. Thus the same proof as in Lemma A.1.17 alongside Prop. A.1.71 shows that universally closed and open are fppf local.

Remark A.1.84. Knowing that our favorite properties of morphisms of algebraic spaces are fppf/étale-local on the source or target is comparatively more useful than the corresponding statement for schemes. This is because we don't have good control over open covers of an algebraic space, but we do have good control over étale covers, since an algebraic space is étale-locally a scheme.

Example A.1.85. Let X be an algebraic space, and suppose that X has an open cover $\{U_i\}$ by schemes. Then X is a scheme. Indeed, let U_{ij} be the scheme $U_i \times_X U_j$, and since $U_i \to X$ is an open immersion it follows that $U_{ij} \to U_i$ is an open immersion. Let $\varphi_{ij} \colon U_{ij} \xrightarrow{\sim} U_{ji}$ be the isomorphism induced by the isomorphism $U_i \times_X U_j \xrightarrow{\sim} U_j \times_X U_i$ of fppf sheaves, which corresponds to an isomorphism of schemes. This satisfies the cocycle condition, so by Lemma A.1.5 we obtain a scheme Y equipped with open immersions $U_i \hookrightarrow Y$ and isomorphisms $U_i \times_Y U_j \xrightarrow{\sim} U_i \times_X U_j$ commuting with the cocycle isomorphisms. Treating $\{U_i \to Y\}$ as an fppf cover, the maps $U_i \to X$ induce $Y \to X$, such that base changing along each U_i produces the identification $\mathrm{Id} \colon U_i \to U_i$. By Prop. A.1.75, this implies $Y \xrightarrow{\sim} X$.

In the remainder of this section, we will briefly discuss epimorphisms and universal effective epimorphisms, fppf-effective descent for quasi-finite separated morphisms of schemes, and finally that étale quotients of schemes are algebraic spaces. The first part of this is general and holds in any sheaf category. So, let $\mathcal{C}$ be a site (with fiber products, so we may use the description with covering families), and denote by $\mathrm{Sh}(\mathcal{C})$ the corresponding sheaf topos of sets.

Lemma A.1.86. *Let $f \colon F \to G$ be a morphism in $\mathrm{Sh}(\mathcal{C})$. The following are equivalent:*

(a) f is locally surjective, meaning that for each $T \in \mathcal{C}$ and $s \in G(T)$, there exists a covering family $\{T_i \to T\}$ such that $s|_{T_i}$ is contained in the image of $F(T_i) \to G(T_i)$ for each i.

(b) f is an epimorphism in $\mathrm{Sh}(\mathcal{C})$.
(c) f is a universally effective epimorphism in $\mathrm{Sh}(\mathcal{C})$.

PROOF. (a) and (b) are equivalent by standard sheaf theory. It remains to show the following two claims: local surjections are effective epis, and local surjections are stable under base change.

1. Effective epi: We have to show the existence part of the defining coequalizer

$$F \times_G F \underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}} F \to G.$$

Let $g\colon F \to H$ be a map such that $g\pi_1 = g\pi_2$. Let $s \in G(T)$, and lift $s|_{T_i}$ to $r_i \in F(T_i)$ for a cover $\{T_i \to T\}$. The assumption says that for any $t, t' \in F(T')$ such that $f(t) = f(t')$, we also have $g(t) = g(t')$. The restrictions of r_i, r_j to $T_i \times_T T_j$ have the same image s in $G(T_i \times_T T_j)$, so $g(r_i) \equiv g(r_j)$ on $T_i \times_T T_j$. By the sheaf condition there is a unique section $\widetilde{g}(s) \in H(T)$ restricting to $g(r_i)$. Here's what we have to check:

1. The section $\widetilde{g}(s)$ is independent of the cover. Indeed if $\{T'_j \to T\}$ is another cover on which local lifts exist, then the resulting section will have to agree on $T_i \times_T T'_j$ with $\widetilde{g}(s)$, and this forms an open cover. Therefore, we obtain a well-defined map of sets $\widetilde{g}\colon G(T) \to H(T)$.
2. As T varies, $\widetilde{g}$ is a morphism of sheaves. Indeed, commutativity of the naturality diagram is checked sectionwise; then this is further checked on a covering family where we already know it holds since g is a morphism.
3. Finally, $g = \widetilde{g}f$. This is also checked sectionwise and locally, where it is true by definition.

Thus we have found the requisite extension to $G \to H$.

2. Stable under base change: Let $g\colon G' \to G$ be a map and consider $F' = F \times_G G'$, which is computed as a presheaf. Let $s \in G'(T)$, consider $g(s) \in G(T)$ and choose a cover $\{T_i \to T\}$ such that $f(r_i) = g(s)|_{T_i}$ for $r_i \in F(T_i)$. Thus $(r_i, s|_{T_i})$ defines an element of $F'(T_i)$, which projects to $s|_{T_i}$. □

Example A.1.87. Suppose $F \to G$ is a representable fppf morphism of fppf presheaves. For each scheme $T \to G$, the fiber product $T' = F \times_G T$ is a scheme and $T' \to T$ is fppf. Thus the section in $G(T)$ lifts along the cover $\{T' \to T\}$ to a section in $F(T')$. Therefore $F \to G$ is locally surjective, hence a universal effective epimorphism.

Now let $X \to Y$ be an fppf morphism of algebraic spaces. Choose $U \to X \to Y$ a representable fppf map, which is a universal effective epimorphism. By Prop. A.1.45 we deduce that $X \to Y$ is a universal effective epimorphism in $\mathrm{Sh}(\mathbf{Sch}_S^{\mathrm{fppf}})$. In particular, it is also clearly a universal effective epimorphism in $\mathbf{AlgSp}_S$, because this subcategory is closed under pullbacks.

Next we discuss equivalence relations in sheaf categories. Recall the following definition:

Definition A.1.88. Let $\mathcal{D}$ be a category with pullbacks.

(a) A *groupoid in* $\mathcal{D}$ is a pair of morphisms $s, t\colon R \rightrightarrows U$ equipped with maps

$$e\colon U \to R \qquad i\colon R \to R \qquad \mu\colon R \times_U R \to R$$

satisfying the following compatibility conditions: $se = te = 1$, $si = t$, $ti = s$, $t\mu = t\pi_1$ and $s\mu = t\pi_2$, where π_i are the pullback projections. Moreover, we have associativity, i.e. $\mu(\mu(h, g), f) = \mu(h, \mu(g, f))$, unitality $\mu(et, 1) = 1 = \mu(1, es)$, and the inverse law $\mu(i, 1) = es$, $\mu(1, i) = et$.

(b) If $R \rightrightarrows U$ is a groupoid in $\mathcal{D}$, then it is an *equivalence relation* if the natural map $R \to U \times U$ is a monomorphism.

Example A.1.89. Since the definition of a groupoid is completely equational except for the pullback projection compatibility, it is preserved by all functors which preserve pullbacks. Since such functors also preserve monomorphisms and products (assuming terminal object), they also preserve equivalence relations.

1. In the definition above, let $U = *$ be a terminal object. Then a groupoid is the same thing as a group object in $\mathcal{D}$, which follows directly from the definition since everything involving s and t now becomes trivial.
2. Suppose $\mathcal{D} = \mathbf{Set}$ and $R \rightrightarrows U$ is a groupoid. Define a category $\mathcal{C}$ with objects U, morphisms R, and s and t the source/target projections. For $u \in U$, we set $1_u = e(u)$. If $f\colon u \to v$ and $g\colon v \to w$ are morphisms in R, then $s(g) = t(f)$, so $(g, f) \in R \times_U R$ and we define $g \circ f \coloneqq \mu(g, f)$. The axioms make clear that this is a category, and the inversion shows that this is a groupoid. From this it is clear that the groupoids in **Set** are the small groupoids in the standard sense. Note that it becomes an equivalence relation iff there's at most one map between any two objects. Thus we can also call an equivalence relation a contractible groupoid.
3. Let S be a set and $R \subset S \times S$ a literal equivalence relation. Denote by $s, t\colon R \to S$ the two projections and i the restriction of the flipping isomorphism, well-defined by the symmetric property. Let $e\colon S \to R$ be the diagonal map, well-defined by the reflexive property. It is clear from the description in (2) that the map μ is uniquely determined, and that it in fact exists by the transitive property. It sends $\big((b, c), (a, b)\big) \mapsto (c, a)$. It is easy to see that the conditions in the definition follow, and hence that this is in fact an equivalence relation in **Set**.
4. Let $R \rightrightarrows U$ be a pair of maps in $\mathcal{D}$ and s, t with e, i, μ having the source and target as given in the definition above. It is clear from the description/Yoneda that this data forms a groupoid/equivalence relation, if and only if the same $\mathrm{Hom}_{\mathcal{D}}(-, R) \rightrightarrows \mathrm{Hom}_{\mathcal{D}}(-, U)$ forms one in $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathbf{Set})$, iff $\mathrm{Hom}_{\mathcal{D}}(T, R) \rightrightarrows \mathrm{Hom}_{\mathcal{D}}(T, U)$ forms one in **Set** for all objects $T \in \mathcal{D}$. Thus we can often reduce to the case of **Set**.
5. Let $f\colon X \to Y$ be a map in $\mathcal{D}$, and consider the pair of arrows $s, t\colon X \times_Y X \rightrightarrows X$. Then we have the diagonal map $e\colon X \to X \times_Y X$, flipping isomorphism $i\colon X \times_Y X \to X \times_Y X$, and projection map

$$(X \times_Y X) \times_X (X \times_Y X) = X \times_Y X \times_Y X \to X \times_Y X$$

to the outer two coordinates. This defines a canonical equivalence relation structure; indeed, we may reduce to the case of **Set** by (4), but then $X \times_Y X \subset X \times X$ defines precisely the relation $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$, so we are done by (3).

Below, we will give a partial converse statement to (5).

Lemma A.1.90. *Let $\mathcal{C}$ be a site and $s, t\colon R \rightrightarrows U$ be an equivalence relation in $\mathrm{Sh}(\mathcal{C})$. If $X = \mathrm{coeq}(R \rightrightarrows U)$, then the natural map $R \to U \times_X U$ is an isomorphism.*

Proof. Since evaluation at $T \in \mathcal{C}$ preserves pullbacks, $R(T) \subset U(T) \times U(T)$ is a set-theoretic equivalence relation for each T. The coequalizer X is the sheafification of $T \mapsto U(T)/R(T)$, and the map given is the inclusion

$$R(T) \subset U(T) \times_{X(T)} U(T).$$

Suppose $u, v \in U(T)$ have the same image in $X(T)$. There is a covering $\{T_i \to T\}$ such that the images of $u|_{T_i}$ and $v|_{T_i}$ coincide in $U(T_i)/R(T_i)$, i.e. these sections are equal modulo R. Thus $(u|_{T_i}, v|_{T_i}) \in R(T_i)$ for each i, which correspond to maps $r_i\colon T_i \to R$ for each i. Since $R(T) \subset U(T) \times U(T)$ is a subsheaf, this implies that $(u, v) \in R(T)$, so we conclude $R(T) = U(T) \times_{X(T)} U(T)$, i.e. $R \xrightarrow{\sim} U \times_X U$. $\square$

Remark A.1.91. Since every scheme has an fppf cover by affine schemes, it is easy to check that if A is a ring, then $\mathbf{Sch}_A^{\mathrm{fppf}}$ has a basis given by $\mathrm{Aff}_A^{\mathrm{fppf}}$, and thus $\mathrm{Sh}(\mathbf{Sch}_A^{\mathrm{fppf}}) = \mathrm{Sh}(\mathrm{Aff}_A^{\mathrm{fppf}})$. In particular, it is clear that we can check whether $R \rightrightarrows U$ is an equivalence relation by checking that $R(B) \rightrightarrows U(B)$ is an equivalence relation for each A-algebra B.

Example A.1.92. Let us give our first example of an algebraic space which isn't a scheme. Let k be a field, $U = \mathbb{A}^1_k$, and $R = \mathbb{A}^1 \sqcup \mathbb{G}_m$, both affine schemes. We have maps $a, b\colon R \rightrightarrows U$, where $a = (1, \iota)$ and $b = (1, -\iota)$. Let us see explicitly that this is an equivalence relation.

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